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ON THE COMMUTATORS OF SINGULAR INTEGRALS RELATED TO BLOCK SPACES

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Abstract. In this paper, the commutators of singular integrals with rough kernels are considered. By the method of block decomposition for kernel function and Fourier transform estimates, some new results about the $L^p(\mathbb{R}^n)$ boundedness for these commutators are obtained.

§1. Introduction

Let $\mathbb{R}^n$, $n \geq 2$, be the n-dimensional Euclidean space and S^{n-1} be the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let $\Omega(x)$ be a homogeneous function of degree zero and have mean value zero on S^{n-1}. Suppose that $h(t) \in L^\infty(0,\infty)$. Define the singular integral operator T by

$$Tf(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) f(y)dy. \tag{1.1}$$

For a positive integer k and $a(x) \in BMO(\mathbb{R}^n)$, define the k-th order commutator $T_{a,k}$ generated by T and a

$$T_{a,k}f(x) = T((a(x)-a(\cdot))^k f)(x), \ f \in C_0^\infty(\mathbb{R}^n). \tag{1.2}$$

It was proved by Coifman, Rochberg and Weiss [4] that if $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})\,(0 < \alpha \leq 1)$ and $h \equiv 1$, then $T_{a,1}$ is bounded on $L^p(\mathbb{R}^n)$ with bound $C\|a\|_{BMO(\mathbb{R}^n)}$ for $1 < p < \infty$. Afterwards, by a well-known result of Duoandikoetxea [6] and the boundedness criterion of Alvarez-Bagby-Kurtz-Pérez for the commutators of linear operator (see [2]), we have obtained the following theorem (see also [10]):

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Theorem A. ([6, 2, 10]) *Let* Ω, a, k *be as above and* $h \equiv 1$, $1 < p < \infty$. *If* $\Omega \in \cup_{q>1} L^q(S^{n-1})$, *then* $T_{a,k}$ *is bounded on* $L^p(\mathbb{R}^n)$.

Recently, to weaken the condition imposed on Ω, Hu Guoen et al. employed the method of Littlewood-Paley theory and Fourier transform estimates from [7] to obtain the following results.

Theorem B. *Let* Ω, a, k *be as above. Then* $T_{a,k}$ *is bounded on* $L^p(\mathbb{R}^n)$ *with bound* $C\|a\|^k_{BMO(\mathbb{R}^n)}$, *if one of the following conditions holds.*

(i) (see [12]). $p = 2$, $h \equiv 1$, $\Omega \in L(\log^+ L)^{k+1}(S^{n-1})$.

(ii) (see [9]). $p = 2$, $h \equiv 1$ *and for some* $\alpha > k + 1$, Ω *satisfies*

$$\sup_{\xi \in S^{n-1}} \int_{S^{n-1}} |\Omega(\theta)| \left(\log \frac{1}{|\theta \cdot \xi|}\right)^{\alpha} d\theta. \tag{1.3}$$

(iii) (see [13] or [9]). *For some* $\alpha > k+1$, $\Omega \in L(\log^+ L)^{\alpha}(S^{n-1})$ *and for some* $s > 1$, h *satisfies* $\sup_{R>0} \int_R^{2R} |h(r)|^s r^{-1} dr < \infty$, $2\alpha/(2\alpha - (k+1)) < p < 2\alpha/(k+1)$ *or* $p = 2$.

Theorem B certainly improve Theorem A since both the condition $\Omega \in L(\log^+ L)^{\alpha}(S^{n-1})$ $(\alpha > k+1)$ and the size condition (1.3) are properly weaker than the condition $\Omega \in \bigcup_{q>1} L^q(S^{n-1})$. Unfortunately, the condition on Ω in Theorem B greatly depends on the order k of $T_{a,k}$. It is natural to ask whether there exists a weaker size condition on Ω, which is independent of k, such that $T_{a,k}$ is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. The main purpose of this paper is to give a positive answer to this problem. Inspired by [1], we shall show that $T_{a,k}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, if $\Omega \in B_q^{0,0}(S^{n-1})$ for some $q > 1$. Here $B_q^{0,0}(S^{n-1})$ denotes certain block spaces introduced by Jiang and Lu(see [15]). We remark that some ideas in the proof of our main results are taken from [7] and [11]. Before stating the main results, we briefly review some pertinent concepts.

Definition 1. ([15]) A q-block on S^{n-1} is an $L^q(1 < q \leq \infty)$ function $b(\cdot)$ that satisfies

$$\text{(i)}\ \ \operatorname{supp}(b) \subseteq Q, \qquad\qquad \text{(ii)}\ \ \|b\|_{L^q(S^{n-1})} \leq |Q|^{\frac{1}{q}-1},$$

where $Q = S^{n-1} \cap \{y \in \mathbb{R}^n : \ |y - \varsigma| < \rho \ \ for \ \ some \ \ \varsigma \in S^{n-1} \ \ and \ \ \rho \in (0, 1]\}$.

Definition 2. ([15]) The block spaces $B_q^{0,0}$ on S^{n-1} are defined by

$$B_q^{0,0}(S^{n-1}) = \{\Omega \in L^1(S^{n-1}) : \Omega(y') = \sum_s C_s b_s(y'),\ M_q^{0,0}(\{C_s\}) < \infty\},$$

where each C_s is a complex number, each b_s is a q-block supported in Q_s, and

$$M_q^{0,0}(\{C_s\}) = \sum_s |C_s| \left\{1 + \log^+ \frac{1}{|Q_s|}\right\}.$$

It should be pointed out that the method of block decomposition for functions was invented by Taibleson and Weiss [17] in the study of the convergence of the Fourier series. Later on, many application of the block decomposition to harmonic analysis were discovered (see [1], [14]–[16] etc.). For further background and information about the theory of spaces generated by blocks and its applications to harmonic analysis, one can consult the book [15]. In [14], Keitoku and Sato showed that for any $q > 1$,

$$\bigcup_{r>1} L^r(S^{n-1}) \subset B_q^{0,0}(S^{n-1}),$$

which is a proper inclusion. And from [14], we easily see that $B_q^{0,0}(S^{n-1})$ is not contained in $L(\log^+ L)^{1+\varepsilon}(S^{n-1})$ for any $\varepsilon > 0$ although the relationship between $B_q^{0,0}(S^{n-1})$ and $L\log^+ L(S^{n-1})$ remains open.

Definition 3. ([3]) A locally integrable function $a(x)$ will be said to belong to $BLO(\mathbb{R}^n)$, if there is a constant C such that for any cube Q

$$m_Q(a) - \inf_{x\in Q} a(x) \leq C,$$

where $m_Q(a) = |Q|^{-1}\int_Q a(x)dx$.

If $a \in BLO(\mathbb{R}^n)$, then we denote $\|a\|_{BLO(\mathbb{R}^n)} = \sup_Q \{m_Q(a) - \inf_{x\in Q} a(x)\}$.

Obviously, $L^\infty(\mathbb{R}^n) \subset BLO(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$ and if $a \in BLO(\mathbb{R}^n)$, then

$$\|a\|_{BMO(\mathbb{R}^n)} \leq 2\|a\|_{BLO(\mathbb{R}^n)}. \tag{1.4}$$

Now let us formulate our main results.

Theorem 1. *Let Ω be homogeneous of degree zero and have mean value zero, k be a positive integer and $a \in BMO(\mathbb{R}^n)$. If $h(t) \in L^\infty(0,\infty)$ and $\Omega \in B_q^{0,0}(S^{n-1})$ for $q > 1$, then the commutator $T_{a,k}$ is bounded on $L^2(\mathbb{R}^n)$ with bound $C\|a\|^k_{BMO(\mathbb{R}^n)}$.*

For the case of $p \neq 2$, $1 < p < \infty$, we need to impose some restrictions on BMO functions $a(x)$ as follows.

Theorem 2. *Let Ω, h, k be as in Theorem 1, $1 < p < \infty$. If $a \in BLO(\mathbb{R}^n)$ and $a(x)$ is subharmonic, then $T_{a,k}$ is bounded on $L^p(\mathbb{R}^n)$ with bound $C\|a\|^k_{BLO(\mathbb{R}^n)}$.*

Remark 1. It is worth pointing out that a BMO function $a(x)$ satisfying the restrictive conditions in Theorem 2 exists. A typical example is $\log|x|$.

Remark 2. $\bigcup_{r>1} L^r(S^{n-1})$ is properly contained in $B_q^{0,0}(S^{n-1})$ for any $q > 1$, and $B_q^{0,0}(S^{n-1})$ is independent of the order of $T_{a,k}$ and is not contained in $L(\log^+ L)^\alpha(S^{n-1})$ $(\alpha > 1)$. Therefore our theorems are an essential improvement on Theorem A and an great extension of the result in Theorem B.

In proving Theorem 2, we shall use the following L^p-boundedness of $M^\Omega_{a,k}$, a maximal operator related to higher order commutators, defined by

$$M^\Omega_{a,k} f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |a(x) - a(y)|^k |h(|x-y|)\Omega(x-y) f(y)| dy.$$

Theorem 3. *Under the same hypothesis as in Theorem 2, the operator $M^\Omega_{a,k}$ satisfies*

$$\|M^\Omega_{a,k} f\|_p \leq C\|a\|^k_{BLO(\mathbb{R}^n)} \|f\|_p.$$

Throughout this paper, C always denotes positive constants that are independent of the essential variables but whose value may vary at each occurrence.

§2. Proof of Theorem 1

Let us begin with some preliminary lemmas.

LEMMA 1. ([11]) *Let* $\phi \in C_0^\infty(\mathbb{R}^n)$ *be a radial function such that* $\operatorname{supp}\phi \subset \{1/4 \le |\xi| \le 4\}$ *and*

$$\sum_{l\in\mathbb{Z}} \phi^3(2^{-l}\xi) = 1, \ |\xi| \neq 0.$$

Denote by S_l *the multiplier operator*

$$\widehat{S_l f}(\xi) = \phi(2^{-l}\xi)\hat{f}(\xi),$$

and $S_l^2 f(x) = S_l(S_l f)(x)$*. For any positive integer* k *and* $a \in BMO(\mathbb{R}^n)$*, consider the* k*-th order commutator of* S_l *and* S_l^2*, respectively, defined by*

$$S_{l;a,k} f(x) = S_l((a(x) - a(\cdot))^k f)(x), \qquad f \in C_0^\infty(\mathbb{R}^n)$$

and

$$S^2_{l;a,k} f(x) = S_l^2((a(x) - a(\cdot))^k f)(x), \qquad f \in C_0^\infty(\mathbb{R}^n).$$

Then for all $1 < p < \infty$*,*

(a) $$\left\| \left(\sum_{l\in\mathbb{Z}} |S_{l;a,k} f|^2 \right)^{\frac{1}{2}} \right\|_p \le C\|a\|^k_{BMO(\mathbb{R}^n)} \|f\|_p;$$

(b) $$\left\| \left(\sum_{l\in\mathbb{Z}} |S^2_{l;a,k} f|^2 \right)^{\frac{1}{2}} \right\|_p \le C\|a\|^k_{BMO(\mathbb{R}^n)} \|f\|_p;$$

(c) $$\left\| \sum_{l\in\mathbb{Z}} S_{l;a,k} f_l \right\|_p \le C(n,k,p)\|a\|^k_{BMO(\mathbb{R}^n)} \left\| \left(\sum_{l\in\mathbb{Z}} |f_l|^2 \right)^{\frac{1}{2}} \right\|_p,$$

$$f_l \in C_0^\infty(\mathbb{R}^n)(l \in \mathbb{Z}).$$

LEMMA 2. ([11]) *Let* $0 < \delta < \infty$*, and take a function* $m_\delta \in C_0^\infty(\mathbb{R}^n)$ *with support contained in* $\{\xi \in \mathbb{R}^n : |\xi| \le \delta\}$*. Suppose that for some positive constant* α*,*

$$\|m_\delta\|_\infty \le C \min\{\delta^\alpha, \delta^{-\alpha}\}, \quad \|\nabla m_\delta\|_\infty \le C.$$

Let T_δ *be the multiplier operator defined by*

$$\widehat{T_\delta f}(\xi) = m_\delta(\xi)\hat{f}(\xi).$$

For a positive integer k and $a \in BMO(\mathbb{R}^n)$, let $T_{\delta;a,k}$ be the k-th order commutator of T_δ. Then for any fixed $0 < v < 1$, there exists a positive constant $C = C(n,k,v)$ such that

$$\|T_{\delta;a,k}f\|_2 \le C \min\{\delta^{\alpha v}, \delta^{-\alpha v}\}\|a\|^k_{BMO(\mathbb{R}^n)}\|f\|_2.$$

LEMMA 3. *Let $\Omega(x') = \sum_s C_s b_s(x')$, $h(t)$ be as in Theorem 1. For $j \in \mathbb{Z}$, set*

$$K_j(x) = \frac{\Omega(x)}{|x|^n} h(|x|)\chi_{\{2^j \le |x| < 2^{j+1}\}}(x),$$

$$K_{j,s}(x) = \frac{b_s(x)}{|x|^n} h(|x|)\chi_{\{2^j \le |x| < 2^{j+1}\}}(x),$$

and $m_j(\xi) = \widehat{K_j}(\xi)$, $m_{j,s}(\xi) = \widehat{K_{j,s}}(\xi)$. Then we have

(i) $|m_j(\xi)| \le C|2^j\xi|$;

(ii) $|m_{j,s}(\xi)| \le |2^j\xi|^{\frac{1}{2\log|Q_s|}}$, *if* $|Q_s| < e^{\frac{q}{1-q}}$;

(iii) $|m_{j,s}(\xi)| \le C|2^j\xi|^{-\omega}$, *if* $|Q_s| \ge e^{\frac{q}{1-q}}$.

Here C and ω are positive constants independent of j, s, ξ and b_s.

Proof. By the mean zero property and the integrability of Ω on S^{n-1}, we have

$$\begin{aligned}
|m_j(\xi)| &= \left| \int_{2^j \le |y| < 2^{j+1}} h(|y|)|y|^{-n}\Omega(y')e^{-2\pi i y\cdot\xi}dy \right| \\
&= \left| \int_{2^j}^{2^{j+1}} h(t)t^{-1} \int_{S^{n-1}} \Omega(y')e^{-2\pi i t y'\cdot\xi} d\sigma(y')dt \right| \\
&\le C \int_{2^j}^{2^{j+1}} t^{-1} \left| \int_{S^{n-1}} \Omega(y')(e^{-2\pi i t y'\cdot\xi} - 1)d\sigma(y') \right| dt \\
&\le C \int_{2^j}^{2^{j+1}} t^{-1} \int_{S^{n-1}} |\Omega(y')||2\pi t y'\cdot\xi| d\sigma(y')dt \\
&\le C\|\Omega\|_{L^1(S^{n-1})}|\xi| \int_{2^j}^{2^{j+1}} dt \le C|2^j\xi|.
\end{aligned}$$

Thus, (i) is proved. (ii) and (iii) are the special cases of (ii) and (iii) Lemma 2.2 in [1]. The proof of Lemma 3 is complete. ∎

Proof of Theorem 1. For $j \in \mathbb{Z}$, let $K_j(\xi)$, $m_j(\xi)$ be as in Lemma 3 and ϕ be as in Lemma 1. Define the multiplier operator S_l by

$$\widehat{S_l f}(\xi) = \phi(2^{-l}\xi)\hat{f}(\xi).$$

Set $m_j^l(\xi) = m_j(\xi)\phi(2^{j-l}\xi)$ and $\widehat{T_j^l f}(\xi) = m_j^l(\xi)\hat{f}(\xi)$. Let

$$U_l f(x) = \sum_{j\in\mathbb{Z}} \left(\left(S_{l-j} T_j^l S_{l-j} \right)_{a,k} f \right)(x).$$

We know from [11] that for f, $g \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} g(x) T_{a,k} f(x) dx = \int_{\mathbb{R}^n} g(x) \sum_{l\in\mathbb{Z}} U_l f(x) dx.$$

Hence

$$\|T_{a,k} f\|_2 \le \sum_{l\in\mathbb{Z}} \|U_l f\|_2. \tag{2.1}$$

With the aid of the formula

$$(a(x) - a(y))^k = \sum_{m=0}^{k} C_k^m \left(a(x) - a(z)\right)^m \left(a(z) - a(y)\right)^{k-m}, \; x, \, y, \, z \in \mathbb{R}^n,$$

we get

$$\begin{aligned} &\int_{\mathbb{R}^n} g(x) U_l f(x) dx \\ &\qquad = \sum_{m=0}^{k} C_k^m \int_{\mathbb{R}^n} g(x) \sum_{j\in\mathbb{Z}} S_{l-j;a,k-m} \left(\left(T_j^l S_{j-l} \right)_{a,m} f \right)(x) dx, \end{aligned}$$

for f, $g \in C_0^\infty(\mathbb{R}^n)$ by a straightforward computation.

By Lemma 1(c), we get

$$\begin{aligned} \|U_l f\|_2 &\le C \sum_{m=0}^{k} \left\| \sum_{j\in\mathbb{Z}} S_{j-l;a,k-m} \left(\left(T_j^l S_{l-j} \right)_{a,m} f \right) \right\|_2 \\ &\le C \sum_{m=0}^{k} \|a\|_{BOM(\mathbb{R}^n)}^{k-m} \left\| \left(\sum_{j\in\mathbb{Z}} \left| \left(T_j^l S_{l-j} \right)_{a,m} f \right|^2 \right)^{\frac{1}{2}} \right\|_2 . \end{aligned} \tag{2.2}$$

Case 1. We first consider the L^2-boundedness of U_l for $l \le 0$.

Let $\tilde{T}_j^l$ be the operator defined by

$$\widehat{\tilde{T}_j^l f}(\xi) = m_j^l(2^{-j}\xi)\hat{f}(\xi).$$

By the vanishing moment and the integrability of Ω, we have

$$|\widehat{K_j}(\xi)| \le C|2^j\xi|, \qquad \|\nabla \widehat{K_j}\|_\infty \le C2^j.$$

Thus

$$\|m_j^l(2^{-j}\cdot)\|_\infty \le C2^l, \qquad \|\nabla m_j^l(2^{-j}\cdot)\|_\infty \le C.$$

Using this and Lemma 2, we obtain that for any fixed $0 < v < 1$ and positive integer i,

$$\|\tilde{T}_{j;a,i}^l f\|_2 \le C2^{vl}\|a\|_{BMO(\mathbb{R}^n)}^i \|f\|_2,$$

which by dilation-invariance implies

$$\|T_{j;a,i}^l f\|_2 \le C2^{vl}\|a\|_{BMO(\mathbb{R}^n)}^i \|f\|_2. \tag{2.3}$$

On the other hand, the Plancherel theorem tells us that

$$\|T_j^l f\|_2 \le C2^l \|f\|_2. \tag{2.4}$$

Observe that for $f,\, g \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} g(x)\left(T_j^l S_{l-j}\right)_{a,m} f(x)dx = \sum_{i=0}^m C_m^i \int_{\mathbb{R}^n} g(x) T_{j;,a,i}^l \left(S_{l-j;a,m-i}f\right)(x)dx.$$

It follows from (2.3), (2.4) and Lemma 1(a) that

$$\begin{aligned} &\left\|\left(\sum_{j\in\mathbb{Z}}\left|\left(T_j^l S_{l-j}\right)_{a,m} f\right|^2\right)^{\frac{1}{2}}\right\|_2^2 \\ &\le C\sum_{i=0}^m \left\|\left(\sum_{j\in\mathbb{Z}}\left|T_{j;a,i}^l \left(S_{l-j;a,m-i}f\right)\right|^2\right)^{\frac{1}{2}}\right\|_2^2 \\ &\le C2^{2vl}\sum_{i=0}^m \|a\|_{BMO(\mathbb{R}^n)}^{2i} \sum_{j\in\mathbb{Z}} \|S_{l-j;a,m-i}f\|_2^2 \\ &\le C2^{2vl}\|a\|_{BMO(\mathbb{R}^n)}^{2m}\|f\|_2^2, \qquad f \in C_0^\infty(\mathbb{R}^n). \end{aligned} \tag{2.5}$$

Therefore

$$\|U_l f\|_2 \le C 2^{vl} \|a\|^k_{BMO(\mathbb{R}^n)} \|f\|_2. \tag{2.6}$$

Case 2. Next we consider the L^2-estimate of U_l for $l > 0$.

Let $K_{j,s}$, $m_{j,s}$ be as in Lemma 3. Then $K_j(\xi) = \sum_s C_s K_{j,s}(\xi)$. Define the operator $T_j^{l,s}$ by

$$\widehat{T_j^{l,s} f}(\xi) = \widehat{K_{j,s}}(\xi)\phi(2^{j-l}\xi)\hat{f}(\xi).$$

Then

$$T_j^l f(\xi) = \sum_s C_s T_j^{l,s} f(\xi),$$

$$\left(T_j^l S_{l-j}\right)_{a,m} f(x) = \sum_s C_s \left(T_j^{l,s} S_{l-j}\right)_{a,m} f(x).$$

And

$$U_l f(x) = \sum_s C_s U_l^s f(x),$$

where

$$U_l^s f(x) = \sum_{j\in\mathbb{Z}} \left(S_{l-j} T_j^{l,s} S_{l-j}\right)_{a,k} f(x).$$

So

$$\|U_l f\|_2 \le \sum_s |C_s| \|U_l^s f\|_2. \tag{2.7}$$

Similarly to (2.2), we have

$$\|U_l^s f\|_2 \le C \sum_{m=0}^{k} \|a\|^{k-m}_{BMO(\mathbb{R}^n)} \left\| \left(\sum_{j\in\mathbb{Z}} \left| \left(T_j^{l,s} S_{l-j}\right)_{a,m} f \right|^2 \right)^{\frac{1}{2}} \right\|_2 . \tag{2.8}$$

In what follows, we estimate $\|U_l^s f\|_2$ for each s. Set

$$m_j^{l,s}(\xi) = \widehat{K_{j,s}}(\xi)\phi(2^{j-l}\xi) = m_{j,s}(\xi)\phi(2^{j-l}\xi).$$

And let $\bar{T}_j^{l,s}$ be the operator defined by

$$\widehat{\bar{T}_j^{l,s} f}(\xi) = m_j^{l,s}(2^{-j}\xi)\hat{f}(\xi).$$

By (ii) and (iii) of Lemma 3, we may assume, without loss of generality, that the support Q_s of b_s are uniformly small such that $|Q_s| < e^{\frac{q}{1-q}}$. Thus

$$|m_{j,s}(\xi)| = |\widehat{K_{j,s}}(\xi)| \leq C|2^j\xi|^{\frac{1}{2\log|Q_s|}}.$$

By a straightforward computation, we get

$$|\nabla m_{j,s}(\xi)| = |\nabla \widehat{K_{j,s}}(\xi)| \leq C2^j.$$

So

$$|m_j^{l,s}(2^{-j}\xi)| = |m_{j,s}(2^{-j}\xi)\phi(2^{-l}\xi)| \leq C2^{\frac{l}{2\log|Q_s|}}$$

and

$$|\nabla m_j^{l,s}(2^{-j}\xi)| = |\nabla(m_{j,s}(2^{-j}\xi)\phi(2^{-l}\xi))| \leq C.$$

By Lemma 2 again, there exists some constant $0 < \theta < 1$ such that

$$\left\|\bar{T}_{j;a,m}^{l,s}f\right\|_2 \leq C2^{\frac{\theta l}{2\log|Q_s|}}\|a\|_{BMO(\mathbb{R}^n)}^m\|f\|_2,$$

which by dilation-invariance implies

$$\left\|T_{j;a,m}^{l,s}f\right\|_2 \leq C2^{\frac{\theta l}{2\log|Q_s|}}\|a\|_{BMO(\mathbb{R}^n)}^m\|f\|_2.$$

From this and Lemma 1(a), we obtain

$$\begin{aligned}
&\left\|\left(\sum_{j\in\mathbb{Z}}\left|\left(T_j^{l,s}S_{l-j}\right)_{a,m}f\right|^2\right)^{\frac{1}{2}}\right\|_2^2 \\
&\qquad \leq C\sum_{i=0}^m\left\|\left(\sum_{j\in\mathbb{Z}}\left|T_{j;a,i}^{l,s}\left(S_{l-j;a,m-i}f\right)\right|^2\right)^{\frac{1}{2}}\right\|_2^2 \\
&\qquad \leq C\sum_{i=0}^m\|a\|_{BMO(\mathbb{R}^n)}^{2i}2^{\frac{\theta l}{\log|Q_s|}}\sum_j\|S_{l-j;a,m-i}f\|_2^2 \\
&\qquad \leq C2^{\frac{\theta l}{\log|Q_s|}}\|a\|_{BMO(\mathbb{R}^n)}^{2m}\|f\|_2^2.
\end{aligned}$$

Thus

$$\|U_l^sf\|_2 \leq C2^{\frac{\theta l}{2\log|Q_s|}}\|a\|_{BMO(\mathbb{R}^n)}^k\|f\|_2. \tag{2.9}$$

This shows that

$$
\begin{aligned}
\sum_{l>0} \|U_l f\|_2 &\le \sum_s |C_s| \sum_{l>0} \|U_l^s f\|_2 \\
&\le C \sum_s |C_s| \sum_{l>0} 2^{\frac{\theta l}{2 \log |Q_s|}} \|a\|^k_{BMO(\mathbb{R}^n)} \|f\|_2 \\
&\le C \sum_s |C_s| \left(\log \frac{1}{|Q_s|} \right) \|a\|^k_{BMO(\mathbb{R}^n)} \|f\|_2.
\end{aligned}
\tag{2.10}
$$

Therefore, it follows from (2.6) and (2.10) that

$$\|T_{a,k} f\|_2 \le \sum_{l\le 0} \|U_l f\|_2 + \sum_{l>0} \|U_l f\|_2 \le C \|a\|^k_{BMO(\mathbb{R}^n)} \|f\|_2.$$

This completes the proof of Theorem 1. □

§3. Proof of Theorem 3

The proof of Theorem 3 is based on the following two lemmas.

LEMMA 4. *Let m be a positive number, $1 < p < \infty$. If $a \in BLO(\mathbb{R}^n)$ and $a(x)$ is a subharmonic function, then the operator $M_{a,m}$ defined by*

$$M_{a,m} f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|\le r} |a(x) - a(y)|^m |f(y)| dy$$

satisfies

$$\|M_{a,m} f\|_p \le C \|a\|^m_{BLO(\mathbb{R}^n)} \|f\|_p.$$

Note that for any cube Q, $|Q|^{-1} \int_Q |a(x) - a_Q|^m dx \le \|a\|^m_{BLO(\mathbb{R}^n)}$. Since a is a subharmonic function, this lemma follows from the same argument as in the proof of Theorems 2.3 and 2.4 in [8]. We omit the details.

LEMMA 5. *Let Ω_0 be homogeneous of degree zero on $\mathbb{R}^n$, $1 < p < \infty$, a and h be as in Theorem 2 . If $\Omega_0 \in L^\lambda(S^{n-1})$, for $\lambda > 1$, then the operator*

$$M^{\Omega_0}_{a,\widetilde{m}} f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|\le r} |a(x) - a(y)|^{\widetilde{m}} |h(|x-y|) \Omega_0(x-y) f(y)| dy$$

satisfies

$$\|M^{\Omega_0}_{a,\widetilde{m}} f\|_p \le C \|a\|^{\widetilde{m}}_{BLO(\mathbb{R}^n)} \|\Omega_0\|_{L^\lambda(S^{n-1})} \|f\|_p,$$

for all integer $\widetilde{m} \ge 0$. Here C is independent of λ.

Proof. For $\widetilde{m}=0$, Lemma 5 was proved by Calderón and Zygmund [5]. Next, we consider the case, $\widetilde{m}>0$. For any $\lambda>1$, write $\lambda'=\frac{\lambda}{\lambda-1}$. Then by a double application of Hölder's inequality, we have

$$\begin{aligned}\|M^{\Omega_0}_{a,\widetilde{m}}f\|_p^p &\le \|h\|_\infty^p \int_{\mathbb{R}^n} \left(M_{a,\lambda'\widetilde{m}}f(x)\right)^{\frac{p}{\lambda'}} \Big(M_{\Omega_0^\lambda}f(x)\Big)^{\frac{p}{\lambda}}\,dx\\ &\le C\|M_{a,\lambda'\widetilde{m}}f\|_p^{\frac{p}{\lambda'}}\|M_{\Omega_0^\lambda}f\|_p^{\frac{p}{\lambda}},\end{aligned}$$

where

$$M_{\Omega_0^\lambda}f(x)=\sup_{r>0}\frac{1}{r^n}\int_{|x-y|\le r}|\Omega_0^\lambda(x-y)f(y)|dy.$$

It follows from Lemma 4 that

$$\|M_{a,\lambda'\widetilde{m}}f\|_p^{\frac{p}{\lambda'}}\le C\|a\|^{\widetilde{m}p}_{BLO(\mathbb{R}^n)}\|f\|_p^{\frac{p}{\lambda'}}.$$

By the method of rotation of Calderón-Zygmund [5], it yields that

$$\|M_{\Omega_0^\lambda}f\|_p^{\frac{p}{\lambda}}\le C\|\Omega_0\|^p_{L^\lambda(S^{n-1})}\|f\|_p^{\frac{p}{\lambda}}.$$

Combining these estimates above, we complete the proof Lemma 5. □

Proof of Theorem 3. By Definitions 1 and 2, we write $\Omega(y')=\sum_s C_s b_s(y')$, where each b_s is a q-block supported in Q_s. Thus

$$\begin{aligned}&M^\Omega_{a,m}f(x)\\ &\quad\le \sum_s |C_s|\sup_{r>0}\int_{|x-y|\le r}|a(x)-a(y)|^m|h(|x-y|)b_s(x-y)f(y)|dy\\ &\quad:=\sum_s|C_s|M^{b_s}_{a,m}f(x).\end{aligned}$$

Consequently,

$$\|M^\Omega_{a,m}f\|_p\le\sum_s|C_s|\|M^{b_s}_{a,m}f\|_p.$$

We now estimate $\|M^{b_s}_{a,m}f\|_p$ for each b_s. It follows from Lemma 5 that for any $\lambda>1$,

$$\|M^{b_s}_{a,m}f\|_p\le C\|b_s\|_{L^\lambda(S^{n-1})}\|a\|^m_{BLO(\mathbb{R}^n)}\|f\|_p.$$

Notice that $\operatorname{supp}(b_s)\subseteq Q_s$ and $\|b_s\|_{L^q(S^{n-1})}\le|Q_s|^{\frac{1}{q}-1}$. If $|Q_s|\ge e^{\frac{q}{1-q}}$, we let $\lambda=q$, to get

$$\begin{aligned}\|M^{b_s}_{a,m}f\|_p &\le C\|b_s\|_{L^q(S^{n-1})}\|a\|^m_{BLO(\mathbb{R}^n)}\|f\|_p\\ &\le C|Q_s|^{\frac{1}{q}-1}\|a\|^m_{BLO(\mathbb{R}^n)}\|f\|_p\le C\|a\|^m_{BLO(\mathbb{R}^n)}\|f\|_p.\end{aligned}$$

If $|Q_s| < e^{\frac{q}{1-q}}$, let $\lambda = \log|Q_s|/(1+\log|Q_s|)$, so that $1 < \lambda < q$ and $\lambda' = -\log|Q_s|$. By Hölder's inequality, we have

$$\begin{aligned}\|M_{a,m}^{b_s}f\|_p &\le C\|b_s\|_{L^q(S^{n-1})}|Q_s|^{\frac{1}{\lambda}-\frac{1}{q}}\|a\|_{BLO(\mathbb{R}^n)}^m\|f\|_p\\ &\le C|Q_s|^{-\frac{1}{\lambda'}}\|a\|_{BLO(\mathbb{R}^n)}^m\|f\|_p \le C\|a\|_{BLO(\mathbb{R}^n)}^m\|f\|_p.\end{aligned}$$

So, we obtain

$$\|M_{a,m}^{\Omega}f\|_p \le C\sum_s |C_s|\|a\|_{BLO(\mathbb{R}^n)}^m\|f\|_p \le C\|a\|_{BLO(\mathbb{R}^n)}^m\|f\|_p$$

and complete the proof of Theorem 3. □

§4. Proof of Theorem 2

To prove Theorem 2, we still need the following auxiliary result.

Let h, a, k and $\Omega(y') = \sum_s C_s b_s(y')$ be as in Theorem 2, $j \in \mathbb{Z}$. Define the following operators:

$$\sigma_{j;a,k}f(x) = \int_{2^j<|x-y|\le 2^{j+1}}[a(x)-a(y)]^k\frac{\Omega(x-y)}{|x-y|^n}h(|x-y|)f(y)dy,$$

$$\sigma_{j;a,k}^s f(x) = \int_{2^j<|x-y|\le 2^{j+1}}[a(x)-a(y)]^k\frac{b_s(x-y)}{|x-y|^n}h(|x-y|)f(y)dy,$$

$$\mu_{j;a,k}f(x) = \int_{2^j<|x-y|\le 2^{j+1}}|a(x)-a(y)|^k\frac{|\Omega(x-y)|}{|x-y|^n}|h(|x-y|)|f(y)dy,$$

$$\mu_{j;a,k}^s f(x) = \int_{2^j<|x-y|\le 2^{j+1}}|a(x)-a(y)|^k\frac{|b_s(x-y)|}{|x-y|^n}|h(|x-y|)|f(y)dy,$$

$$\mu_{a,k}^*f(x) = \sup_{j\in\mathbb{Z}}|\mu_{j;a,k}f(x)| \quad \text{and} \quad \mu_{a,k}^{s*}f(x) = \sup_{j\in\mathbb{Z}}|\mu_{j;a,k}^s f(x)|.$$

Clearly, we have

$$\mu_{a,k}^*f(x) \le CM_{a,k}^{\Omega}f(x) \ \ \textit{and} \ \ \mu_{a,k}^{s*}f(x) \le CM_{a,k}^{b_s}f(x).$$

By Lemma 5 and Theorem 3, it is easy to see that for all $1 < p < \infty$,

$$\|\mu_{a,k}^*f\|_p \le C\|a\|_{BLO(\mathbb{R}^n)}^k\|f\|_p \tag{4.1}$$

and

$$\|\mu_{a,k}^{s*}f\|_p \le C\|b_s\|_{L^\lambda(S^{n-1})}\|a\|_{BLO(\mathbb{R}^n)}^k\|f\|_p, \tag{4.2}$$

and the bounds are independent of b_s.

By applying (4.1) and (4.2), we can obtain the following lemma.

Lemma 6. *Under the same assumptions as in Theorem 2, for arbitrary functions f_j,*

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\sigma_{j;a,k} f_j|^2 \right)^{\frac{1}{2}} \right\|_p \le C\|a\|^k_{BLO(\mathbb{R}^n)} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^2 \right)^{\frac{1}{2}} \right\|_p \tag{4.3}$$

and

$$\begin{aligned} &\left\| \left(\sum_{j\in\mathbb{Z}} |\sigma^s_{j;a,k} f_j|^2 \right)^{\frac{1}{2}} \right\|_p \\ &\qquad \le C\|b_s\|_{L^\lambda(S^{n-1})}\|a\|^k_{BLO(\mathbb{R}^n)} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^2 \right)^{\frac{1}{2}} \right\|_p \end{aligned} \tag{4.4}$$

for all $1 < p < \infty$ and for any $\lambda > 1$.

Proof. We prove only (4.3) because the other is essentially similar. The ideas in our proof are taken from those in Lemma of [7] and Lemma 2 of [11]. In fact, it suffices to consider the case $p > 2$ so that $q = (\frac{p}{2})'$, and there exists $g \in L^q_+$ of unit norm such that

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\sigma_{j;a,k} f_j|^2 \right)^{\frac{1}{2}} \right\|_p^2 = \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} |\sigma_{j;a,k} f_j(x)|^2 g(x) dx.$$

Also, by Hölder's inequality and a simple computation, we have

$$|\sigma_{j;a,k} f(x)|^2 \le C\mu_{j;a,2k}(|f|^2)(x)$$

and

$$\int_{\mathbb{R}^n} \mu_{j;a,k}(|f|^2)(x)g(x)dx = \int_{\mathbb{R}^n} f^2(x)\mu_{j;\widetilde{a},2k}\widetilde{g}(-x)dx,$$

where $\widetilde{a}(x) = a(-x)$ and $\widetilde{g}(x) = g(-x)$. Therefore

$$\begin{aligned}\left\| \left(\sum_{j\in\mathbb{Z}} |\sigma_{j;a,k} f_j|^2 \right)^{\frac{1}{2}} \right\|_p^2 &\le C \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} \mu_{j;a,2k} \left(|f_j|^2 \right)(x) g(x) dx \\ &= C \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} f_j^2(x) \mu_{j;\widetilde{a},2k} \widetilde{g}(-x) dx \\ &\le C \int_{\mathbb{R}^n} \sup_{j\in\mathbb{Z}} \left| \mu_{j;\widetilde{a},2k} \widetilde{g}(-x) \right| \sum_{j\in\mathbb{Z}} f_j^2(x) dx \\ &\le C \left\| \mu^*_{\widetilde{a},2k} \widetilde{g} \right\|_q \left\| \sum_{j\in\mathbb{Z}} |f_j|^2 \right\|_{\frac{p}{2}}.\end{aligned}$$

By (4.1), we obtain

$$\begin{aligned}\left\| \left(\sum_{j\in\mathbb{Z}} |\sigma_{j;a,k} f_j|^2 \right)^{\frac{1}{2}} \right\|_p^2 &\le C \|a\|^{2k}_{BLO(\mathbb{R}^n)} \|g\|_q \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^2 \right)^{\frac{1}{2}} \right\|_p^2 \\ &= C \|a\|^{2k}_{BLO(\mathbb{R}^n)} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^2 \right)^{\frac{1}{2}} \right\|_p^2,\end{aligned}$$

which proves Lemma 6. ∎

Proof of Theorem 2. Let U_l, T_j^l, S_{l-j} be the same as that in the proof of Theorem 1. Then for $1 < p < \infty$, similarly to (2.1) and (2.2), we have

$$\|T_{a,k} f\|_p \le \sum_{l\in\mathbb{Z}} \|U_l f\|_p \tag{4.5}$$

and

$$\|U_l f\|_p \le C \sum_{m=0}^{k} \|a\|^{k-m}_{BLO(\mathbb{R}^n)} \left\| \left(\sum_{j\in\mathbb{Z}} \left| \left(T_j^l S_{l-j} \right)_{a,m} f \right|^2 \right)^{\frac{1}{2}} \right\|_p. \tag{4.6}$$

Now we estimate $\|U_l f\|_p$ in two cases as follows:

Case 1. First we show the L^p-boundedness of U_l for $l \le 0$.

For $p = 2$, by the same arguments as to (2.6), we obtain

$$\|U_l f\|_2 \le C 2^{vl} \|a\|^k_{BLO(\mathbb{R}^n)} \|f\|_2. \tag{4.7}$$

Next we turn to estimate L^p-boundedness of $U_l f$. Write

$$\left(T^l_j S_{l-j}\right)_{a,m} f(x) = \sum_{i=0}^{m} C^i_m \sigma_{j;a,i}\left(S^2_{l-j;a,m-i} f\right)(x).$$

We know from Lemma 6 and Lemma 1(b) that for all $1 < p < \infty$,

(4.8)

$$\begin{aligned}\|U_l f\|_p &\le C \sum_{m=0}^{k} \|a\|^{k-m}_{BLO(\mathbb{R}^n)} \sum_{i=0}^{m} \left\| \left(\sum_{j\in\mathbb{Z}} \left|\sigma_{j;a,i}\left(S^2_{l-j;a,m-i} f\right)\right|^2 \right)^{\frac12} \right\|_p \\ &\le C \sum_{m=0}^{k} \sum_{i=0}^{m} C^m_i \|a\|^{k-m+i}_{BLO(\mathbb{R}^n)} \left\| \left(\sum_{j\in\mathbb{Z}} \left|S^2_{l-j;a,m-i} f\right|^2 \right)^{\frac12} \right\|_p \\ &\le C \|a\|^k_{BLO(\mathbb{R}^n)} \|f\|_p.\end{aligned}$$

Using interpolation between (4.7) and (4.8), we obtain

$$\sum_{l\le 0} \|U_l f\|_p \le C \|a\|^k_{BLO(\mathbb{R}^n)} \|f\|_p. \tag{4.9}$$

Case 2. We next consider the L^p-estimate of U_l for $l > 0$.

Let $T^l_{j,s}$, S_{l-j}, U^s_l be as that in the proof of Theorem 1. Similarly to (2.7) and (2.8), we have for $1 < p < \infty$,

$$\|U_l f\|_p \le \sum_s |C_s| \|U^s_l f\|_p, \tag{4.10}$$

and

$$\|U^s_l f\|_p \le C \sum_{m=0}^{k} \|a\|^{k-m}_{BLO(\mathbb{R}^n)} \left\| \left(\sum_{j\in\mathbb{Z}} \left| \left(T^{l,s}_j S_{l-j}\right)_{a,m} f \right|^2 \right)^{\frac12} \right\|_p. \tag{4.11}$$

For each b_s, without loss of generality, we may assume that the support Q_s of b_s are uniformly small such that $|Q_s| < e^{\frac{q}{1-q}}$. Similarly to (2.9), we can get that for some $0 < \theta < 1$,

$$\|U_l^s f\|_2 \leq C 2^{\frac{\theta l}{2\log|Q_s|}} \|a\|_{BLO(\mathbb{R}^n)}^k \|f\|_2. \tag{4.12}$$

For $1 < p < \infty$, noting that

$$\left(T_j^{l,s} S_{l-j}\right)_{a,m} f(x) = \sum_{i=0}^{m} C_m^i \sigma_{j;a,i}^s \left(S_{l-j;a,m-i}^2 f\right)(x)$$

and invoking (4.4) and Lemma 1(b) with $\lambda = \frac{\log|Q_s|}{1+\log|Q_s|}$, we have

(4.13)

$$\begin{aligned} \|U_l^s f\|_p &\leq C \sum_{m=0}^{k} \|a\|_{BLO(\mathbb{R}^n)}^{k-m} \sum_{i=0}^{m} \left\| \left(\sum_{j\in\mathbb{Z}} \left|\sigma_{j;a,i}^s \left(S_{l-j;a,m-i}^2 f\right)\right|^2 \right)^{\frac{1}{2}} \right\|_p \\ &\leq C \|b_s\|_{L^\lambda(S^{n-1})} \sum_{m=0}^{k} \sum_{i=0}^{m} \|a\|_{BLO(\mathbb{R}^n)}^{k-m+i} \left\| \left(\sum_{j\in\mathbb{Z}} \left|S_{l-j;a,m-i}^2 f\right|^2 \right)^{\frac{1}{2}} \right\|_p \\ &\leq C \|a\|_{BLO(\mathbb{R}^n)}^k \|f\|_p. \end{aligned}$$

Using interpolation between (4.12) and (4.13) again, we obtain

$$\|U_l^s f\|_p \leq C 2^{\frac{\theta_1 \theta l}{2\log|Q_s|}} \|a\|_{BLO(\mathbb{R}^n)}^k \|f\|_p, \tag{4.14}$$

for some $0 < \theta_1 \leq 1$. This shows that

$$\begin{aligned} \sum_{l>0} \|U_l f\|_p &\leq \sum_s |C_s| \sum_{l>0} \|U_l^s f\|_p \\ &\leq C \sum_s |C_s| \sum_{l>0} 2^{\frac{\theta_1 \theta l}{2\log|Q_s|}} \|a\|_{BLO(\mathbb{R}^n)}^k \|f\|_p \\ &\leq C \sum_s |C_s| \left(\log \frac{1}{|Q_s|}\right) \|a\|_{BLO(\mathbb{R}^n)}^k \|f\|_p. \end{aligned} \tag{4.15}$$

Therefore, (4.9) and (4.15) now imply

$$\|T_{a,k} f\|_p \leq \sum_{l\leq 0} \|U_l f\|_p + \sum_{l>0} \|U_l f\|_p \leq C \|a\|_{BLO(\mathbb{R}^n)}^k \|f\|_p,$$

which completes the proof of Theorem 2. ∎

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