ON THE EXISTENCE AND BOUNDEDNESS OF SQUARE FUNCTION OPERATORS ON CAMPANATO SPACES

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Abstract. Let $g(f)$ be a Littlewood-Paley square function of f, which belongs to Campanato spaces $L^{p,\alpha}(1 < p < \infty, -\frac{n}{p} \leq \alpha < 1)$. We prove that if $g(f)(x_0)$ exists (i.e. $g(f)(x_0) < \infty$) for a single point $x_0 \in R^n$, then $g(f)(x)$ exists almost everywhere in R^n and $\|g(f)\|_{L^{p,\alpha}} \leq C\|f\|_{L^{p,\alpha}}$. Thus we give an improvement of some earlier results such as in [8], where it is always needed to assume $g(f)(x)$ exists in a set of positive measure in order to get the a.e. existence and boundedness of $g(f)(x)$.

§1. Introduction

In this note we study the existence and boundedness property of square function operators on Campanato spaces. First we introduce the definition of square functions, which are the generalization of classical ones. See [4].

DEFINITION 1.1. $\psi \in C^1(R^n)$ is called a Littlewood-Paley function, if there is a positive constant c such that:

(i) $\psi \in L^1$ and $\int\limits_{R^n} \psi(x)dx = 0$;

(ii) $|\psi(x)| \leq c(1 + |x|)^{-n-1}$;

(iii) $|\nabla\psi(x)| \leq c(1 + |x|)^{-n-2}$.

For a given Littlewood-Paley function, one may define three kinds of square functions.

DEFINITION 1.2. Let ψ be a Littlewood-Paley function and $f \in L_{loc}(R^n)$, define

1. Radial square function.

$$g_\psi(f)(x) = \left(\int_0^\infty |\psi_t * f(x)|^2 \, \frac{dt}{t} \right)^{\frac{1}{2}} ;$$

2 Nontangential square function.

$$s_\psi(f)(x) = \left(\int_{|x-y|<t} |\psi_t * f(y)|^2 \, \frac{dtdy}{t^{n+1}} \right)^{\frac{1}{2}} ;$$

3. Tangential square function.

$$g_{\lambda,\psi}^*(f)(x) = \left(\int_0^\infty \int_{R^n} |\psi_t * f(x-y)|^2 \left(1 + \frac{|y|}{t} \right)^{-\lambda n} \frac{dtdy}{t^{n+1}} \right)^{\frac{1}{2}} ;$$

where $\lambda > 0$ is a real number, $\psi_t(x) = \frac{1}{t^n}\psi(\frac{x}{t})$.

To be convenient, we omit the subscript ψ and use $g(f), s(f), g_\lambda^*(f)$ to denote three square functions respectively. Next we introduce the concept of Campanato spaces, which are the generalization of BMO and Lipschitz spaces.

DEFINITION 1.3. ([1]) Suppose f is a locally integrable function and $1 \le p < \infty, -\frac{n}{p} \le \alpha < 1$. Define

$$[f]_{L^{p,\alpha}} = \sup_B \left(|B|^{-1-\frac{p\alpha}{n}} \int_B |f(x) - f_B|^p \, dx \right)^{\frac{1}{p}} .$$

Campanato space $L^{p,\alpha} = \{f \in L_{Loc} : \quad [f]_{L^{p,\alpha}} < \infty\}$.

An useful property of a Campanato function is the following

LEMMA 1.1. ([2]) *Suppose* $f \in L^{p,\alpha}, 1 \le p < \infty$. *If* $\beta > 0$ *and* $\alpha < \min\{\beta, 1\}$, *then for any ball* $B = B_r(x_0)$

$$\int_{R^n} \frac{|f(y) - f_B|}{r^{n+\beta} + |x_0 - y|^{n+\beta}} dy \le Cr^{\alpha-\beta} \, [f]_{L^{p,\alpha}},$$

where C *is a constant depending only on* n, α *and* β.

In [6] the author studied the behavior of classical square functions acting on L^∞ and BMO space. He pointed out that the value of square function of an L^∞ function may be infinite everywhere in R^n. Under the assumption of the existence (finiteness) on a set of positive measure, he got the a.e. finiteness and BMO-boundedness of square functions. Soon after, such kind of results was generalized to other kind of square functions such as Lusin's area functions, Stein's g_λ^* functions and so on. See [3], for example. Later, in [7] the authors improved aforementioned results in the sense that in order to guarantee the a.e. existence and BMO-boundedness of square function, only the assumption of **one point finiteness** is enough. Similar results also hold for Lipschitz function spaces. See [5]. Since the generalization of BMO and Lipschitz space is Campanato space, a natural question is that if there are similar answers for the square function operators acting on Campanato spaces. Recently, K.Yabuta and others obtained some satisfactory results, see [8] and the references therein. But their results still have a not so satisfied point, that is, in some cases in order to get the a.e. finiteness it is always needed to assume the finiteness of square functions on a positive measure set (such as the first occurrence in [6]). The main results in this note point out that the **one point existence** of square function is enough. More precisely, we have

THEOREM 1. *Suppose* $f \in L^{p,\alpha}(1 < p < \infty, -\frac{n}{p} \leq \alpha < 1)$. *If* $g(f)(x_0) < +\infty$ *for a single point* $x_0 \in R^n$, *then* $g(f)(x)$ *exists almost everywhere in* R^n *and there is a constant* C *depending on* ψ, n, p, α *such that*

$$\|g(f)\|_{L^{p,\alpha}} \leq C\|f\|_{L^{p,\alpha}}.$$

THEOREM 2. *Suppose* $f \in L^{p,\alpha}(1 < p < \infty, -\frac{n}{p} \leq \alpha < 1)$. *If there is a point* $x_0 \in R^n$ *such that* $s(f)(x_0) < +\infty$, *then* $s(f)(x) < +\infty$ *almost everywhere in* R^n *and there is a constant* C *depending only on* ψ, n, p, α *such that*

$$\|s(f)\|_{L^{p,\alpha}} \leq C\|f\|_{L^{p,\alpha}}.$$

THEOREM 3. *Suppose* $f \in L^{p,\alpha}(1 < p < \infty, -\frac{n}{p} \leq \alpha < 1), \lambda > 3 + \frac{2}{n}$. *If there is a point* $x_0 \in R^n$ *such that* $g_\lambda^*(f)(x_0) < +\infty$, *then* $g_\lambda^*(f)(x)$ *exists almost everywhere in* R^n *and there is a constant* C *depending only on* $\psi, n, p, \lambda, \alpha$ *such that*

$$\|g_\lambda^*(f)\|_{L^{p,\alpha}} \leq C\|f\|_{L^{p,\alpha}}.$$

Obviously, our results improve and generalize some earlier results, especially those in [7] and [8].

Finally, We need to point out that though our results are new and the approach is different from others, many of our idea come from [6, 7] and [8].

§2. Proof of Theorem 1

We first deal with radial square function, which is relatively simple.

Let $B = B_r(x_0)$ be any ball with centre x_0. As usual, we decompose f in the following way,

$$f(x) = f_{4B} + (f(x) - f_{4B})\chi_{4B}(x) + (f(x) - f_{4B})\chi_{(R^n \backslash 4B)}(x)$$
$$= f_1 + f_2(x) + f_3(x).$$

Define

$$g_0(f)(x) = \left(\int_0^r |\psi_t * f(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}};$$

$$g_\infty(f)(x) = \left(\int_r^\infty |\psi_t * f(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

Obviously,

$$g_0(f)(x), g_\infty(f)(x) \le g(f)(x) \le g_0 f(x) + g_\infty f(x).$$

By property (i) of ψ,

(2.1) $g(f_1) = g_0(f_1) = g_\infty(f_1) \equiv 0;$

Since $g(f_2)(x) \in L^p$, $g(f_2)(x)$ exists a.e. in R^n and

(2.2) $\|g(f_2)\|_{L^p(B)} \le C\|f_2\|_{L^p} \le C|B|^{\frac{1}{p} + \frac{\alpha}{n}}[f]_{L^{p,\alpha}};$

In fact we have the following pointwise estimate for $g_\infty(f_2)$:

LEMMA 2.1.   *For any $x \in R^n$,*

(2.3) $g_\infty(f_2)(x) \le C|B|^{\frac{\alpha}{n}}[f]_{L^{p,\alpha}}.$

Proof. By condition (ii) of ψ,

$$g_\infty(f_2)(x) = \left(\int_r^\infty \left| \int_{R^n} \psi_t(x-y)f_2(y)dy \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

$$\leq \left(\int_r^\infty \left(\int_{4B} |\psi_t(x-y)||f(y)-f_{4B}|dy \right)^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

$$\leq C \left(\int_r^\infty \left| \int_{4B} t^{-n} \left(1 + \frac{|x-y|}{t} \right)^{-n-1} |f(y)-f_{4B}|dy \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

$$\leq C \left(\int_{4B} |f(y)-f_{4B}|dy \right) \left(\int_r^{+\infty} t^{-2n-1}dt \right)^{\frac{1}{2}}$$

$$\leq Cr^{-n} \int_{4B} |f(y)-f_{4B}|dy \leq C|B|^{\frac{\alpha}{n}}[f]_{L^{p,\alpha}}.$$

We need the following two lemmas.

LEMMA 2.2.   *For any $x \in B$, $g_0(f_3)(x) < \infty$ and*

(2.4) $$g_0(f_3)(x) \leq C|B|^{\frac{\alpha}{n}}[f]_{L^{p,\alpha}}.$$

Proof.   Note that if $t < r, x \in B, y \notin 4B$, then $(t+|x-y|) \sim (r+|x_0-y|)$.

$$g_0(f_3)(x) = \left(\int_0^r \left| \int_{R^n} \psi_t(x-y)f_3(y)dy \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

$$\leq C \left(\int_0^r \left| \int_{R^n} \frac{1}{t^n} \frac{t^{n+1}}{t^{n+1} + |x-y|^{n+1}} |f_3(y)|dy \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

$$\leq C \left(\int_0^r \left| \int_{R^n \backslash 4B} \frac{|f(y)-f_{4B}|}{r^{n+1} + |x-y|^{n+1}} dy \right|^2 tdt \right)^{\frac{1}{2}}$$

$$\leq Cr^{\alpha-1}[f]_{L^{p,\alpha}} \left(\int_0^r tdt \right)^{\frac{1}{2}}$$

$$\leq Cr^\alpha[f]_{L^{p,\alpha}} = C|B|^{\frac{\alpha}{n}}[f]_{L^{p,\alpha}};$$

The next to the last inequality comes from Lemma 1.1.

The following is a key lemma.

LEMMA 2.3.   *For any $x \in B$, $g_\infty(f_3)(x) < \infty$ and*

$$(2.5) \qquad |g_\infty(f_3)(x) - g_\infty(f)(x_0)| \leq C|B|^{\frac{\alpha}{n}}[f]_{L^{p,\alpha}}.$$

Note that by assumption, $g_\infty(f)(x_0) \leq g(f)(x_0) < \infty$.

Proof.   By property (i) of ψ,

$$|g_\infty(f_3)(x) - g_\infty(f)(x_0)|$$

$$= \left| \left(\int_r^{+\infty} \left| \int_{R^n} \psi_t(x-y)f_3(y)dy \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right.$$

$$\left. - \left(\int_r^{+\infty} \left| \int_{R^n} \psi_t(x_0-y)f(y)dy \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right|$$

$$\leq \left(\int_r^{+\infty} \left| \int_{R^n} [\psi_t(x-y)f_3(y) - \psi_t(x_0-y)f(y)] dy \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

$$\leq \left(\int_r^{+\infty} \left| \int_{R^n} [\psi_t(x-y) - \psi_t(x_0-y)] f_3(y)dy \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} + g_\infty(f_2)(x_0)$$

$$= I(x) + g_\infty(f_2)(x_0);$$

$$I(x) \leq \int_{R^n \backslash 4B} |f(y) - f_{4B}| \left(\int_r^{+\infty} |\psi_t(x-y) - \psi_t(x_0-y)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} dy;$$

For $\forall x \in B, y \notin 4B$, $|\psi(x-y) - \psi(x_0-y)| \leq C|x-x_0| (1 + |x_0-y|)^{-(n+2)}$.

$$\left(\int_r^{+\infty} |\psi_t(x-y) - \psi_t(x_0-y)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

$$\leq C \left(\int_r^{+\infty} \frac{1}{t^{2n}} \left[\frac{t^{-1}|x-x_0|}{(1 + t^{-1}|x_0-y|)^{n+2}} \right]^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

$$\leq Cr \left(\int_0^{+\infty} \left(1 + \frac{|x_0 - y|}{t} \right)^{-2(n+2)} \frac{dt}{t^{2n+3}} \right)^{\frac{1}{2}}$$

$$= C \frac{r}{|x_0 - y|^{n+1}} \left(\int_0^{+\infty} \frac{s}{(1+s)^{2(n+2)}} ds \right)^{\frac{1}{2}}$$

$$= C \frac{r}{|x_0 - y|^{n+1}};$$

Hence by Lemma 1.1,

$$I(x) \leq Cr \int_{R^n \setminus 4B} \frac{|f(y) - f_{4B}|}{|x_0 - y|^{n+1}} dy \leq Cr^\alpha [f]_{L^{p,\alpha}}.$$

Finally, using Lemma 2.1,

$$|g_\infty(f_3)(x) - g_\infty(f)(x_0)| \leq I(x) + g_\infty(f_2)(x_0) \leq C|B|^{\frac{\alpha}{n}} [f]_{L^{p,\alpha}}.$$

Remark 2.1. In fact, in the estimate of the term I we proved

$$\left(\int_0^{+\infty} |\psi_t(x - y) - \psi_t(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \leq C|y||x|^{-(n+1)}.$$

We continue to prove the theorem. By Lemmas 2.2 and 2.3, we know $g(f_3)(x)$ is finite everywhere in B. By the a.e. existence of $g(f_2)(x)$ we know $g(f)(x)$ is finite a.e. in B. By the arbitrarity of the radius of $B = B_r(x_0)$, we get the a.e. existence of $g(f)(x)$ in R^n. Next we prove the boundedness of $g(f)$.

Let $E = \{x \in R^n | g(f)(x) < \infty\}$. Obviously, one only needs to prove that for any ball $B^* = B_\rho(x^*)$ with centre $x^* \in E$,

$$(2.6) \qquad \left(\int_{B^*} |g(f)(x) - (g(f))_{B^*}|^p dx \right)^{\frac{1}{p}} \leq C|B^*|^{\frac{1}{p} + \frac{\alpha}{n}} [f]_{L^{p,\alpha}}.$$

In fact, let

$$f(x) = f_{4B^*} + (f(x) - f_{4B^*})\chi_{4B^*}(x) + (f(x) - f_{4B^*})\chi_{(R^n \setminus 4B^*)}(x)$$
$$= h_1 + h_2(x) + h_3(x);$$

and denote

$$g_0(f)(x) = \left(\int_0^\rho |\psi_t * f(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}};$$

$$g_\infty(f)(x) = \left(\int_\rho^\infty |\psi_t * f(x)|^2 \frac{dt}{t}\right)^{\frac{1}{2}}.$$

Repeating the argument of proving (2.1), (2.2), (2.4) and (2.5), we have

(2.7) $g(h_1) = g_0(h_1) = g_\infty(h_1) \equiv 0;$

(2.8) $\|g(h_2)\|_{L^p(B^*)} \le C\|h_2\|_{L^p} \le C|B^*|^{\frac{1}{p}+\frac{\alpha}{n}}[f]_{L^{p,\alpha}};$

(2.9) $\sup_{x\in B^*} g_0(h_3)(x) \le C|B^*|^{\frac{\alpha}{n}}[f]_{L^{p,\alpha}};$

(2.10) $\sup_{x\in B^*} |g_\infty(h_3)(x) - g_\infty(f)(x^*)| \le C|B^*|^{\frac{\alpha}{n}}[f]_{L^{p,\alpha}}.$

Hence

$$\left(\int_{B^*} |g(f)(x) - [g(f)]_{B^*}|^p\, dx\right)^{\frac{1}{p}}$$

$$\le 2\left(\int_{B^*} |g(f)(x) - g_\infty(f)(x^*)|^p\, dx\right)^{\frac{1}{p}}$$

$$\le 2\left(\int_{B^*} |g(f)(x) - g(h_3)(x)|^p\, dx\right)^{\frac{1}{p}}$$

$$+2\left(\int_{B^*} |g(h_3)(x) - g_\infty(f)(x^*)|^p\, dx\right)^{\frac{1}{p}}$$

$$\le 2\left(\int_{B^*} |g(h_2)(x)|^p\right)^{\frac{1}{p}} + 2\left(\int_{B^*} |g(h_3)(x) - g_\infty(h_3)(x)|^p\, dx\right)^{\frac{1}{p}}$$

$$+2\left(\int_{B^*} |g_\infty(h_3)(x) - g_\infty(f)(x^*)|^p\, dx\right)^{\frac{1}{p}}$$

$$\le 2\|g(h_2)\|_{L^p(B^*)} + 2\left(\int_{B^*} |g_0(h_3)(x)|^p\, dx\right)^{\frac{1}{p}}$$

$$+2|B^*|^{\frac{1}{p}}\sup_{x\in B^*} |g_\infty(h_3)(x) - g_\infty(f)(x^*)|$$

$$\le C|B^*|^{\frac{1}{p}+\frac{\alpha}{n}}[f]_{L^{p,\alpha}}.$$

§3. Proof of Theorems 2, 3

We only give the proof of Theorem 3. The proof of Theorem 2 is similar and we leave it to the interesting reader.

As in the proof of Theorem 1, we only need to prove the following three lemmas concerning on g_λ^*. Use the same notation as the last section, define

$$g_{\lambda,0}^*(f)(x) = \left(\int_0^r \int_{R^n} |\psi_t * f(x-u)|^2 \, d\mu \right)^{\frac{1}{2}};$$

$$g_{\lambda,\infty}^*(f)(x) = \left(\int_r^\infty \int_{R^n} |\psi_t * f(x-u)|^2 \, d\mu \right)^{\frac{1}{2}};$$

where $d\mu = \left(1 + \frac{|u|}{t}\right)^{-\lambda n} \frac{du\,dt}{t^{n+1}}$.

LEMMA 3.1.   *For any $x \in R^n$, $g_{\lambda,\infty}^*(f_2)(x) < \infty$ and*

(3.1) $$g_{\lambda,\infty}^*(f_2)(x) \le C|B|^{\frac{\alpha}{n}}[f]_{L^{p,\alpha}};$$

where the constant C depends only on ψ, n, p, α and λ.

Proof. By noting condition (ii) of ψ,

$$g_{\lambda,\infty}^*(f_2)(x) = \left(\int_r^\infty \int_{R^n} \left| \int_{R^n} \psi_t(x-y-u) f_2(y) dy \right|^2 d\mu \right)^{\frac{1}{2}}$$

$$\le \left(\int_r^\infty \int_{R^n} \left[\int_{4B} |\psi_t(x-u-y)| \, |f(y) - f_{4B}| \, dy \right]^2 d\mu \right)^{\frac{1}{2}}$$

$$\le C \left(\int_r^\infty \int_{R^n} \left[\int_{4B} \frac{|f(y) - f_{4B}| \, dy}{t^n \left(1 + t^{-1}|x-y-u|\right)^{n+1}} \right]^2 d\mu \right)^{\frac{1}{2}}$$

$$\le C \int_{4B} |f(y) - f_{4B}| dy \left(\int_r^{+\infty} \int_{R^n} \left(\frac{t}{t+|u|} \right)^{\lambda n} \frac{du\,dt}{t^{3n+1}} \right)^{\frac{1}{2}}$$

$$\le C \int_{4B} |f(y) - f_{4B}| dy \left(\int_r^{+\infty} \int_{R^n} \left(\frac{1}{1+|v|} \right)^{\lambda n} \frac{dv\,dt}{t^{2n+1}} \right)^{\frac{1}{2}}$$

$$\le C \frac{1}{r^n} \int_{4B} |f(y) - f_{4B}| dy \le C|B|^{\frac{\alpha}{n}}[f]_{L^{p,\alpha}}.$$

The following lemma is essentially contained in Lemma 4 in [7].

LEMMA 3.2.   *If $\lambda > 3 + \frac{2}{n}$, then for any $x \in B, g_{\lambda,0}^*(f_3)(x)$ exists and*

(3.2) $$g_{\lambda,0}^*(f_3)(x) \le C|B|^{\frac{\alpha}{n}}[f]_{L^{p,\alpha}}.$$

Proof. By condition (ii) of ψ and $(1+|x|) \leq (1+|x-y|)(1+|y|), \forall x, y \in R^n$,

$$g^*_{\lambda,0}(f_3)(x)$$

$$= \left(\int_0^r \int_{R^n} \left| \int_{R^n} \psi_t(x-y-u) f_3(y) dy \right|^2 d\mu \right)^{\frac{1}{2}}$$

$$\leq C \left[\int_0^r \int_{R^n} \left(\int_{R^n \backslash 4B} \frac{t|f(y) - f_{4B}|}{(t+|x-u-y|)^{n+1}} dy \right)^2 \left(\frac{t}{t+|u|} \right)^{\lambda n} \frac{dudt}{t^{n+1}} \right]^{\frac{1}{2}}$$

$$\leq C \int_{R^n \backslash 4B} |f(y) - f_{4B}| \left(\int_0^r \int_{R^n} \frac{t\left(1+\frac{|u|}{t}\right)^{-\lambda n + 2n + 2}}{(t+|x-y|)^{2(n+1)}} \frac{dudt}{t^n} \right)^{\frac{1}{2}} dy;$$

Noting that for $t \in (0, r), x \in B, y \notin 4B$, one has $t + |x - y| \sim r + |x_0 - y|$ and $\lambda > 3 + \frac{2}{n}$,

$$g^*_{\lambda,0}(f_3)(x) \leq Cr \int_{R^n \backslash 4B} \frac{|f(y) - f_{4B}|}{(r+|x_0-y|)^{n+1}} dy$$

$$\leq Cr^{\alpha}[f]_{L^{p,\alpha}} = C|B|^{\frac{\alpha}{n}}[f]_{L^{p,\alpha}};$$

The last inequality comes from Lemma 1.1.

LEMMA 3.3. *If* $\lambda > 3 + \frac{2}{n}$, *then for any* $x \in B, g^*_{\lambda,\infty}(f_3)(x) < \infty$ *and*

$$(3.3) \qquad \left| g^*_{\lambda,\infty}(f_3)(x) - g^*_{\lambda,\infty}(f)(x_0) \right| \leq C|B|^{\frac{\alpha}{n}}[f]_{L^{p,\alpha}}.$$

Proof. By Lemma 3.1 and using the same argument as in the proof of Lemma 2.3, one only needs to prove

$$(3.4) \quad I(x) = \left(\int_{R^{n+1}_+} \left| \int_{R^n} [\psi_t(x-u-y) - \psi_t(x_0-u-y)] f_3(y) dy \right|^2 d\mu \right)^{\frac{1}{2}}$$

$$\leq Cr \int_{R^n \backslash 4B} \frac{|f(y) - f_{4B}| dy}{|x_0 - y|^{n+1}};$$

By the Minkowski inequality, we only need to estimate the following term $II(x)$. By using condition (ii) of ψ,

$$II(x) = \left(\int_{R^{n+1}_+} |\psi_t(x-u-y) - \psi_t(x_0-u-y)|^2 d\mu \right)^{\frac{1}{2}}$$

$$= \left(\int_{R_+^{n+1}} \left| \int_0^1 (x - x_0) \cdot \nabla \psi_t(x_0 - u - y + \theta(x - x_0)) d\theta \right|^2 d\mu \right)^{\frac{1}{2}}$$

$$\leq C \int_0^1 \left(\int_{R_+^{n+1}} \frac{|x - x_0|^2}{t^{2n+2} \left(1 + t^{-1}|x_0 - u - y + \theta(x - x_0)| \right)^{2(n+2)}} d\mu \right)^{\frac{1}{2}} d\theta$$

$$\leq Cr \int_0^1 \left(\int_{R_+^{n+1}} \frac{(1 + |v|)^{-\lambda n} dv dt}{t^{2n+3} \left(1 + \left| \frac{x_0 - y + \theta(x - x_0)}{t} - v \right| \right)^{2(n+2)}} \right)^{\frac{1}{2}} d\theta$$

$$= Cr \int_0^1 |z|^{-n-1} \left(\int_{R_+^{n+1}} \frac{(1 + |v|)^{-\lambda n} dv ds}{s^{2n+3} \left(1 + |s^{-1}z' - v| \right)^{2(n+2)}} \right)^{\frac{1}{2}} d\theta$$

$$= Cr \int_0^1 |z|^{-n-1} \left(\int_0^{+\infty} \frac{h_{z'}(s) ds}{s^{2n+3}} \right)^{\frac{1}{2}} d\theta;$$

where $z = |x_0 - y + \theta(x - x_0)|$, $z' = z|z|^{-1}$, $h_{z'}(s) = \int_{R^n} \frac{(1+|v|)^{-\lambda n} dv}{(1+|s^{-1}z'-v|)^{2(n+2)}}$.
Next we estimate $h_{z'}(s)$.

If $s \geq 1$, then

$$h_{z'}(s) \leq \int_{R^n} \frac{dv}{(1 + |v|)^{\lambda n}} = C(n, \lambda);$$

If $s < 1$, then

$$h_{z'}(s) \leq \int_{|v| < \frac{1}{2s}} + \int_{\frac{1}{2s} \leq |v| \leq \frac{2}{s}} + \int_{|v| > \frac{2}{s}};$$

Note that if $|v| < \frac{1}{2s}$, then $|s^{-1}z' - v| \geq \frac{1}{2s}$,

$$\int_{|v| < \frac{1}{2s}} \frac{(1 + |v|)^{-\lambda n} dv}{(1 + |s^{-1}z' - v|)^{2(n+2)}} \leq 2^n \int_{R^n} \frac{s^{2(n+2)}(1 + |v|)^{-\lambda n} dv}{(1 + 2s)^{2(n+2)}} \leq C s^{2(n+2)};$$

If $\frac{1}{2s} \leq |v| \leq \frac{2}{s}$, then $1 + |v| \sim \frac{1}{s}$,

$$\int_{\frac{1}{2s} \leq |v| \leq \frac{2}{s}} \frac{(1 + |v|)^{-\lambda n} dv}{(1 + |s^{-1}z' - v|)^{2(n+2)}}$$

$$\leq C \int_{\frac{1}{2s} \leq |v| \leq \frac{2}{s}} \frac{s^{\lambda n} dv}{(1 + |s^{-1}z' - v|)^{2(n+2)}} \leq C s^{(\lambda-1)n};$$

If $|v| > \frac{2}{s}$, then$|s^{-1}z' - v| \geq |v| - \frac{1}{s} \geq \frac{1}{2}|v|$,

$$\int\limits_{|v|>\frac{2}{s}} \frac{(1+|v|)^{-\lambda n}dv}{(1+|s^{-1}z'-v|)^{2(n+2)}} \leq C \int\limits_{|v|>\frac{2}{s}} |v|^{-\lambda n-2n-4}dv \leq Cs^{\lambda n+2n+3};$$

Finally by noting $\lambda > 3 + \frac{2}{n}, (\lambda - 1)n > 2n + 2,$

$$\int_0^{+\infty} \frac{h_{z'}(s)ds}{s^{2n+3}} \leq C;$$

hence

$$II(x) \leq Cr \int_0^1 |z|^{-n-1}d\theta \leq Cr|x_0 - y|^{-n-1};$$

and (3.5) is proved.

Remark 3.1.   We need the assumption of $\lambda > 3 + \frac{2}{n}$, while in [8] only $\lambda > 1 + \frac{2}{n}$ is assumed.

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