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LIOUVILLE SEQUENCES*

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Abstract. The new concept of a Liouville sequence is introduced in this paper by mean of the related Liouville series. Main results are two criteria for when certain sequences are Liouville. Several applications are presented. A counterexample is included for the case that we substantially weaken the hypotheses in the main results.

§1. Introduction

We can define the Liouville numbers in the following way. Let α be a real number. If for every positive real number r there exist integers p and q such that $0 < |\alpha - p/q| < 1/q^r$ then the number α is Liouville.

There are many results concerning the Liouville numbers. Bundschuh in [3] presents a Liouville-type estimate. This paper also contains a list of references including [6] and [8] which present the criteria for algebraic independence of certain Liouville series. A survey of these types of results can be found in the book of Nishioka [7]. Also the result of Petruska [9] establishes several interesting criteria concerning the strong Liouville numbers. The latter was first defined by Erdös in [4].

If the sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers tends to infinity very fast then the continued fraction $[a_1, a_2, a_3, \dots]$ is a Liouville number. The algebraic independence of certain Liouville continued fractions is to be found in [1] or [2].

It is relatively easy to prove that the number $\sum_{n=1}^{\infty} 1/(n!)^n$ is Liouville. This suggests similar results for another infinite series. In 1975 Erdös [5] proved a very interesting criterion for Liouville series.

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THEOREM 1.1. (Erdös) *Let $a_1 < a_2 < a_3 < \cdots$ be an infinite sequence of integers satisfying*

$$\limsup_{n\to\infty} a_n^{\frac{1}{t^n}} = \infty$$

for every $t > 0$, and

$$a_n > n^{1+\epsilon}$$

for some fixed $\epsilon > 0$ and $n > n_0(\epsilon)$. Then

$$\alpha = \sum_{n=1}^{\infty} \frac{1}{a_n}$$

is a Liouville number.

We define Liouville sequences and present two criteria for them in Theorem 2.1 and Theorem 2.2. The former generalizes the above result of Erdös. Several examples of Liouville series are included.

§2. Liouville sequences

DEFINITION 2.1. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. If, for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers, the sum $\sum_{n=1}^{\infty} 1/a_n c_n$ is a Liouville number, then the sequence is called Liouville.*

THEOREM 2.1. *Let ϵ, ϵ_1 and ϵ_2 be three positive real numbers satisfying $\epsilon_1 < \epsilon/(1+\epsilon) = \epsilon_2$. Let s be a nonnegative integer, and $\{L_j(x)\}_{j=0}^{s+2}$ be a sequence of functions defined by $L_0(x) = x$, $L_{j+1}(x) = \log_2(L_j(x))$, $j = 0, 1, \ldots, s+1$, for all sufficiently large positive real number x. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of positive integers such that $\{a_n\}_{n=1}^{\infty}$ is nondecreasing,*

$$\limsup_{n\to\infty} \frac{1}{n} L_{s+2}(a_n) = \infty, \tag{1}$$

$$a_n > \Big(\prod_{j=0}^{s} L_j(n)\Big) L_s^{\epsilon}(n) \tag{2}$$

and

$$b_n < L_s^{\epsilon_1}(a_n) \tag{3}$$

hold for every sufficiently large positive integer n. Then the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ is Liouville.

LEMMA 2.1. *Let s, ϵ, ϵ_2 and $L_j(x)$, $j = 0, 1, \ldots, s+2$ satisfy all conditions stated in Theorem 2.1. Denote*

$$y = f(x) = \Big(\prod_{j=0}^{s} L_j(x)\Big) L_s^{\epsilon}(x).$$

Let $x = F(y)$ be the inverse function for $y = f(x)$. Then

$$F(y) \geq y^{\frac{1}{1+\epsilon}} \tag{4}$$

and

$$y L_s^{-\epsilon_2}(y) \geq F(y) \tag{5}$$

for every sufficiently large positive real number y.

Proof (*of Lemma* 2.1). Let $s = 0$. Then $y = f(x) = x^{1+\epsilon}$ and $x = F(y) = y^{1/(1+\epsilon)}$. Thus (4) and (5) hold.

Assume $s > 0$. Then for sufficiently large x

$$f(x) = \Big(\prod_{j=0}^{s} L_j(x)\Big) L_s^{\epsilon}(x) \leq x^{1+\epsilon}.$$

From this we obtain

$$F(y) \geq y^{\frac{1}{1+\epsilon}}$$

for sufficiently large y. Thus (4) holds.

From $s > 0$ and (4) we obtain that for every sufficiently large y there is a constant b which does not depend on y such that

$$y \geq x L_s^{1+\epsilon}(x) \geq b x L_s^{1+\epsilon}(y) \geq x L_s^{\epsilon_2}(y).$$

Hence (5) holds and the proof of Lemma 2.1 is completed. ∎

LEMMA 2.2. *Let R be a real number such that $R > 1$. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a nondecreasing sequence of positive integers. Let k be a positive integer such that $a_k > 2^{R^{3k}}$. Then there is a positive integer t not greater than k such that*

$$\prod_{a_n \leq 2^{R^{t+k}}} a_n \leq 2^{\frac{1}{R-1}R^{t+k}}.$$

Proof (*of Lemma* 2.2). Denote M the number of a_n such that $a_n \leq 2^{R^k}$. Let P_j ($j = 0, 1, 2, \ldots, k$) be the number of a_n such that $a_n \in (2^{R^k}, 2^{R^{j+k}}]$ and denote $Q_j = j - P_j - M$ ($j = 0, 1, \ldots, k$). From this and the fact that $a_k > 2^{R^{3k}}$ we obtain $Q_0 = -M$, Q_j is an integer, $Q_{j+1} - Q_j \leq 1$ ($j = 0, 1, \ldots, k$) and $Q_k = k - P_k - M \geq 1$. Then there is a least positive integer $t > M$ such that $Q_t = t - P_t - M = 1$. Thus $Q_{t-1} = 0$ and there is no a_n such that $a_n \in (2^{R^{t-1+k}}, 2^{R^{t+k}}]$. In addition, for every $v = 1, 2, \ldots, t$ the number of a_n such that $a_n \in (2^{R^{t-v+k}}, 2^{R^{t+k}}]$ is less than v, otherwise the number t will not be the least; and the number of a_n such that $a_n \in (2^{R^k}, 2^{R^{t+k}}]$ is equal to $t - M - 1$. It follows that

$$\begin{aligned}
\prod_{a_n \leq 2^{R^{t+k}}} a_n &= \prod_{n=1}^{M} a_n \prod_{a_n \in (2^{R^k}, 2^{R^{t+k}}]} a_n \\
&\leq \prod_{n=1}^{M} a_n \prod_{n=M+1}^{t-1} 2^{R^{n+k}} \\
&\leq \prod_{n=0}^{t-1} 2^{R^{n+k}} \\
&= 2^{\frac{R^{t+k}-R^k}{R-1}} \\
&\leq 2^{\frac{1}{R-1}R^{t+k}}.
\end{aligned}$$

The proof of Lemma 2.2 is complete. ∎

Proof (*of Theorem* 2.1). Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of positive integers. Then the sequences $\{a_n c_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ also satisfy conditions (1)–(3) and if in addition we reorder the sequence $\{a_n c_n\}_{n=1}^{\infty}$ to be nondecreasing then the new sequence together with the relevant reordered sequence

$\{b_n\}_{n=1}^{\infty}$ will satisfy (1)–(3) also. Thus it suffices to prove Theorem 2.1 for nondecreasing sequence $\{a_n\}_{n=1}^{\infty}$ and arbitrary sequence $\{b_n\}_{n=1}^{\infty}$ of positive integers satisfying (1)–(3). So it suffices to prove that the series $\alpha = \sum_{n=1}^{\infty} b_n/a_n$ is a Liouville number. To establish this we find positive integer n for every $r > 2$ such that

$$\Big(\prod_{j=1}^{n} a_j\Big)^r \sum_{j=1}^{\infty} \frac{b_{n+j}}{a_{n+j}} < 1. \tag{6}$$

Let R be a sufficiently large positive real number. Equation (1) implies that there is the least positive integer k such that

$$L_{s+2}(a_k) > 3k \log_2 R.$$

From this we obtain

$$L_s(a_k) > 2^{R^{3k}}. \tag{7}$$

Lemma 2.2 and (7) imply that there is a positive integer t such that $k \geq t$ and

$$\prod_{a_n \leq 2^{R^{t+k}}} a_n \leq 2^{\frac{1}{R-1} R^{t+k}}. \tag{8}$$

Now we have

$$\sum_{a_n > 2^{R^{t+k}}} \frac{b_n}{a_n} = \sum_{a_k > a_n > 2^{R^{t+k}}} \frac{b_n}{a_n} + \sum_{j=0}^{\infty} \frac{b_{k+j}}{a_{k+j}}. \tag{9}$$

We will estimate both sums on the right hand side of equation (9).

Let us consider the first sum. Inequality (3) and $\epsilon_1 < 1$ imply

$$\begin{aligned} \sum_{a_k > a_n > 2^{R^{t+k}}} \frac{b_n}{a_n} &\leq \sum_{a_k > a_n > 2^{R^{t+k}}} \frac{L_s^{\epsilon_1}(a_n)}{a_n} \\ &\leq \sum_{a_k > a_n > 2^{R^{t+k}}} \frac{1}{a_n^{1-\epsilon_1}} \\ &\leq k 2^{-R^{t+k}(1-\epsilon_1)} \\ &\leq 2^{-\frac{1-\epsilon_1}{2} R^{t+k}}. \end{aligned} \tag{10}$$

Now we will estimate the second sum.

$$\sum_{j=0}^{\infty}\frac{b_{k+j}}{a_{k+j}}=\sum_{k+j\le F(a_k)}\frac{b_{k+j}}{a_{k+j}}+\sum_{k+j>F(a_k)}\frac{b_{k+j}}{a_{k+j}},\tag{11}$$

where $F(y)$ is the inverse function of $y=f(x)=(\prod_{j=0}^{s}L_j(x))L_s^{\epsilon}(x)$. Then inequality (3), Lemma 2.1 and the fact that the function $x^{-1}L_s^{\epsilon_1}(x)$ is decreasing imply

$$\begin{aligned}\sum_{k+j\le F(a_k)}\frac{b_{k+j}}{a_{k+j}}&\le\sum_{k+j\le F(a_k)}\frac{L_s^{\epsilon_1}(a_{k+j})}{a_{k+j}}\\&\le\frac{F(a_k)L_s^{\epsilon_1}(a_k)}{a_k}\\&\le\frac{a_kL_s^{\epsilon_1}(a_k)}{L_s^{\epsilon_2}(a_k)a_k}\\&=\frac{1}{L_s^{\epsilon_2-\epsilon_1}(a_k)}.\end{aligned}\tag{12}$$

Inequalities (2), (3) and Lemma 2.1 imply

$$\begin{aligned}\sum_{k+j>F(a_k)}\frac{b_{k+j}}{a_{k+j}}&\le\sum_{k+j>F(a_k)}\frac{L_s^{\epsilon_1}(a_{k+j})}{a_{k+j}}\\&\le\sum_{k+j>F(a_k)}\frac{L_s^{\epsilon_1}((\prod_{i=0}^{s}L_i(k+j))L_s^{\epsilon}(k+j))}{(\prod_{i=0}^{s}L_i(k+j))L_s^{\epsilon}(k+j)}\\&\le\sum_{k+j>F(a_k)}\frac{BL_s^{\epsilon_1(1+\epsilon)}(k+j)}{(\prod_{i=0}^{s}L_i(k+j))L_s^{\epsilon}(k+j)}\\&=\sum_{k+j>F(a_k)}\frac{B}{(\prod_{i=0}^{s}L_i(k+j))L_s^{(1+\epsilon)(\epsilon_2-\epsilon_1)}(k+j)}\\&\le C\int_{F(a_k)}^{\infty}\frac{dx}{(\prod_{i=0}^{s}L_i(x))L_s^{(1+\epsilon)(\epsilon_2-\epsilon_1)}(x)}\\&\le\frac{D}{L_s^{(1+\epsilon)(\epsilon_2-\epsilon_1)}(F(a_k))}\\&\le\frac{E}{L_s^{\epsilon_2-\epsilon_1}(a_k)},\end{aligned}\tag{13}$$

where B, C, D and E are suitable positive real constants depending only on s, ϵ and ϵ_1. From (11), (12) and (13) we obtain

$$\sum_{j=0}^{\infty} \frac{b_{k+j}}{a_{k+j}} \leq \frac{E+1}{L_s^{\epsilon_2-\epsilon_1}(a_k)} \tag{14}$$

Now (7), (9), (10), (14) and the fact that R is sufficiently large imply

$$\begin{aligned}\sum_{a_n > 2^{R^{t+k}}} \frac{b_n}{a_n} &\leq 2^{-\frac{1-\epsilon_1}{2} R^{t+k}} + \frac{E+1}{L_s^{\epsilon_2-\epsilon_1}(a_k)} \\ &\leq 2^{-\frac{1-\epsilon_1}{2} R^{t+k}} + \frac{E+1}{2^{(\epsilon_2-\epsilon_1)R^{3k}}} \\ &\leq 2^{-\frac{1-\epsilon_1}{3} R^{t+k}}.\end{aligned}$$

From this and (8) we obtain

$$\begin{aligned}\Big(\prod_{a_k \leq 2^{R^{t+k}}} a_n\Big)^r \sum_{a_n > 2^{R^{t+k}}} \frac{b_n}{a_n} &\leq \frac{2^{\frac{r}{R-1} R^{t+k}}}{2^{\frac{1-\epsilon_1}{3} R^{t+k}}} \\ &= 2^{-(\frac{1-\epsilon_1}{3} - \frac{r}{R-1})R^{t+k}}.\end{aligned}$$

This implies (6) for a sufficiently large R. The proof of Theorem 2.1 is complete. ∎

Corollary 2.1. *Let ϵ, ϵ_1 be two positive real numbers satisfying $\epsilon_1 < \epsilon/(1+\epsilon)$, and $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of positive integers such that $\{a_n\}_{n=1}^{\infty}$ is nondecreasing,*

$$\limsup_{n\to\infty} \frac{\log\log a_n}{n} = \infty,$$

$$a_n > n^{1+\epsilon}$$

and

$$b_n < a_n^{\epsilon_1}$$

hold for every sufficiently large positive integer n. Then the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ is Liouville.

Corollary 2.1 is an immediate consequence of Theorem 2.1. It is enough to put $s = 0$.

Corollary 2.2. *Let $\epsilon > 0$ be a real number, s be a nonnegative integer, $L_j(x)$ be as in Theorem 2.1, and $\{a_n\}_{n=1}^{\infty}$ be a nondecresing sequence of positive integers such that*

$$\limsup_{n\to\infty} \frac{L_{s+2}(a_n)}{n} = \infty$$

and

$$a_n > \Big(\prod_{j=0}^{s} L_j(n)\Big) L_s^{\epsilon}(n)$$

hold for every sufficiently large positive integer n. Then the sequence $\{a_n\}_{n=1}^{\infty}$ is Liouville.

Corollary 2.2 is an immediate consequence of Theorem 2.1. It is enough to put $b_n = 1$ for every positive integer n.

Theorem 2.2. *Let a, b and c be three real numbers satisfying $0 < a < 1$ and $0 < c < b$. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of positive integers, $\{a_n\}_{n=1}^{\infty}$ is nondecreasing, such that*

$$\limsup_{n\to\infty} \frac{1}{n} \log_2 \log_2 a_n = \infty, \tag{15}$$

$$a_n > n 2^{b(\log_2 n)^a} \tag{16}$$

and

$$b_n < 2^{c(\log_2 a_n)^a} \tag{17}$$

hold for every sufficiently large positive integer n. Then the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ is Liouville.

LEMMA 2.3. *Let a and b be two positive real numbers such that $0 < a < 1$. Denote $y = g(x) = x2^{b(\log_2 x)^a}$. Let $x = G(y)$ be the inverse function of $y = g(x)$. Then for every real number $d < b$*

$$y2^{-d(\log_2 y)^a} > G(y) > y2^{-b(\log_2 y)^a} \tag{18}$$

hold for every sufficiently large positive real number y.

Proof (*of Lemma* 2.3). In fact,

$$G(y) = y2^{-b(\log_2 x)^a}$$

with $y \geq x$ ($x \geq 1$) implies the right hand side of inequality (18). On the other hand, by using the fact that $y < x^{1+\delta}$ with an arbitrary given $\delta > 0$ holds for every sufficiently large x,

$$\begin{aligned} G(y) &= y2^{-b(\log_2 x)^a} \\ &< y2^{-(1+\delta)^{-a} b(\log_2 y)^a}, \end{aligned}$$

which implies the left hand side of inequality (18). ∎

Proof (*of Theorem* 2.2). As in Theorem 2.1 it suffices to prove that the series $\alpha = \sum_{n=1}^{\infty} b_n/a_n$ is a Liouville number for the nondecreasing sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers. To establish this we find a positive integer n for every $r > 2$ such that (6) holds.

Let R be a sufficiently large positive real number. Equation (15) implies that there is the least positive integer k such that

$$\log_2 \log_2 a_k > \frac{3}{a} k \log_2 R.$$

From this we obtain

$$a_k > 2^{R^{\frac{3}{a}k}}. \tag{19}$$

Lemma 2.2 and (19) imply that there is a positive integer t such that $k \geq t$ and

$$\prod_{a_n \leq 2^{R^{t+k}}} a_n \leq 2^{\frac{1}{R-1} R^{t+k}}. \tag{20}$$

As in the proof of Theorem 2.1 we have

$$\sum_{a_n>2^{R^{t+k}}} \frac{b_n}{a_n} = \sum_{a_k>a_n>2^{R^{t+k}}} \frac{b_n}{a_n} + \sum_{j=0}^{\infty} \frac{b_{k+j}}{a_{k+j}}. \tag{21}$$

From (17) we obtain

$$\begin{aligned} \sum_{a_k>a_n>2^{R^{t+k}}} \frac{b_n}{a_n} &\le \sum_{a_k>a_n>2^{R^{t+k}}} \frac{2^{c(\log_2 a_n)^a}}{a_n} \\ &\le \frac{k2^{c(\log_2(2^{R^{t+k}}))^a}}{2^{R^{t+k}}} \\ &= \frac{k2^{cR^{(t+k)a}}}{2^{R^{t+k}}} \\ &\le 2^{-\frac{1}{2}R^{t+k}}. \end{aligned} \tag{22}$$

Now we will estimate the second sum of the right-hand side of equation (21).

$$\sum_{j=0}^{\infty} \frac{b_{k+j}}{a_{k+j}} = \sum_{k+j\le G(a_k)} \frac{b_{k+j}}{a_{k+j}} + \sum_{k+j>G(a_k)} \frac{b_{k+j}}{a_{k+j}}, \tag{23}$$

where $G(y)$ is the inverse function of $y = g(x) = x2^{b(\log_2 x)^a}$. Then (17), Lemma 2.3 and the fact that the function $x^{-1}2^{c(\log_2 x)^a}$ is decreasing imply

$$\begin{aligned} \sum_{k+j\le G(a_k)} \frac{b_{k+j}}{a_{k+j}} &\le \sum_{k+j\le G(a_k)} \frac{2^{c(\log_2 a_{k+j})^a}}{a_{k+j}} \\ &\le \frac{G(a_k)2^{c(\log_2 a_k)^a}}{a_k} \\ &\le \frac{a_k 2^{-\frac{b+c}{2}(\log_2 a_k)^a}2^{c(\log_2 a_k)^a}}{a_k} \\ &= 2^{-\frac{b-c}{2}(\log_2 a_k)^a}. \end{aligned} \tag{24}$$

Inequalities (16), (17) and Lemma 2.3 imply

$$\sum_{k+j>G(a_k)} \frac{b_{k+j}}{a_{k+j}} \le \sum_{k+j>G(a_k)} \frac{2^{c(\log_2 a_{k+j})^a}}{a_{k+j}} \tag{25}$$

$$\begin{aligned}
&\le \sum_{k+j>G(a_k)} \frac{2^{c(\log_2((k+j)2^{b(\log_2(k+j))^a}))^a}}{(k+j)2^{b(\log_2(k+j))^a}} \\
&\le \sum_{k+j>G(a_k)} \frac{1}{(k+j)2^{\frac{b-c}{2}(\log_2(k+j))^a}(\log_2(k+j))^{1-a}} \\
&\le J\int_{G(a_k)}^{\infty} \frac{dx}{x2^{\frac{b-c}{2}(\log_2 x)^a}(\log_2 x)^{1-a}} \\
&\le \frac{L}{2^{\frac{b-c}{2}(\log_2 G(a_k))^a}} \\
&\le \frac{1}{2^{\frac{b-c}{3}(\log_2 a_k)^a}},
\end{aligned}$$

where J and L are suitable positive real constants. From (23), (24) and (25) we obtain

$$\sum_{j=0}^{\infty} \frac{b_{k+j}}{a_{k+j}} \le 2\cdot 2^{-\frac{b-c}{3}(\log_2 a_k)^a}. \tag{26}$$

Now (19), (21), (22) and (26) imply

$$\begin{aligned}
\sum_{a_n>2^{R^{t+k}}} \frac{b_n}{a_n} &\le 2^{-\frac{1}{2}R^{t+k}} + 2\cdot 2^{-\frac{b-c}{3}(\log_2 a_k)^a} \\
&\le 2^{-\frac{1}{2}R^{t+k}} + 2\cdot 2^{-\frac{b-c}{3}(\log_2 2^{R^{\frac{3}{a}k}})^a} \\
&\le 2^{-\frac{1}{2}R^{t+k}} + 2\cdot 2^{-\frac{b-c}{3}R^{3k}} \\
&\le 2^{-\frac{1}{3}R^{t+k}}.
\end{aligned}$$

From this and (20) we obtain

$$\begin{aligned}
\Big(\prod_{a_n\le 2^{R^{t+k}}} a_n\Big)^r \sum_{a_n>2^{R^{t+k}}} \frac{b_n}{a_n} &\le 2^{\frac{r}{R-1}R^{t+k}} 2^{-\frac{1}{3}R^{t+k}} \\
&= 2^{-(\frac{1}{3}-\frac{r}{R-1})R^{t+k}}.
\end{aligned}$$

This implies (6) for a sufficiently large R and the proof of Theorem 2.2 is complete.

§3. Examples and comments

Remark 3.1. Put $b_n = 1$ for every positive integer n in Corollary 2.1 or $s = 0$ in Corollary 2.2. Then we obtain the Erdös theorem.

EXAMPLE 3.1. The sequences

$$\left\{\frac{2^{n^n}+1}{n^n}\right\}_{n=1}^{\infty}, \left\{\frac{2^{n!}+1}{n!}\right\}_{n=1}^{\infty}, \left\{\frac{2^{n!}+1}{3^{3^n}}\right\}_{n=1}^{\infty},$$
$$\left\{\frac{3^{n^n}+1}{2^{3^n}}\right\}_{n=1}^{\infty}, \left\{\frac{5^{n!}+1}{4^{n!}+1}\right\}_{n=1}^{\infty}$$

and

$$\left\{\frac{3^{n!}+1}{2^{n!}}\right\}_{n=1}^{\infty}$$

are Liouville.

EXAMPLE 3.2. Let $a_1 = 2$,

$$\begin{aligned}
a_k &= 2^{2^{2^2}} + k - 2, & k &= 2, 3, \dots, 2^{2^{2^2}} 2^{-2\cdot 2^2} = n_1 - 1,\\
a_k &= 2^{2^{{n_1}^{n_1}}} + k - n_1, & k &= n_1, \dots, 2^{2^{{n_1}^{n_1}}} 2^{-2{n_1}^{n_1}} = n_2 - 1,\\
a_k &= 2^{2^{{n_2}^{n_2}}} + k - n_2, & k &= n_2, \dots, 2^{2^{{n_2}^{n_2}}} 2^{-2{n_2}^{n_2}} = n_3 - 1
\end{aligned}$$

and so on. Then the sequence $\{a_n\}_{n=1}^{\infty}$ is Liouville.

Remark 3.2. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} a_n = \infty$ and $\lim_{n\to\infty} a_n = 0$. Then, for every positive real number x, there is a subsequence $\{B_n\}_{n=1}^{\infty}$ of the sequence $\{a_n\}_{n=1}^{\infty}$ such that $x = \sum_{n=1}^{\infty} B_n$. Indeed, let us put $x_0 = 0$ and

$$x_{n+1} = \begin{cases} x_n & \text{if } a_{n+1} + x_n \ge x,\\ x_n + a_{n+1} & \text{if } a_{n+1} + x_n < x. \end{cases}$$

Then $\{x_n\}_{n=1}^{\infty}$ is a nondecreasing sequence. From $\sum_{n=1}^{\infty} a_n = \infty$, $\lim_{n\to\infty} a_n = 0$ and the definition of the sequence $\{x_n\}_{n=1}^{\infty}$ we obtain $\lim_{n\to\infty} x_n = x$. Let $\{B_n\}_{n=1}^{\infty}$ consist of positive terms of the sequence

$\{x_n - x_{n-1}\}_{n=1}^{\infty}$. Then $\{B_n\}_{n=1}^{\infty}$ is a subsequence of the sequence $\{a_n\}_{n=1}^{\infty}$ and

$$\begin{aligned}\sum_{n=1}^{\infty} B_n &= \sum_{n=1}^{\infty}(x_n - x_{n-1}) \\ &= \lim_{n\to\infty} x_n - x_0 \\ &= x.\end{aligned}$$

EXAMPLE 3.3. Let $a_1 = 2$ and

$$a_k = \max\bigl(2^{2^{2^2}} + k - 2, [\![k(\log_2 k)\log_2\log_2 k]\!]\bigr), \qquad k = 2, \dots, n_1 - 1,$$

where $[\![x]\!]$ is the greatest integer not greater than x and n_1 is the least positive integer such that $\sum_{j=1}^{n_1-1} 1/a_j > 1$. (Such n_1 must exist because $\sum_{k=1}^{\infty} 1/(k(\log_2 k)\log_2\log_2 k) = \infty$.) Let

$$a_k = \max\bigl(2^{2^{n_1^{n_1}}} + k - n_1, [\![k(\log_2 k)\log_2\log_2 k]\!]\bigr), \qquad k = n_1, n_1+1, \dots, n_2 - 1,$$

where n_2 is the least positive integer such that $\sum_{j=1}^{n_2-1} 1/a_j > 2$ and so on. The series $\sum_{n=1}^{\infty} 1/a_n$ is divergent but $\limsup_{n\to\infty}((1/n)\log_2\log_2 a_n) = \infty$. This and Remark 3.2 imply that the sequence $\{a_n\}_{n=1}^{\infty}$ contains a subsequence $\{d_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} 1/d_n$ is a rational number. On the other side $d_n > n(\log_2 n)\log_2\log_2 n$ and $\limsup_{n\to\infty}((1/n)\log_2\log_2 d_n) = \infty$.

OPEN PROBLEM 3.1. *It is not known if the infinite sequence* $\{(2^{n^n}+1)2^{n!}/2^{n^n}\}_{n=1}^{\infty}$ *is Liouville or not. Let us note that the infinite series* $\sum_{n=1}^{\infty} 2^{n^n}/(2^{n^n}+1)2^{n!}$ *converges very rapidly.*

EXAMPLE 3.4. Let $h_1 = 1$ and $h_{n+1} = 2^{(h_n!)^2}$ for every positive integer n. Put $a_k = k2^{h_n!}$ for every $k = h_n, h_n+1, \dots, h_{n+1}-1$ and $n = 1, 2, \dots$. Denote $r(n)$ the number of primes which divide n. Let j be a positive integer. Then the sequence $\{a_n/r^j(n)\}_{n=1}^{\infty}$ is Liouville. It is a great surprise because there are infinitely many n such that $\sum_{j=n}^{\infty} 1/a_j > 1/(2n2^{\sqrt{\log_2 n}})$.

EXAMPLE 3.5. Let $\{G_n\}_{n=1}^{\infty}$ be the linear recurrence sequence of the k-th order such that $G_1, G_2, \dots, G_k, b_0, \dots, b_k$ are positive integers and for every positive integer n, $G_{n+k} = G_n b_0 + G_{n+1} b_1 + \cdots + G_{n+k-1} b_{k-1}$. If the roots $\alpha_1, \dots, \alpha_s$ of the equation $x^k = b_0 + b_1 x + \cdots + b_{k-1} x^{k-1}$ satisfy $|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_s|$, $|\alpha_1| > 1$ and α_1/α_j is not the root of unity for every $j = 2, 3, \dots, s$, then the sequence $\{G_{n^n}/G_{n!}\}_{n=1}^{\infty}$ is Liouville.

This is an immediate consequence of Corollary 2.1 and the inequality

$$|\alpha_1|^{n(1-\epsilon)} < G_n < |\alpha_1|^{n(1+\epsilon)}$$

which can be found in [10], for instance.

Remark 3.3. Put $b_n = 1$ in Theorem 2.2. Then we obtain the Erdös theorem with the weaker condition $a_n > n 2^{(\log_2 n)^a}$ instead of $a_n > n^{1+\epsilon}$. On the other hand Example 3.3 demonstrates that we cannot greatly improve condition (2) in Theorem 2.1, or condition (16) in Theorem 2.2, to give a negative answer to the Erdös problem if it is possible to substantially weaken the condition $a_n > n^{1+\epsilon}$. For more details see [5], for instance.

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