O. Fujino
Nagoya Math. J.
Vol. 172 (2003), 129–171

A CANONICAL BUNDLE FORMULA FOR CERTAIN ALGEBRAIC FIBER SPACES AND ITS APPLICATIONS

OSAMU FUJINO

Abstract. We investigate period maps of polarized variations of Hodge structures of weight one or two. We treat the case when the period domains are bounded symmetric domains. We deal with a relationship between canonical extensions of some Hodge bundles and automorphic forms. As applications, we obtain a canonical bundle formula for certain algebraic fiber spaces, such as Abelian fibrations, $K3$ fibrations, and solve Iitaka's famous conjecture $C_{n,m}$ for some algebraic fiber spaces.

§1. Introduction

We start in recalling Kodaira's canonical bundle formula:

Theorem 1.1. ([Kod]) *If $f : X \to C$ is a minimal elliptic surface over $\mathbb{C}$, then the relative canonical divisor $K_{X/C}$ is expressed as*

$$K_{X/C} = f^*L + \sum_P \frac{m_P - 1}{m_P} f^*(P),$$

where L is a nef divisor on C and P runs over the set of points such that $f^*(P)$ is a multiple fiber with multiplicity $m_P > 1$.

Furthermore, $12L$ is expressed as

$$12K_{X/C} = f^*J^*\mathcal{O}_{\mathbb{P}^1}(1) + 12\sum_P \frac{m_P - 1}{m_P} f^*(P) + \sum \sigma_Q f^*(Q),$$

where σ_Q is an integer $\in [0, 12)$ and $J : C \to \mathbb{P}^1$ is the j-function [Ft, Section 2] (see [Ft, (2.9)] and [U1]).

In our notation, the above formula means that

$$\mathcal{O}_C(12L^{ss}_{X/C}) \simeq J^*\mathcal{O}_{\mathbb{P}^1}(1),$$

Received July 16, 2001.
1991 Mathematics Subject Classification: Primary 14J15, 14J10; Secondary 14N30, 14J40.

where the nef $\mathbb{Q}$-divisor $L^{ss}_{X/C}$ is the *semistable part* of $K_{X/C}$. For the definition of $L^{ss}_{X/Y}$, see Definition 3.5.

The main purpose of this paper is to generalize the above formula for $K3$ and Abelian fibrations $f : X \to Y$. The following is one of the main theorems (Theorem 5.1 and Remark 5.2) of this paper (see also Theorems 2.10, 2.11).

THEOREM 1.2. *Let $f : X \to Y$ be a surjective morphism between nonsingular projective varieties X and Y. Let $\mathcal{L}$ be an f-ample line bundle on X. Assume that there exists a simple normal crossing divisor Σ on Y such that every fiber of f over $Y_0 := Y \setminus \Sigma$ is a $K3$ surface* (*resp. an n-dimensional Abelian variety*)*. Then we obtain a polarized variation of Hodge structures of weight two* (*resp. one*)*. Let D be the period domain and Γ the arithmetic subgroup which fixes the polarized Hodge structure* (*for the details, see* 2.4 (*Period domains*))*. We put $\mathcal{S} := D/\Gamma$. Let $p_0 : Y_0 \to \mathcal{S}$ be the period map of the weight two* (*resp. one*) *polarized Hodge structures and $p : Y \to \bar{\mathcal{S}}$ be the extension of p_0. We note that the projective variety $\bar{\mathcal{S}}$ is the Baily-Borel-Satake compactification of $\mathcal{S}$, which is embedded into a projective space by automorphic forms of the same weight, say k, and that p always exists by Borel's extension theorem. We define $a = 19k$* (*resp. $a = k(n+1)$*)*. Then $aL^{ss}_{X/Y}$ is a Weil divisor and*

$$\mathcal{O}_Y(aL^{ss}_{X/Y}) \simeq p^*\mathcal{O}_{\bar{\mathcal{S}}}(1).$$

We note that a is decided only by the polarized Hodge structures of the fibers of f and independent of Y.

We further assume that f is semistable in codimension one. Then we obtain

$$(f_*\omega_{X/Y})^{\otimes a} \simeq \mathcal{O}_Y(aL^{ss}_{X/Y}) \simeq p^*\mathcal{O}_{\bar{\mathcal{S}}}(1).$$

In Section 2, we treat the period map. The variation of Hodge structures in Section 2 doesn't need to be geometric. We give a purely Hodge theoretic proof of Theorem 1.2 in Section 5. Theorems 2.10 and 2.11 are much stronger than the above stated theorem in some sense. However, in order to state Theorems 2.10 and 2.11, we need various notation and assumptions. So, we omit them here. Theorem 2.10 seems to have some applications to the study of symplectic manifolds (see Theorem 5.6).

Combining this theorem with [FM, Proposition 2.8, Theorem 3.1], we have the following formula.

COROLLARY 1.3. *Under the notation and assumptions of Theorem* 1.2, *we have a canonical bundle formula*

$$K_X = f^*(K_Y + L^{ss}_{X/Y}) + \sum_P s_P f^*P + B,$$

where P, B *and* $L^{ss}_{X/Y}$ *are as follows.*

(0) P *runs through all the irreducible components of* Σ.

(1) $f_*\mathcal{O}_X(\lfloor iB_+ \rfloor) = \mathcal{O}_Y$ *for all* $i \geq 0$, *where* B_+ *is the positive part of* B.

(2) $\operatorname{codim}_Y f(\operatorname{Supp} B_-) \geq 2$, *where* B_- *is the negative part of* B.

(3) $\mathcal{O}_Y(aL^{ss}_{X/Y}) \simeq p^*\mathcal{O}_{\bar{S}}(1)$ *as in Theorem* 1.2.

(4) *Let*

$$N := \operatorname{lcm}\Big\{ y \in \mathbb{Z}_{>0} \;\Big|\; \varphi(y) \leq 22 \; \big(\textit{resp. } \varphi(y) \leq \frac{(2n)!}{n!n!}\big) \Big\},$$

where φ *is the Euler function. Then* $NL^{ss}_{X/Y}$ *is a Weil divisor, and for each* P, *there exist* $u_P, v_P \in \mathbb{Z}_{>0}$ *such that* $0 < v_P \leq N$ *and* $s_P = (Nu_P - v_P)/(Nu_P)$.

For various applications of the (log-)canonical bundle formula, see [FM, Sections 5, 6].

We note that the semistable part $L^{ss}_{X/Y}$ is semi-ample under certain weaker assumptions. For the precise statement, see Section 6.

By an application of the semi-ampleness of the semistable part, we deal with Iitaka's conjecture $C_{n,m}$. The following is a very special case of Theorem 7.4 (see also Corollary 7.6).

THEOREM 1.4. *Let* $f : X \to Y$ *be a surjective morphism with connected fibers between non-singular projective varieties* X *and* Y. *Assume that the Kodaira dimension of the generic fiber of* f *is one, that is,* $\kappa(X_\eta) = 1$, *where* η *is the generic point of* Y. *Let*

$$f : X \xrightarrow{\;g\;} Z \xrightarrow{\;h\;} Y$$

be the relative Iitaka fiber space. Assume that general fibers of g *are Abelian varieties. Then the inequality* $\kappa(X) \geq \kappa(Y) + \kappa(X_\eta)$ *holds.*

The proof of Theorem 1.4 (Theorem 7.4) is essentially the same as the arguments in [Ka3, Section 5]. Since some modifications are needed, we will explain the details in Section 7.

We summarize the contents of this paper: In Section 2, we treat polarized variations of Hodge structures of weight one and two whose period domains are bounded symmetric domains. We investigate the relation between the canonical extension of some Hodge bundles and automorphic forms on period domains. The main purpose of this section is to prove Theorems 2.10 and 2.11. In Section 3, we review the basic definitions and properties of the semistable part $L^{ss}_{X/Y}$, which was introduced in [FM]. Section 4 deals with the behavior of $L^{ss}_{X/Y}$ under pull-backs. It will play important roles in various applications of a canonical bundle formula. In Section 5, we apply Theorems 2.10 and 2.11 to algebraic fiber spaces. In Section 6, we collect the results about the semi-ampleness of $L^{ss}_{X/Y}$. Theorem 6.3 is stated in [Mo, (5.15.9)(ii)] with the idea of the proof. One of the starting points of this paper is to give a precise proof of Theorem 6.3. Finally, in Section 7, we explain an application of the semi-ampleness of the semistable part. We prove Iitaka's conjecture $C_{n,m}$ for special fiber spaces.

Notation. Let $\mathbb{Z}_{>0}$ be the set of positive integers. We work over $\mathbb{C}$, the complex number field, in this paper.

We denote by $\Delta := \{t \in \mathbb{C} \mid |t| < 1\}$ a unit disc and by $\Delta^* := \Delta \setminus \{0\}$ a punctured disc.

Let X be a normal variety and B, B' $\mathbb{Q}$-divisors on X. If $B - B'$ is effective, we write $B \succ B'$ or $B' \prec B$. We write $B \sim B'$ if $B - B'$ is a principal divisor on X (linear equivalence of $\mathbb{Q}$-divisors).

Let B_+, B_- be the effective $\mathbb{Q}$-divisors on X without common irreducible components such that $B_+ - B_- = B$. They are called the *positive* and the *negative* parts of B.

Let $f : X \to Y$ be a surjective morphism. Let B^h, B^v be the $\mathbb{Q}$-divisors on X with $B^h + B^v = B$ such that an irreducible component of $\operatorname{Supp} B$ is contained in $\operatorname{Supp} B^h$ iff it is mapped onto Y. They are called the *horizontal* and the *vertical* parts of B over Y. B is said to be *horizontal* (resp. *vertical*) over Y if $B = B^h$ (resp. $B = B^v$). The phrase "over Y" might be suppressed if there is no danger of confusion.

An *algebraic fiber space* $f : X \to Y$ is a surjective morphism between non-singular projective varieties X and Y with connected fibers. We denote $\dim f := \dim X - \dim Y$.

Assume that there exists a simple normal crossing divisor Σ on Y such that $f_0 : X_0 \to Y_0$ is smooth, where $Y_0 := Y \setminus \Sigma$, $X_0 := f^{-1}(Y_0)$, and $f_0 := f|_{X_0}$. Let $\pi : Y' \to Y$ be a projective morphism from a non-singular variety Y' such that $\Sigma' := \pi^{-1}(\Sigma)$ is a simple normal crossing divisor. Let $X' \to (X \times_Y Y')_{main}$ be an arbitrary projective resolution such that $f' : X' \to (X \times_Y Y')_{main} \to Y'$ is smooth over $Y' \setminus \Sigma'$, where $(X \times_Y Y')_{main}$ is the main component of $X \times_Y Y'$. We call $f' : X' \to Y'$ an algebraic fiber space *induced* by π. We often use the notation f', X', etc. without mentioning "induced by π" if there is no confusion.

We freely use covering constructions and base change theorems [Mo, Section 4], [Ka1, Section 2] throughout this paper.

When we treat Iitaka's fiber spaces, we always apply the elimination of indeterminacy without mentioning it. For the basic results about Iitaka fiberings, see [Mo, Sections 1, 2].

Acknowledgements. I would like to express my sincere gratitude to Professors Shigefumi Mori and Noboru Nakayama, who gave me various advice and useful comments. I would like to thank the following people, who gave me some comments or answered my questions: Professors Hiroki Aoki, Yoshio Fujimoto, Shigeyuki Kondo, Yoichi Miyaoka, Shigeru Mukai, Atsushi Moriwaki, Kenji Ueno, and my colleague: Hiromichi Takagi.

I also like to thank Professor Daisuke Matsushita for reading a part of the preliminary version of this paper and pointing out some ambiguities.

After this paper was circulated as a RIMS preprint: RIMS-1325, I talked about it in the following places: Kyoto University, Osaka University, the Newton Institute in the University of Cambridge, and Nagoya University. I am grateful to the audiences and appreciate their comments.

The papers [F1] and [F2] are continuations of this paper, in particular, Section 7. Recently, Florin Ambro treats related topics in [Am]. I recommend the readers to see these preprints.

§2. Hodge structures

In this section, we use the following notation.

NOTATION. Let $\mathfrak{h}$ be the *upper half-plane*, that is,

$$\mathfrak{h} := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}.$$

We define

$$\mathbb{H}_g := \left\{ Z \;\middle|\; \begin{array}{l} Z \text{ is a } g \text{ times } g \text{ symmetric matrix and} \\ \operatorname{Im} Z \text{ is positive definite} \end{array} \right\}.$$

We call it the *Siegel upper half-plane* of degree g. In our notation, $\mathfrak{h} = \mathbb{H}_1$.

Let S be an n times n symmetric matrix and Z be an n-dimensional column vector. Then we denote $S[Z] := {}^tZSZ$.

First, we recall the definition of Hodge structures and variations of Hodge structures (see [Sd, Sections 2, 3] or [GS, **1, 3**]).

2.1. (Hodge structures) Let $H_{\mathbb{R}}$ be a finite dimensional real vector space with a $\mathbb{Q}$-structure by a lattice $H_{\mathbb{Z}} \subset H_{\mathbb{R}}$, and let $H_{\mathbb{C}}$ denote the complexification of $H_{\mathbb{R}}$, $H_{\mathbb{Q}} = H_{\mathbb{Z}} \otimes \mathbb{Q}$.

A *Hodge structure* is a decomposition

$$H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}, \quad \text{with} \quad H^{q,p} = \bar{H}^{p,q}, \tag{1}$$

where $^-$ means the complex conjugation. The integers $h^{p,q} = \dim H^{p,q}$ are the *Hodge numbers*.

The Hodge structure (1) is said to have *weight* k if the subspace $H^{p,q}$ are nonzero only when $p + q = k$. To each Hodge structure of weight k one assigns the *Hodge filtration*

$$H_{\mathbb{C}} \supset \cdots \supset F^{p-1} \supset F^p \supset F^{p+1} \supset \cdots \supset 0, \tag{2}$$

where $F^p = \bigoplus_{i \geq p} H^{i,k-i}$.

A *polarization* for a Hodge structure of weight k consists of the datum of a bilinear form S on $H_{\mathbb{C}}$, which is defined over $\mathbb{Q}$, and which is symmetric for even k, skew for odd k, such that

$$S(H^{p,q}, H^{r,s}) = 0 \quad \text{unless} \quad p = s,\ q = r,$$

$$(\sqrt{-1}\,)^{p-q} S(v, \bar{v}) > 0 \quad \text{if} \quad v \in H^{p,q},\ v \neq 0.$$

We define

$$G_{\mathbb{C}} = \{ g \in \mathrm{GL}(H_{\mathbb{C}}) \mid S(gu, gv) = S(u, v) \text{ for all } u, v \in H_{\mathbb{C}} \},$$

$$G_{\mathbb{R}} = \{ g \in \mathrm{GL}(H_{\mathbb{R}}) \mid S(gu, gv) = S(u, v) \text{ for all } u, v \in H_{\mathbb{R}} \}.$$

The bilinear form S was assumed to take rational values on the lattice $H_{\mathbb{Z}}$. In particular, then,

$$G_{\mathbb{Z}} := \{g \in G_{\mathbb{R}} \mid gH_{\mathbb{Z}} = H_{\mathbb{Z}}\}$$

lies in $G_{\mathbb{R}}$ as an *arithmetic subgroup*.

2.2. (Variation of Hodge structures) We introduce the notion of a (*polarized*) *variation of Hodge structure*. The ingredients are:

(a) a connected complex manifold M;

(b) a flat complex vector bundle $\mathcal{H} = \mathcal{H}_{\mathbb{C}} \to M$, with a flat real structure $\mathcal{H}_{\mathbb{R}} \subset \mathcal{H}_{\mathbb{C}}$, and with a flat bundle of lattices $\mathcal{H}_{\mathbb{Z}} \subset \mathcal{H}_{\mathbb{R}}$;

(c) an integer k;

(d) a flat, non-degenerate bilinear form S on $\mathcal{H}_{\mathbb{C}}$, which is rational with respect to the lattice bundle $\mathcal{H}_{\mathbb{Z}}$, and which is symmetric or skew, depending on whether k is even or odd;

(e) and a descending filtration

$$\mathcal{H}_{\mathbb{C}} \supset \cdots \supset \mathcal{F}^{p-1} \supset \mathcal{F}^{p} \supset \mathcal{F}^{p+1} \supset \cdots \supset 0$$

of $\mathcal{H}_{\mathbb{C}}$ by holomorphic subbundles.

These objects are to satisfy the following two conditions:

(i) For each point $t \in M$, the fibers $\mathcal{F}^p_t$ of the bundles $\mathcal{F}^p$ constitute the Hodge filtration of Hodge structure of weight k on the fiber of $\mathcal{H}_{\mathbb{C}}$ at t, and S polarizes this Hodge structure.

(ii) For each p,

$$\nabla \mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes \Omega^1_M,$$

where ∇ is the canonical flat connection of $\mathcal{H}_{\mathbb{C}}$ and Ω^1_M is the holomorphic cotangent bundle.

From now on, we mainly treat the following types of polarized Hodge structures.

2.3. (Assumptions) We treat polarized Hodge structures of weight k with the following numerical conditions;

(WII) $k = 2$, $h^{1,1} = g \geq 3$, $h^{2,0} = h^{0,2} = 1$ and $h^{p,q} = 0$ for other (p, q)'s, or

(WI) $k = 1$, $h^{1,0} = h^{0,1} = g \geq 1$ and $h^{p,q} = 0$ for other (p, q)'s.

2.4. (Period domains) We recall the *period domain* of Hodge structures.

Case (WII). We define the bilinear form $Q := -S$. We put

$$\check{D} = \{[v] \in \mathbb{P}(H_{\mathbb{C}}) \mid Q(v, v) = 0\}$$

and

$$\mathcal{D} = \{[v] \in \mathbb{P}(H_{\mathbb{C}}) \mid Q(v, v) = 0 \text{ and } Q(v, \bar{v}) > 0\}.$$

Then $\mathcal{D}$ consists of two connected components, which are mapped into each other by complex conjugation.

We choose and fix D either one component of $\mathcal{D}$, which is a bounded symmetric domain of type IV and of dimension g. We call D a *period domain* and $\check{D}$ the *compact dual* of D. For the details of bounded symmetric domains of type IV, see [Ba, Chapter 6, Section 3, B], [Gr, Section 2], [Sa, Appendix, Section 6], [Od, **1**], or [Kon, **2**].

The group $\mathrm{G}_{\mathbb{R}}$ acts on $\mathcal{D}$ as automorphisms. We denote by $\mathrm{G}^0_{\mathbb{R}}$ the subgroup of $\mathrm{G}_{\mathbb{R}}$ of index two that consists of isometries preserving the connected components of $\mathcal{D}$. The stabilizer of the lattice $H_{\mathbb{Z}}$ in the subgroup $\mathrm{G}^0_{\mathbb{R}}$ is an arithmetic subgroup $\Gamma = \mathrm{G}_{\mathbb{Z}} \cap \mathrm{G}^0_{\mathbb{R}}$.

Let $\{h_1, \ldots, h_n\}$ be a basis of $H_{\mathbb{R}}$ such that the bilinear form Q with respect to $\{h_i\}$ is as follows;

$$Q = \begin{pmatrix} 0 & 0 & H \\ 0 & -I_{g-2} & 0 \\ H & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -S_1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

where I_{g-2} is the unit matrix of degree $g-2$, $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ the hyperbolic lattice, and

$$S_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_{g-2} & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

We realize D as a tube domain $\mathcal{H}_g$ in $\mathbb{C}^g$. We consider the domain

$$\mathcal{H}_g = \left\{ {}^tZ = (\omega, {}^t\zeta, \tau) \in \mathfrak{h} \times \mathbb{C}^{g-2} \times \mathfrak{h} \;\middle|\; \frac{1}{2} S_1[\operatorname{Im} Z] > 0 \right\}$$

in $\mathbb{C}^g$, where $\operatorname{Im} Z$ is the imaginary part of a column vector Z. The domain is embedded in the projective space as follows:

$$\begin{aligned} pr(Z) &= pr({}^t(\omega, \zeta_1, \dots, \zeta_{g-2}, \tau)) \\ &= {}^t\Big(-\frac{1}{2}S_1[Z] : \omega : \zeta_1 : \cdots : \zeta_{g-2} : \tau : 1\Big). \end{aligned}$$

Let γ be an element of the orthogonal group $\mathrm{G}^0_{\mathbb{R}}$. Then a *holomorphic automorphy factor* $J(\gamma, Z)$ on $\mathrm{G}^0_{\mathbb{R}} \times \mathcal{H}_g$ is the n-th coordinate of

$$\gamma \begin{pmatrix} -\frac{1}{2}S_1[Z] \\ \omega \\ \zeta_1 \\ \vdots \\ \zeta_{g-2} \\ \tau \\ 1 \end{pmatrix}.$$

If we denote $\gamma = (\gamma_{ij})_{i,j=1}^n \in \mathrm{G}^0_{\mathbb{R}}$, then

$$J(\gamma, Z) = -\frac{1}{2}\gamma_{n,1}S_1[Z] + \gamma_{n,2}\omega + \sum_{j=3}^{g} \gamma_{n,j}\zeta_{j-2} + \gamma_{n,n-1}\tau + \gamma_{n,n}.$$

For the details about $J(\gamma, Z)$ and the actions of $\mathrm{G}^0_{\mathbb{R}}$ on $\mathcal{H}_g$, see [Gr, Section 2] and [Od, **1**].

Case (WI). (cf. [Ca, Section 1]) We put $Q = S$, where S is the bilinear form in 2.1. The period domain of all such polarized Hodge structures of weight one on $H_{\mathbb{C}}$ is then

$$D = \{F^1 \in \mathrm{G}(g, H_{\mathbb{C}}) \mid Q(F^1, F^1) = 0,\ \sqrt{-1}\,Q(F^1, \bar{F}^1) > 0\},$$

where $\mathrm{G}(g, H_{\mathbb{C}})$ is a Grassmannian that parameterizes g-dimensional subspace of $H_{\mathbb{C}}$. We denote

$$\check{D} = \{F^1 \in \mathrm{G}(g, H_{\mathbb{C}}) \mid Q(F^1, F^1) = 0\}$$

the *compact dual* of D. It is well-known that D is isomorphic to $\mathbb{H}_g$, the *Siegel upper half-plane*. In this case, we put $\Gamma = G_{\mathbb{Z}}$.

We recall that the choice of a symplectic basis, that is, a rational basis $\{h_1, \dots, h_{2g}\}$ of $H_{\mathbb{Q}}$ relative to which;

$$Q = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

gives rise to the usual realization of D as a Siegel upper half-plane. Let $F^1 \in D$ and $\{\omega_1, \dots, \omega_g\}$ a basis of F^1; let

$$\begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix}$$

be the matrix whose columns give the coefficients of the ω_i's relative to the basis $\{h_j\}$. The second bilinear relation $\sqrt{-1}\, Q(F^1, \bar{F}^1) > 0$ guarantees that $\det \Omega_2 \neq 0$, and therefore F^1 has a (unique) basis of the form

$$\omega_i = \sum_j z_{ji} h_j + h_{g+i}.$$

It is easy to check that the bilinear relations imply that, denoting by Z the matrix (z_{ij}), $Z \in \mathbb{H}_g$.

2.5. (Baily-Borel-Satake compactification) Let Γ and D be as in 2.4. Then Γ acts on D properly discontinuously, and hence $\mathcal{S} := D/\Gamma$ has a canonical structure of normal analytic space by Cartan's theorem.

Let D^* be the union of D and the *rational boundary components* with the *Satake topology*. The arithmetic subgroup Γ acts on D^* and we obtain the *Baily-Borel-Satake* compactification $\bar{\mathcal{S}} := D^*/\Gamma$ of $\mathcal{S}$. It is well-known that $\bar{\mathcal{S}}$ is a normal projective variety. In our situation, D is *simple* and $\operatorname{codim}_{\bar{\mathcal{S}}}(\bar{\mathcal{S}} \setminus \mathcal{S}) \geq 2$ except for the case when $g = 1$ in (WI). It is well-known that $\mathcal{S} = \mathbb{C}$ and $\bar{\mathcal{S}} = \mathbb{P}^1$ if $g = 1$ in (WI).

The following is [BB, 10.11, Theorem] (for the precise statement, see [BB]).

THEOREM 2.6. *There are finitely many integral automorphic forms of the same weight, say l, which induce an embedding of $\bar{\mathcal{S}}$ into a projective space.*

Let $\bar{\mathcal{S}} \hookrightarrow \mathbb{P}$ be the above embedding. Then we define $\mathcal{O}_{\bar{\mathcal{S}}}(1) = \mathcal{O}_{\mathbb{P}}(1)|_{\bar{\mathcal{S}}}$.

Let us recall automorphic forms. We adopt the following definition, which is familiar to us.

2.7. (Automorphic forms, cf. [BB, 7.3] or [AMRT, Section 1.1]) An *automorphic form* f on D of weight l for Γ is a holomorphic section, invariant under Γ, of the canonical line bundle lK_D.

If D is realized as a domain in $\mathbb{C}^{\dim D}$, then K_D is canonically trivialized, and f is represented by a holomorphic function f' on D that verifies

$$(\bigstar) \qquad f'(\gamma z) j(\gamma, z)^l = f'(z)$$

for $z \in D$ and $\gamma \in \Gamma$, where j is the Jacobian of the map $\gamma : D \to D$ at z.

In this paper, an automorphic form means not a holomorphic section of K_D but a holomorphic function on D with the above property ($\bigstar$).

We omit the precise definition of *integral automorphic forms* (see [BB, 8.5]). Note that in our situation every automorphic form is integral, unless $D = \mathbb{H}_1$ and $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ (see [BB, 10.14]). When $D = \mathbb{H}_1$ and $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, we assume that an automorphic form f is holomorphic at infinity. This definition coincides with the definition of *modular forms* in [Se, Chapter VII, Definition 4]. Roughly speaking, integral automorphic forms on D extends to D^* continuously and if we choose k sufficiently large and divisible, there are many integral automorphic forms of weight k that separate points on D^*/Γ.

2.8. (Universal subbundles) We will regard automorphic forms as a global section of some line bundles.

Case (WII). Let $\mathbb{F}^2$ be the universal subbundle on $\mathbb{P}(H_{\mathbb{C}})$. We note that $\mathbb{P}(H_{\mathbb{C}})$ parameterizes lines in $H_{\mathbb{C}}$. We will omit the restriction symbols like $|_D$ if there is no danger of confusion.

By adjunction, we obtain an isomorphism $K_{\check{D}} \simeq \mathcal{O}_{\check{D}}(-g) \simeq (\mathbb{F}^2)^{\otimes g}$ since $\check{D}$ is a smooth quadric hypersurface in $\mathbb{P}(H_{\mathbb{C}})$. Let $\{u_i\}$ be homogeneous coordinates of $\mathbb{P}(H_{\mathbb{C}})$ with respect to the basis $\{h_i\}$ in 2.4. We define a *canonical free basis* of $\mathbb{F}^2$ on D as follows;

$$\Omega = \frac{u_1}{u_n} h_1 + \cdots + \frac{u_{n-1}}{u_n} h_{n-1} + h_n.$$

Of course, $\mathbb{F}^2 \simeq \mathcal{O}_D \cdot \Omega$ on D. Then $\gamma \in \mathrm{G}^0_{\mathbb{R}}$ acts on Ω as a multiplication by $J(\gamma, Z)^{-1}$. On the other hand, $\gamma \in \mathrm{G}^0_{\mathbb{R}}$ acts on the form $\varpi = \mathrm{d}\omega \wedge \mathrm{d}\zeta \wedge \mathrm{d}\tau$ as a multiplication by $j(\gamma, Z)$, that is, $\gamma^* \varpi = j(\gamma, Z)\varpi$. Since $j(\gamma, Z) = J(\gamma, Z)^{-g}$, we obtain a $\mathrm{G}^0_{\mathbb{R}}$-equivariant isomorphism $K_D \simeq (\mathbb{F}^2)^{\otimes g}$ on D. (cf. [Gr, Section 5, p. 1201], [Od, **1**, pp. 100–101].)

So, we will consider an automorphic form f of weight k as a holomorphic section of $(\mathbb{F}^2)^{\otimes a}$, where $a = gk$. That is, if f is an automorphic form of weight k, then

$$f \cdot \Omega^{\otimes a} \in H^0(D, (\mathbb{F}^2)^{\otimes a}).$$

Case (WI). Let m be a natural number such that mQ is integral on $H_{\mathbb{Z}}$. By choosing the basis $\{e_1, \ldots, e_n\}$ of $H_{\mathbb{Z}}$ suitably, we can assume that

$$Q = \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix},$$

where

$$\delta := \frac{1}{m} \begin{pmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_g \end{pmatrix}$$

is a $g \times g$ diagonal matrix such that $\delta_i \in \mathbb{Z}_{>0}$ for every i and $\delta_i | \delta_{i+1}$ for $1 \leq i \leq g-1$. We define

$$\mathrm{Sp}(\delta, \mathbb{Z}) := \left\{ M \in \mathrm{GL}(2g, \mathbb{Z}) \;\middle|\; M \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix} {}^tM = \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix} \right\},$$

and

$$\mathrm{Sp}(2g, \mathbb{Q}) := \left\{ M \in \mathrm{GL}(2g, \mathbb{Q}) \;\middle|\; M \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} {}^tM = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \right\},$$

where I means the $g \times g$ unit matrix. If $\delta = I$, we write $\mathrm{Sp}(\delta, \mathbb{Z}) = \mathrm{Sp}(2g, \mathbb{Z})$ the *symplectic group*.

In this realization, $\Gamma = \mathrm{Sp}(\delta, \mathbb{Z})$. Replacing the basis $\{e_i\}$ with the $\mathbb{Q}$-basis $\{h_i\}$ such that

$$(h_1, \ldots, h_n) = (e_1, \ldots, e_n) \begin{pmatrix} 1 & 0 \\ 0 & \delta^{-1} \end{pmatrix},$$

we regard Γ as a subgroup of $\mathrm{Sp}(2g, \mathbb{Q})$ by the following embedding;

$$\begin{pmatrix} 1 & 0 \\ 0 & \delta^{-1} \end{pmatrix} \Gamma \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \subset \mathrm{Sp}(2g, \mathbb{Q}).$$

Let $\mathbb{F}^1$ be the universal subbundle on the Grassmannian $\mathrm{G}(g, H_{\mathbb{C}})$. Let $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{R})$. Then the Jacobian of γ at $Z \in \mathbb{H}_g$ is $j(\gamma, Z) =$

$\det(CZ+D)^{-(g+1)}$. We define

$$(\widetilde{\omega}_1,\dots,\widetilde{\omega}_g)=(h_1,\dots,h_n)\begin{pmatrix}Z\\I\end{pmatrix},$$

where $Z\in\mathbb{H}_g$, and

$$\Omega=\widetilde{\omega}_1\wedge\cdots\wedge\widetilde{\omega}_g$$

a free basis of $\bigwedge^g\mathbb{F}^1$ on D, that is, $\bigwedge^g\mathbb{F}^1\simeq\mathcal{O}_D\cdot\Omega$. We call Ω a *canonical free basis* of $\bigwedge^g\mathbb{F}^1$. On the other hand, γ operates on Ω as a multiplication by $\det(CZ+D)^{-1}$.

Therefore, we will consider an automorphic form f of weight k as a holomorphic section of $\big(\bigwedge^g\mathbb{F}^1\big)^{\otimes a}$, where $a=k(g+1)$. That is, if f is an automorphic form of weight k, then

$$f\cdot\Omega^{\otimes a}\in H^0\big(D,\big(\textstyle\bigwedge^g\mathbb{F}^1\big)^{\otimes a}\big).$$

2.9. (Main Theorems) The following two theorems are the main results of this section. In the theorems, $\mathcal{F}^2(\overline{\mathcal{H}})$ (resp. $\mathcal{F}^1(\overline{\mathcal{H}})$) is the canonical extension of $\mathcal{F}^2$ (resp. $\mathcal{F}^1$) (see 2.14 below).

THEOREM 2.10. *Let $\mathcal{H}_{\mathbb{C}}\to M$ be a polarized variation of Hodge structures of type* (WII). *Assume that M is a Zariski open set of a complex manifold $\overline{M}$ such that $\overline{M}\setminus M$ is a simple normal crossing divisor in $\overline{M}$. We further assume that local monodromies around every irreducible component of $\overline{M}\setminus M$ are unipotent. Then*

$$p^*\mathcal{O}_{\bar{\mathcal{S}}}(1)\simeq\mathcal{F}^2(\overline{\mathcal{H}})^{\otimes a},$$

where $\bar{\mathcal{S}}$ is embedded into a projective space by automorphic forms of weight k and $a=gk$. We note that $p:\overline{M}\to\bar{\mathcal{S}}$ is Borel's extension of the period map. We also note that a depends only on polarized Hodge structures.

THEOREM 2.11. *Let $\mathcal{H}_{\mathbb{C}}\to M$ be a polarized variation of Hodge structures of type* (WI). *Assume that M is a Zariski open set of a complex manifold $\overline{M}$ such that $\overline{M}\setminus M$ is a simple normal crossing divisor. We further assume that local monodromies around every irreducible component of $\overline{M}\setminus M$ are unipotent. Then*

$$p^*\mathcal{O}_{\bar{\mathcal{S}}}(1)\simeq\big(\textstyle\bigwedge^g\mathcal{F}^1(\overline{\mathcal{H}})\big)^{\otimes a},$$

where $\bar{\mathcal{S}}$ is embedded into a projective space by automorphic forms of weight k and $a=k(g+1)$. We note that $p:\overline{M}\to\bar{\mathcal{S}}$ is Borel's extension of the period map. We also note that a depends only on polarized Hodge structures.

2.12. (Local analysis) Let $\mathcal{H}_{\mathbb{C}}$ be a polarized variation of Hodge structures of type (WII) or (WI) over $S_0 = (\Delta^*)^l \times \Delta^{d-l}$. Let $\mathbf{e} : U := \mathfrak{h}^l \times \Delta^{d-l} \to (\Delta^*)^l \times \Delta^{d-l}$ be a universal cover. It is given by

$$\mathbf{e}(z_1, \ldots, z_l, t_{l+1}, \ldots, t_d) = (t_1, \ldots, t_d),$$

where $t_i = \exp(2\pi\sqrt{-1}\, z_i)$ for $1 \le i \le l$. We put

$$\mathcal{F} := \begin{cases} \mathcal{F}^1 & \text{Case (WI)} \\ \mathcal{F}^2 & \text{Case (WII).} \end{cases}$$

Since the pull-back $\mathbf{e}^*\mathcal{H}_{\mathbb{C}}$ is trivial, $\mathbf{e}^*\mathcal{F} \subset \mathbf{e}^*\mathcal{H}_{\mathbb{C}}$ induces a holomorphic mapping (the *period mapping*) $\Phi : U \to D$ of the Hodge structures and a group representation $\rho : \pi_1(S_0) \to \Gamma$ that satisfy $\Phi(\gamma u) = \rho(\gamma)\Phi(u)$, where $u \in U$ and $\gamma \in \pi_1(S_0)$.

We consider the following commutative diagram of the period map of the Hodge structures

$$\begin{array}{ccc} U & \xrightarrow{\Phi} & D \\ {\scriptstyle \mathbf{e}}\downarrow & & \downarrow \\ S_0 & \xrightarrow[p_0]{} & D/\Gamma = \mathcal{S}. \end{array}$$

It is well-known that p_0 can be extended to $p : S \to \bar{\mathcal{S}}$ by Borel's extension theorem [Bo, Theorem A]. By patching together, we obtain $p : \overline{M} \to \bar{\mathcal{S}}$.

Enlarge $\mathfrak{h}$ to $\mathfrak{h}^* = \mathfrak{h} \cup \{i\infty\}$, where a fundamental system of neighborhoods of $i\infty$ is given by

$$W_c = \{z \in \mathfrak{h} \mid \operatorname{Im} z > c\} \cup \{i\infty\}.$$

Extend the map $\mathbf{e} : \mathfrak{h} \to \Delta^*$ to $\mathbf{e} : \mathfrak{h}^* \to \Delta$ by $\mathbf{e}(i\infty) = 0$.

By [AMRT, p. 278, Proposition], the map p lifts to a continuous map $\widetilde{\Phi} : U^* \to D^*$, extending a map $\Phi : U \to D$, where $U^* = (\mathfrak{h}^*)^l \times \Delta^{d-l}$.

$$\begin{array}{ccc} U^* & \xrightarrow{\widetilde{\Phi}} & D^* \\ {\scriptstyle \mathbf{e}}\downarrow & & \downarrow \\ S & \xrightarrow[p]{} & D^*/\Gamma = \bar{\mathcal{S}} \end{array}$$

Here, we put on D^* the Satake topology. We note that Γ is assumed to be *neat* in [AMRT, Chapter III, Section 7], but the above claim is true without this assumption (see the proof of [AMRT, p. 278]).

2.13. (Monodromies) The fundamental group $\pi_1(S_0)$ is isomorphic to $\mathbb{Z}^{\oplus l}$. We note that $\pi_1(S_0) = \langle \gamma_1, \ldots, \gamma_l \rangle$, where γ_i corresponds to a path that circles around i-th coordinate counter-clockwise.

We define the monodromy matrix T_γ corresponding to $\gamma \in \pi_1(S_0)$ as follows;

$$\gamma(e_1, \ldots, e_n) = (e_1, \ldots, e_n)T_\gamma^{-1},$$

where $\{e_1, \ldots, e_n\}$ is a basis of $H_\mathbb{Z}$. Let $\{f_1, \ldots, f_n\}$ be another basis of $H_\mathbb{R}$ such that

$$(f_1, \ldots, f_n) = (e_1, \ldots, e_n)P^{-1}$$

for $P \in \mathrm{GL}(H_\mathbb{R})$. Then the monodromy matrix with respect to the basis $\{f_i\}$ is $PT_\gamma P^{-1}$. We will omit $P \cdot P^{-1}$ for simplicity if there is no danger of confusion.

From now on, we put the following assumption:

(**U**) $\qquad T_i := T_{\gamma_i}$ is unipotent for every i.

Under this assumption (**U**), we define $N_i = \log T_i$ for $1 \le i \le l$.

2.14. (Canonical extensions) We define

$$\mathbf{v}_j := (f_1, \ldots, f_n) \exp\left(\sum_{i=1}^{l} z_i N_i \right) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

for $1 \le j \le n$, where the column vector on the right hand side is the j-th unit vector with respect to the basis $\{f_i\}$. Then, by the definition of $\mathbf{v}_j$, $\mathbf{v}_j$ is invariant under the monodromy actions. So there exists $v_j \in H^0(S_0, \mathcal{H}_\mathbb{C})$ such that $\mathbf{v}_j = \mathbf{e}^* v_j$ for $1 \le j \le n$. We define

$$\overline{\mathcal{H}} = \mathcal{O}_S v_1 \oplus \cdots \oplus \mathcal{O}_S v_n.$$

Then we call $\overline{\mathcal{H}}$ the *canonical extension* of $\mathcal{H}$. Of course, $\overline{\mathcal{H}}$ does not depend on the choice of the basis $\{f_i\}$.

We define $\mathcal{F}(\overline{\mathcal{H}})^p := j_*\mathcal{F}^p \cap \overline{\mathcal{H}}$ the *canonical extension* of $\mathcal{F}^p$, where $j : S_0 \to S$.

The local canonical extensions are patched together. Thus we can define globally the canonical extensions on $\overline{M}$.

2.15. (Monodromy weight filtrations) In this paragraph, we will investigate the monodromies T_i and N_i.

Cutting the base space S_0 generally, we can assume that S_0 is one-dimensional. So we omit the subscript i.

By the monodromy theorem, $N^{k+1} = 0$, where k is the weight of the Hodge structure (see [Sd, Section 6]).

In this case, there exists a unique ascending filtration $\{W_l\}$ of $H_{\mathbb{Q}}$, called the *monodromy weight filtration*,

$$0 \subset W_0 \subset W_1 \subset \cdots \subset W_{2k} = H_{\mathbb{Q}},$$

satisfying

$$N : W_i \longrightarrow W_{i-2},$$
$$N^l : W_{k+l}/W_{k+l+1} \simeq W_{k-l}/W_{k-l-1}$$

for each $l \geq 0$ (see [Sd, (6.4) Lemma]). On the other hand, by the nilpotent orbit theorem, $\Psi := \exp(-zN)\Phi$ descends to a single-valued map $\Psi : S_0 \to \check{D}$ and extends across the origin to a map $\Psi : S \to \check{D}$.

Definition 2.16. The filtration $\Psi(0) \in \check{D}$ will be called the *limiting filtration* and will be denoted by $\{F^p_\infty\}$

Theorem 2.17. ([Sd, (6.16) Theorem]) *The limiting filtration $\{F^p_\infty\}$ together with the monodromy weight filtration $\{W_l\}$ gives a mixed Hodge structure on the vector space $H_{\mathbb{Q}}$.*

We describe the monodromy weight filtration.

Case (WII). (cf. [Ku, Section 2].) Setting

$$F^i_k := F^i_\infty \cap W_k / F^i_\infty \cap W_{k-1},$$

and

$$f^i_k := \dim F^i_k,$$

we have $f^2_4 + f^2_3 + f^2_2 = h^{2,0} = 1$. We note that $f^i_k = 0$ for $i > k$ since the filtration induces a pure Hodge structure of weight k on $\mathrm{Gr}^W_k = W_k/W_{k-1}$. Therefore,

$$f^2_4 = \dim W_4/W_3 = \dim W_0$$

since $N^2 : W_4/W_3 \simeq W_0$ is an isomorphism and $\mathrm{Gr}^W_4 = F^2_4$. Similarly,

$$f^2_3 = \frac{1}{2} \dim W_3/W_2 = \dim W_1/W_0$$

since $N : W_3/W_2 \simeq W_1/W_0$ is an isomorphism and $\mathrm{Gr}^W_3 = F^2_3 \oplus \bar{F}^2_3$.

Type (I). If $N = 0$, then $T = I$ and $\Psi(0) = \Phi(0) \in D$.

Type (II). Assume that $N \neq 0, N^2 = 0$. By the assumption and the definition of the monodromy weight filtration, $W_0 = 0$ and $W_1 \neq 0$. The subspace W_1 is totally isotropic and $\dim W_1 \leq 2$ since the signature of Q is $(2, g)$. Therefore, $\dim W_1 = 2$ since $\dim W_1$ is even by the above argument. Thus,

$$0 = W_0 \subsetneq W_1 \subset W_2 = W_1^{\perp} \subset W_3 = W_4 = H_{\mathbb{Q}}.$$

Type (III). Assume that $N^2 \neq 0, N^3 = 0$. Since $N^2 \neq 0$, $W_0 \neq 0$. By the above relations, $\dim W_0 = f_4^2 = 1$ and $f_3^2 = f_2^2 = 0$. So, we get the filtration;

$$0 \subsetneq W_0 = W_1 \subsetneq W_2 = W_3 = W_1^{\perp} \subset W_4 = H_{\mathbb{Q}}.$$

Case (WI). By the monodromy theorem, $N^2 = 0$. The monodromy weight filtration $\{W_l\}$ of N is the filtration

$$0 \subset W_0 \subset W_1 \subset W_2 = H_{\mathbb{Q}}$$

given by: $W_0 = \operatorname{Im} N$; $W_1 = \operatorname{Ker} N$. It is easy to check that W_0 is a totally isotropic $\mathbb{Q}$-subspace and $W_1 = W_0^{\perp}$.

2.18. (Rational boundary components) Let F be a rational boundary component of D. We denote by $N(F)$ $(\subset G_{\mathbb{R}})$, $W(F)$ and $U(F)$, the stabilizer subgroup of F, the unipotent radical of $N(F)$, and the center of $W(F)$, respectively. We define $D(F) := U(F)_{\mathbb{C}} D \subset \check{D}$.

There are various ways of characterizing the rational boundary components. For our purpose, the following is the most convenient. The rational boundary components correspond to totally isotropic $\mathbb{Q}$-subspaces of $H_{\mathbb{Q}}$ bijectively.

Case (WII). We recall the above correspondence in the case (WII) (cf. [Sc, 2.1], [Kon, **2**]).

Proposition 2.19. *The set of all rational boundary components of D corresponds to the set of all totally isotropic $\mathbb{Q}$-subspaces of $H_{\mathbb{Q}}$. If E is a totally isotropic $\mathbb{Q}$-subspace of $H_{\mathbb{Q}}$, then the corresponding rational boundary component is defined by $\mathbb{P}(E \otimes \mathbb{C}) \cap \overline{D}$, where $\overline{D}$ is the topological closure of D in $\check{D}$.*

Remark 2.20. By the theorem of Mayer [Se, p. 43, Corollary 2], there exists an isotropic $\mathbb{Q}$-vector since $\dim H_{\mathbb{Q}} \geq 5$ (see 2.3 (Assumptions) (WII)). Therefore, D/Γ is not compact.

Type (II). Let F_1 be a 1-dimensional rational boundary component corresponding to the totally isotropic $\mathbb{Q}$-subspace W_1. We take a $\mathbb{Q}$-basis $\{f_1, \dots, f_n\}$ of $H_{\mathbb{Q}}$ such that F_1 corresponds to the totally isotropic subspace $\mathbb{Q}f_1 \oplus \mathbb{Q}f_2$ and the intersection matrix Q with respect to $\{f_i\}$ is

$$Q = \begin{pmatrix} 0 & 0 & H \\ 0 & K & 0 \\ H & 0 & 0 \end{pmatrix},$$

where K is a negative definite matrix of degree $g-2$, $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ the hyperbolic lattice. By changing the basis $\{f_3, \dots, f_g\}$ linearly on $\mathbb{R}$, we obtain a basis $\{f_i\}$ of $H_{\mathbb{R}}$ such that $K = -I_{g-2}$.

An elementary calculation shows;

$$N(F_1) = \left\{ \begin{pmatrix} U & V & W \\ 0 & X & Y \\ 0 & 0 & Z \end{pmatrix} \middle| \begin{array}{l} \det U > 0,\ {}^tUHZ = H,\ {}^tXKX = K, \\ {}^tXKY + {}^tVHZ = 0, \\ {}^tYKY + {}^tZHW + {}^tWHZ = 0 \end{array} \right\},$$

where U, W, Z are 2 by 2 matrices, X is $(g-2)$ by $(g-2)$ matrix and ${}^tV$, Y are $(g-2)$ by 2 matrices;

$$W(F_1) = \left\{ \begin{pmatrix} I_2 & V & W \\ 0 & I_{g-2} & Y \\ 0 & 0 & I_2 \end{pmatrix} \middle| \begin{array}{l} KY + {}^tVH = 0, \\ {}^tYKY + {}^tHW + {}^tWH = 0 \end{array} \right\},$$

$$U(F_1) = \left\{ \begin{pmatrix} I_2 & 0 & W \\ 0 & I_{g-2} & 0 \\ 0 & 0 & I_2 \end{pmatrix} \middle| W = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix},\ c \in \mathbb{R} \right\}.$$

By the construction of the monodromy weight filtration and $T = I + N$, we obtain that $T \in U(F_1)$.

We put $D(F_1) = U(F_1)_{\mathbb{C}}D$. It can be checked easily that $D \subset D(F_1) \subset \mathbb{C}^{g+1} = \{(t_1 : \cdots : t_{g+2}) \in \mathbb{P}(H_{\mathbb{C}}) \mid t_{g+2} \neq 0\}$.

We define $\Psi = \exp(-zN)\Phi$ as above. Then Ψ descends to a single-valued map $\Psi : S_0 \to \check{D}$. By the nilpotent orbit theorem, Ψ is holomorphically extended to $\Psi : S \to \check{D}$. Since $\exp(-zN) \in U(F_1)_{\mathbb{C}}$ and $\exp(zN)\Psi(0) \in D$ for $\operatorname{Im} z \gg 0$, we obtain $\Psi : S \to D(F_1) \subset \check{D}$.

Type (III). Let F_0 be a 0-dimensional rational boundary component corresponding to the totally isotropic $\mathbb{Q}$-subspace W_0. We take a $\mathbb{Q}$-basis $\{f_1, \dots, f_n\}$ of $H_{\mathbb{Q}}$ such that F_0 corresponds to the totally isotropic subspace $\mathbb{Q}f_1$ and the intersection matrix Q with respect to $\{f_i\}$ is

$$Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & K' & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

where K' has the signature $(1, g-1)$. As in the type (II), by replacing $\{f_2, \dots, f_{g+1}\}$, we obtain a basis $\{f_i\}$ of $H_{\mathbb{R}}$ such that

$$K' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -I_g & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

with respect to the basis $\{f_i\}$.

A direct calculation shows;

$$N(F_0) = \left\{ \begin{pmatrix} U & V & W \\ 0 & X & Y \\ 0 & 0 & Z \end{pmatrix} \;\middle|\; \begin{array}{l} UZ = 1,\ {}^tXK'X = K', \\ {}^tXK'Y + {}^tVZ = 0, \\ {}^tYK'Y + 2ZW = 0 \end{array} \right\},$$

where $U, W, Z \in \mathbb{R}$, X is g by g matrix and ${}^tV$, Y are g by 1 matrices;

$$W(F_0) = U(F_0) = \left\{ \begin{pmatrix} 1 & V & W \\ 0 & I_g & Y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; K'Y + {}^tV = 0,\ {}^tYK'Y + 2W = 0 \right\}.$$

By the construction of the filtration and $T = I + N + \frac{1}{2}N^2$, we can check that $T \in U(F_0)$.

By the same argument as in the type (II), we obtain $D(F_0) = U(F_0)_{\mathbb{C}}D \subset \mathbb{C}^{g+1} = \{t_{g+2} \neq 0\}$ and $\Psi = \exp(-zN)\Phi : S \to D(F_0) \subset \check{D}$.

Case (WI). By choosing the $\mathbb{Q}$-basis $\{f_i\}$ of $H_{\mathbb{Q}}$ suitably, we can assume that $W_0 = \mathbb{Q}f_{g'+1} \oplus \cdots \oplus \mathbb{Q}f_g$ and

$$Q = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

We denote the rational boundary component corresponding to W_0 by $F_{g'}$. Then

$$N(F_{g'}) = \left\{ \begin{pmatrix} A' & 0 & B' & * \\ * & u & * & * \\ C' & 0 & D' & * \\ 0 & 0 & 0 & {}^t u^{-1} \end{pmatrix} \middle| \begin{array}{l} \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in \mathrm{Sp}(2g', \mathbb{R}), \\ u \in \mathrm{GL}(g'', \mathbb{R}) \end{array} \right\},$$

where $g'' = g - g'$, and

$$W(F_{g'}) = \left\{ \begin{pmatrix} I_{g'} & 0 & 0 & n \\ {}^t m & I_{g''} & {}^t n & b \\ 0 & 0 & I_{g'} & -m \\ 0 & 0 & 0 & I_{g''} \end{pmatrix} \middle| {}^t n m + b = {}^t m n + {}^t b \right\},$$

$$U(F_{g'}) = \left\{ \begin{pmatrix} I_{g'} & 0 & 0 & 0 \\ 0 & I_{g''} & 0 & b \\ 0 & 0 & I_{g'} & 0 \\ 0 & 0 & 0 & I_{g''} \end{pmatrix} \middle| {}^t b = b \right\}.$$

For the details, see [Nm2, Section 4], [Nm1, **2**].

Then, we can check that the monodromy matrix T is contained in $U(F)$, where F is the rational boundary component corresponding to the totally isotropic $\mathbb{Q}$-subspace W_0.

So, by the nilpotent orbit theorem, we obtain $\Psi : S \to D(F) \subset \check{D}$ as in the above case.

2.21. (Proof of theorems) Let $\pi : U_M \to M$ be a universal cover of M. Then we obtain a commutative diagram of the period map;

$$\begin{array}{ccc} U_M & \xrightarrow{\Phi_M} & D \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ M & \xrightarrow[p_0]{} & D/\Gamma = \mathcal{S}. \end{array}$$

Let $\{\varphi_0, \ldots, \varphi_b\}$ be a system of integral automorphic forms of weight k which induces the embedding of $\bar{\mathcal{S}} \subset \mathbb{P}^b$ (see Theorem 2.6). We note that the projective embedding of $\bar{\mathcal{S}}$ on $\mathcal{S}$ is defined by

$$Z \bmod \Gamma \longrightarrow (\varphi_0(Z) : \cdots : \varphi_b(Z)) \in \mathbb{P}^b$$

for $Z \in D$.

CASE (WII). Let Ω be the canonical free basis of $\mathbb{F}^2$ on D. Then we have

$$\varphi_i \cdot \Omega^{\otimes a} \in H^0(D, (\mathbb{F}^2)^{\otimes a})$$

for $a = gk$, and

$$\Phi_M^*(\varphi_i \cdot \Omega^{\otimes a}) \in H^0(U_M, \pi^*(\mathcal{F}^2)^{\otimes a})$$

by the universality of $\mathbb{F}^2$. Since $\varphi_i \cdot \Omega^{\otimes a}$ is Γ-invariant by the definition of automorphic forms, $\Phi_M^*(\varphi_i \cdot \Omega^{\otimes a})$ is invariant under the monodromy actions and hence

$$\Phi_M^*(\varphi_i \cdot \Omega^{\otimes a}) = \pi^*\xi_i \quad \text{for} \quad \xi_i \in H^0(M, (\mathcal{F}^2)^{\otimes a}).$$

Then, $\{\xi_0, \dots, \xi_b\}$ induces a morphism

$$(\xi_0 : \cdots : \xi_b) : M \longrightarrow \mathbb{P}^b$$

that factors through $\mathcal{S}$. We note that

$$\begin{aligned}
&(\Phi_M^*\varphi_0 : \cdots : \Phi_M^*\varphi_b) \\
&\quad = (\Phi_M^*(\varphi_0 \cdot \Omega^{\otimes a}) : \cdots : \Phi_M^*(\varphi_b \cdot \Omega^{\otimes a})) \\
&\quad = (\pi^*\xi_0 : \cdots : \pi^*\xi_b) \qquad\qquad : U_M \to \mathcal{S} \subset \mathbb{P}^b.
\end{aligned}$$

Therefore, $(\mathcal{F}^2)^{\otimes a} \simeq p_0^*\mathcal{O}_{\mathcal{S}}(1)$. We note that in the above argument, we didn't use the monodromy condition (**U**) (cf. Proof of Theorem 5.1).

If we prove that ξ_i extends to a holomorphic section $\bar{\xi}_i$ of $\mathcal{F}^2(\overline{\mathcal{H}})^{\otimes a}$ on $\overline{M}$ for every i and $\{\bar{\xi}_i\}$ generates $\mathcal{F}^2(\overline{\mathcal{H}})^{\otimes a}$, then we obtain a holomorphic map

$$(\bar{\xi}_0 : \cdots : \bar{\xi}_b) : \overline{M} \longrightarrow \mathbb{P}^b$$

that is an extension of $(\xi_0 : \cdots : \xi_b) : M \to \mathbb{P}^b$. So, we obtain

$$\mathcal{F}^2(\overline{\mathcal{H}})^{\otimes a} \simeq p^*\mathcal{O}_{\bar{\mathcal{S}}}(1).$$

Therefore, it is sufficient to check that ξ_i can be extended to a holomorphic section of $\mathcal{F}^2(\overline{\mathcal{H}})^{\otimes a}$ and $\{\xi_i\}$ generates $\mathcal{F}^2(\overline{\mathcal{H}})^{\otimes a}$.

CASE (WI). In the above argument, it is sufficient to replace $\mathbb{F}^2$ with $\bigwedge^g \mathbb{F}^1$ and $a = gk$ with $a = k(g+1)$.

2.22. (Proof of theorems continued) We go back to the local setting;

$$\begin{array}{ccc} U & \xrightarrow{\Phi} & D \\ {\scriptstyle \mathbf{e}}\downarrow & & \downarrow \\ S_0 & \xrightarrow[p_0]{} & D/\Gamma = \mathcal{S}. \end{array}$$

For the proof of theorem, it is obvious that we can assume that $l = 1$.

Case (WII). By the definition and construction of the canonical extension, $\Phi^*\Omega$ is a free basis of $\mathbf{e}^*\mathcal{F}^2$ on U and $\Psi^*\Omega$ is a free basis of $\mathcal{F}^2(\overline{\mathcal{H}})$ on S such that $\mathbf{e}^*\Psi^*\Omega = \Phi^*\Omega$.

More concretely, we choose an $\mathbb{R}$-basis $\{f_i\}$ as in 2.18 such that the corresponding rational boundary component is $F_1 = \mathbb{Q}f_1 \oplus \mathbb{Q}f_2$ or $F_0 = \mathbb{Q}f_1$. Then, we can write

$$\Phi = {}^t(Z_1, \ldots, Z_n)$$

with respect to the basis $\{f_i\}$ and

$$\Psi = \exp(-zN)\Phi = {}^t(W_1, \ldots, W_n),$$

where W_i is holomorphic on S, $W_n = Z_n$ on S_0, and $W_n = 1$ on S. Thus,

$$(f_1, \ldots, f_n)\begin{pmatrix} Z_1 \\ \vdots \\ Z_{n-1} \\ 1 \end{pmatrix} = (f_1, \ldots, f_n)\exp(zN)\begin{pmatrix} W_1 \\ \vdots \\ W_{n-1} \\ 1 \end{pmatrix}.$$

Therefore, $\Phi^*\Omega = \sum Z_i f_i = \sum W_i \mathbf{e}^* v_i = \mathbf{e}^*\Psi^*\Omega$.

On the other hand, let φ be one of the $\{\varphi_i\}$. The pull-back $\Phi^*\varphi$ is a holomorphic function on U and

$$\Phi^*\varphi \cdot \Phi^*\Omega^{\otimes a} \in H^0(U, (\mathbf{e}^*\mathcal{F}^2)^{\otimes a}),$$

where $a = gk$.

Since the local monodromies are unipotent, $\Phi^*\varphi$ is invariant under the monodromy actions. Thus, $\Phi^*\varphi = \mathbf{e}^*\widetilde{\varphi}$, where $\widetilde{\varphi}$ is a holomorphic function on S_0. Then $\xi = \widetilde{\varphi} \cdot \Psi^*\Omega^{\otimes a}$. Let $(i\infty, t_2^0, \ldots, t_d^0) \in U^*$ be a point. We define

$$\widetilde{\Phi}(i\infty, t_2^0, \ldots, t_d^0) = x \in D^*$$

and

$$\mathbf{e}(i\infty, t_2^0, \ldots, t_d^0) = (0, t_2^0, \ldots, t_d^0) =: y.$$

Let $N(x)$ be a *good neighborhood* of x. For the definition of the good neighborhood, see [BB, 8.1, 4.9(iv), and 4.10]. Since $\widetilde{\Phi}$ is continuous with respect to the Satake topology, the image of a small neighborhood of $(i\infty, t_2^0, \ldots, t_d^0)$ by $\widetilde{\Phi}$ is in $N(x)$. Since φ is integral, $\widetilde{\varphi}$ is bounded around the point y and extends to a holomorphic function on S by Riemann's extension theorem. Thus, we obtain an extension $\bar{\xi}$ of ξ, that is,

$$\bar{\xi} = \widetilde{\varphi} \cdot (\Psi^*\Omega)^{\otimes a} \in H^0(S, \mathcal{F}^2(\overline{\mathcal{H}})^{\otimes a}).$$

By the choice of $\{\varphi_i\}$, there exists i such that $\varphi_i(x) \neq 0$ for the prescribed point $x \in D^*$. Therefore, $\{\bar{\xi}_i\}$ generates $\mathcal{F}^2(\overline{\mathcal{H}})^{\otimes a}$ on S. Thus, by the above argument, we obtain the required results.

Case (WI). We choose a $\mathbb{Q}$-basis $\{f_i\}$ as in 2.18. Let

$$\Phi = \begin{pmatrix} Z \\ I \end{pmatrix}$$

be the period map with respect to the basis $\{f_i\}$. Then

$$\Psi = \begin{pmatrix} W \\ I \end{pmatrix} = \exp(-zN)\Phi.$$

Therefore,

$$\begin{aligned} (f_1, \ldots, f_n) \begin{pmatrix} Z \\ I \end{pmatrix} &= (f_1, \ldots, f_n) \exp(zN) \begin{pmatrix} W \\ I \end{pmatrix} \\ &= (\mathbf{e}^* v_1, \ldots, \mathbf{e}^* v_n) \begin{pmatrix} W \\ I \end{pmatrix}. \end{aligned}$$

Then, we obtain $\Phi^*\Omega = \mathbf{e}^*\Psi^*\Omega$. By the same argument as in the type (WII), we obtain the required results. Details are left to the readers.

§3. Semistable part $L_{X/Y}^{ss}$

We review the basic definitions and properties of the semistable part $L_{X/Y}^{ss}$ without proof. For details, we recommend the reader to see [FM, Sections 2, 4].

3.1. Let $f : X \to Y$ be an algebraic fiber space with $\dim X = n + d$ and $\dim Y = d$ such that the Kodaira dimension of the generic fiber of f is zero, that is, $\kappa(X_\eta) = 0$. We fix the smallest $b \in \mathbb{Z}_{>0}$ such that the b-th plurigenus $P_b(X_\eta)$ is non-zero.

Proposition 3.2. ([FM, Proposition 2.2]) *There exists one and only one $\mathbb{Q}$-divisor D modulo linear equivalence on Y with a graded $\mathcal{O}_Y$-algebra isomorphism*

$$\bigoplus_{i\geq 0}\mathcal{O}_Y(\lfloor iD\rfloor) \simeq \bigoplus_{i\geq 0}(f_*\mathcal{O}_X(ibK_{X/Y}))^{**},$$

where M^{**} denotes the double dual of M.

Furthermore, the above isomorphism induces the equality

$$bK_X = f^*(bK_Y + D) + B,$$

where B is a $\mathbb{Q}$-divisor on X such that $f_*\mathcal{O}_X(\lfloor iB_+\rfloor) = \mathcal{O}_Y$ $(\forall i > 0)$ and $\operatorname{codim}_Y f(\operatorname{Supp} B_-) \geq 2$.

If furthermore $b = 1$ and fibers of f over codimension one points of Y are all reduced, then the divisor D is a Weil divisor.

Remark 3.3. In Proposition 3.2, we note that for an arbitrary open set U of Y, $D|_U$ and $B|_{f^{-1}(U)}$ depend only on $f|_{f^{-1}(U)}$.

Definition 3.4. Under the notation of 3.2, we denote D by $L_{X/Y}$. It is obvious that $L_{X/Y}$ depends only on the birational equivalence class of X over Y.

The following definition is a special case of [FM, Definition 4.2] (see also [FM, Proposition 4.7]).

Definition 3.5. We set $s_P := b(1 - t_P)$, where t_P is the log-canonical threshold of f^*P with respect to $(X, -(1/b)B)$ over the generic point η_P of P, where P is an irreducible Weil divisor on Y:

$$t_P := \max\{t \in \mathbb{R} \mid (X, -(1/b)B + tf^*P) \text{ is log-canonical over } \eta_P\}.$$

Note that $t_P \in \mathbb{Q}$ and that $s_P \neq 0$ only for a finite number of codimension 1 points P because there exists a nonempty Zariski open set $U \subset Y$ such that $s_P = 0$ for every prime divisor P with $P \cap U \neq \emptyset$. We note that s_P depends only on $f|_{f^{-1}(U)}$ where U is an open set containing P.

We set $L^{ss}_{X/Y} := L_{X/Y} - \sum_P s_P P$ and call it the *semistable part* of $K_{X/Y}$.

We note that D, $L_{X/Y}$, s_P, t_P and $L^{ss}_{X/Y}$ are birational invariants of X over Y.

Putting the above symbols together, we have *the canonical bundle formula* for X over Y:

$$bK_X = f^*(bK_Y + L^{ss}_{X/Y}) + \sum_P s_P f^* P + B,$$

where B is a $\mathbb{Q}$-divisor on X such that $f_*\mathcal{O}_X(\lfloor iB_+ \rfloor) = \mathcal{O}_Y$ ($\forall i > 0$) and $\operatorname{codim}_Y f(\operatorname{Supp} B_-) \geq 2$.

The next proposition follows from the definition of s_P easily.

Proposition 3.6. *Under the notation and the assumptions of* 3.1*, $s_P = 0$ if $f : X \to Y$ has a semistable resolution in a neighborhood of P. Furthermore, if $b = 1$, then $L_{X/Y} = L^{ss}_{X/Y}$ is a Weil divisor and $\mathcal{O}_Y(L^{ss}_{X/Y}) \simeq (f_*\omega_{X/Y})^{**}$.*

The following proposition explains the Hodge theoretic properties of $L_{X/Y}$ and $L^{ss}_{X/Y}$ (see [FM, Corollay 2.5]). For the definition and properties of canonical extensions, see [Kol, **2**], [Mw, **2**], and [Ny1].

Proposition 3.7. (See also Corollary 4.5) *Under the notation and the assumptions of* 3.1*, we assume that there exists a simple normal crossing divisor Σ on Y such that $f_0 : X_0 \to Y_0$ is smooth and that $b = 1$. Then $\lfloor L_{X/Y} \rfloor$ is the* upper canonical extension *and $\lfloor L^{ss}_{X/Y} \rfloor$ is the* lower canonical extension *of $f_{0*}\omega_{X_0/Y_0}$. Therefore, if the local monodromies around Σ are unipotent, then $\lfloor L_{X/Y} \rfloor = \lfloor L^{ss}_{X/Y} \rfloor$.*

Definition 3.8. (Canonical cover of the generic fiber) Under the notation of 3.1, consider the following construction. Since $\dim |bK_{X_\eta}| = 0$, there exists a Weil divisor W on X such that

(i) W^h is effective and $f_*\mathcal{O}_X(iW^h) = \mathcal{O}_Y$ for all $i > 0$, and
(ii) $bK_X - W$ is a principal divisor (ψ) for some non-zero rational function ψ on X.

Let $s : Z \to X$ be the normalization of X in $\mathbb{C}(X)(\psi^{1/b})$. We call $Z \to X \to Y$ a *canonical cover* of $X \to Y$. We often call $Z' \to X$ a canonical cover after replacing Z with its resolution Z'.

The following lemma is easy but useful. We will use it in Section 7.

LEMMA 3.9. *Let $f : X \to Y$ be as in* 3.1. *Let $\pi : Y' \to Y$ be a birational morphism from a non-singular projective variety Y' such that $(X \times_Y Y')_{main} \to Y'$ is equi-dimensional. Let $X' \to (X \times_Y Y')_{main}$ be any resolution. Applying the canonical bundle formula for $f' : X' \to Y'$, we obtain*

$$bK_{X'} = f'^*(bK_{Y'} + L^{ss}_{X'/Y'}) + \sum_{P'} s_{P'} f'^* P' + B'.$$

Since $\operatorname{codim}_{Y'} f'(\operatorname{Supp} B'_-) \geq 2$, *$B'_-$ is μ-exceptional, where $\mu : X' \to X$.*

§4. Pull-back of $L^{ss}_{X/Y}$

In this section, we investigate the behavior of the semistable part $L^{ss}_{X/Y}$ under pull-backs.

The following lemma is essentially the same as [Mo, (5.15.8)].

LEMMA 4.1. *Let $f : X \to Y$ and $h : W \to Y$ be algebraic fiber spaces such that*

(i) *the Kodaira dimension of the generic fiber of f is zero and b is the smallest positive integer such that the b-th plurigenus $P_b(X_\eta)$ is non-zero,*

(ii) *h factors as*

$$h : W \xrightarrow{\ g\ } X \xrightarrow{\ f\ } Y,$$

where g is generically finite,

(iii) *there is a simple normal crossing divisor Σ on Y such that f and h are smooth over $Y_0 := Y \setminus \Sigma$,*

(iv) *the Kodaira dimension of the generic fiber W_η is zero and the geometric genus $p_g(W_\eta) = 1$, where η is the generic point of Y,*

Then $bL^{ss}_{W/Y} = L^{ss}_{X/Y}$.

Sketch of the proof. By the same argument as in [Mo, (5.15.5)], that is, taking a finite covering of Y, we can assume that $h : W \to Y$ is semistable in codimension one and the singularities of X over codimension one point of Y are at most rational Gorenstein and $\mathcal{O}_Y(L^{ss}_{X/Y}) = (f_* \omega^{\otimes b}_{X/Y})^{**}$ (see [Mo, (5.15.6)(i)]).

Let $Y^\dagger$ be an open dense subset of Y with $\mathrm{codim}(Y - Y^\dagger) \geq 2$ such that h is flat over $Y^\dagger$, $h_*\omega_{W/Y}^{\otimes b} = \mathcal{O}_Y(bL_{W/Y}^{ss})$ on $Y^\dagger$, the singularities of $f^{-1}(Y^\dagger)$ are at most rational Gorenstein, and $\mathcal{O}_Y(L_{X/Y}^{ss}) = f_*\omega_{X/Y}^{\otimes b}$ on $Y^\dagger$. Let $X^\dagger$ be an open dense subset of $f^{-1}(Y^\dagger)$ such that $\mathrm{codim}(f^{-1}(Y^\dagger) - X^\dagger) \geq 2$ and g is finite flat over $X^\dagger$. Let $W^\dagger := g^{-1}(X^\dagger)$ and let $g^\dagger : W^\dagger \to X^\dagger$ be the induced morphism. We note that $g^{\dagger *}\omega_{X^\dagger/Y^\dagger} = \omega_{W^\dagger/Y^\dagger}(-R)$ for an effective divisor R that is a sum of irreducible divisors dominating $Y^\dagger$ (see [Mo, (5.15.6)(ii)]).

Since $\kappa(W_\eta) = \kappa(X_\eta) = 0$, $bR|_{W_\eta}$ is the fixed part of $|bK_{W_\eta}|$. Then the same argument as in [Mo, (5.15.8)] holds. Therefore, we obtain $L_{X/Y}^{ss} = bL_{W/Y}^{ss}$. ∎

The following is very useful for applications. We will often use it in Section 7.

Proposition 4.2. *Under the notation and assumptions of Lemma* 4.1*, let $\pi : Y' \to Y$ be a morphism from a non-singular projective variety Y' such that $\Sigma' := \pi^{-1}(\Sigma)$ is a simple normal crossing divisor on Y'. We assume that $\kappa(W'_\eta) = \kappa(X'_\eta) = 0$, where*

$$h' : W' \xrightarrow{\ g'\ } X' \xrightarrow{\ f'\ } Y'$$

is induced by π and W'_η (resp. X'_η) is the generic fiber of h' (resp. f'). We note that, when π is surjective, $\kappa(W'_\eta) = \kappa(X'_\eta) = 0$ is always true.

Then we obtain $\pi^*L_{X/Y}^{ss} = L_{X'/Y'}^{ss}$. In particular, if $L_{X/Y}^{ss}$ is semi-ample, then so is $L_{X'/Y'}^{ss}$. Furthermore, we assume that π is surjective. Then $L_{X/Y}^{ss}$ is semi-ample if and only if so is $L_{X'/Y'}^{ss}$.

We note that when π is not surjective, we have to choose $L_{W/Y}^{ss}$, which is unique modulo linear equivalence, such that $\mathrm{Supp}\lfloor L_{W/Y}^{ss}\rfloor$ *does not contain $\pi(Y')$.*

Proof. By Lemma 4.1, it is sufficient to prove that $\pi^*L_{W/Y}^{ss} = L_{W'/Y'}^{ss}$.

Once we fix an embedding $f_*\omega_{W/Y} \subset \mathbb{C}(Y)$ such that $\mathrm{Supp}\lfloor L_{W/Y}^{ss}\rfloor$ does not contain $\pi(Y')$, we obtain $f'_*\omega_{W'/Y'} \subset \pi^*f_*\omega_{W/Y} \subset \mathbb{C}(Y')$. We note that $f_*\omega_{W/Y}$ (resp. $f'_*\omega_{W'/Y'}$) is the upper canonical extension of $f_{0*}\omega_{W_0/Y_0}$ (resp. $f'_{0*}\omega_{W'_0/Y'_0}$). Therefore, $\pi^*L_{W/Y}^{ss} - L_{W'/Y'}^{ss}$ is determined as $\mathbb{Q}$-divisor.

Let $\alpha : \widetilde{Y} \to Y$ be a finite covering that induces a semistable resolution in codimension one to $W \times_Y \widetilde{Y} \to \widetilde{Y}$ such that the union of the branch locus and Σ is simple normal crossing. We can also assume that the union of Σ' and the branch locus of $\widetilde{Y} \times_Y Y' \to Y'$ is simple normal crossing. We take a finite cover $\beta : \widetilde{Y}' \to \widetilde{Y} \times_Y Y' \to Y'$ such that $\widetilde{Y}'$ is non-singular and the union of the branch locus of β and Σ' is simple normal crossing. We can further assume that $W' \times_{Y'} \widetilde{Y}' \to \widetilde{Y}'$ has a semistable resolution in codimension one. We have $\alpha^* L^{ss}_{W/Y} = L^{ss}_{\widetilde{W}/\widetilde{Y}}$, $\beta^* L^{ss}_{W'/Y'} = L^{ss}_{\widetilde{W}'/\widetilde{Y}'}$ by [FM, Corollary 2.5 (ii)]. By the following lemma, we have $\gamma^* \widetilde{f}_* \omega_{\widetilde{W}/\widetilde{Y}} = \widetilde{f}'_* \omega_{\widetilde{X}'/\widetilde{Y}'}$, where $\gamma : \widetilde{Y}' \to \widetilde{Y} \times_Y Y' \to \widetilde{Y}$. Therefore, we get $\beta^* L^{ss}_{W'/Y'} = \beta^* \pi^* L^{ss}_{W/Y}$. Thus, we obtain the required equality $L^{ss}_{W'/Y'} = \pi^* L^{ss}_{W/Y}$. ∎

Remark 4.3. In Proposition 4.2, the assumption $\kappa(W'_\eta) = \kappa(X'_\eta) = 0$ is unnecessary if the invariance of the Kodaira dimension under smooth deformations is true (cf. [Ny2, Theorem (5.6, 5.10)]).

It is not difficult to see that $\kappa(W'_\eta) = \kappa(X'_\eta) = 0$ if $\pi(Y')$ is in a sufficiently general position in Y, that is, $\pi(Y')$ is not contained in the countable union of certain proper Zariski closed subsets of Y.

The next lemma is well-known. It is an easy consequence of the theory of the canonical extension of VHS.

LEMMA 4.4. *Let $f : X \to Y$ be an algebraic fiber space with $\dim f = n$. Let Σ be a simple normal crossing divisor such that f is smooth over $Y_0 := Y \setminus \Sigma$. Assume that all the local monodromies of $\mathcal{H} := \mathcal{O}_{Y_0} \otimes R^n f_{0*} \mathbb{Z}_{X_0}$ around Σ are unipotent. Let $\pi : Y' \to Y$ be a projective morphism from a non-singular projective variety Y' such that $\pi^{-1}(\Sigma)$ is a simple normal crossing divisor. Then $\pi^*(f_* \omega_{X/Y}) \simeq f'_* \omega_{X'/Y'}$, where $f' : X' \to Y'$ is an algebraic fiber space induced by π.*

Proof. Under the assumption of this lemma, the canonical extension and the pull-back by π are commutative (see [Ka3, Proposition 1]). Hence, $\pi^*(f_* \omega_{X/Y}) \simeq f'_* \omega_{X'/Y'}$. ∎

The following is a supplement to Proposition 3.7.

COROLLARY 4.5. *Under the same notation and assumptions as in Proposition 3.7, we assume that the local monodromies around Σ are unipotent, then $L^{ss}_{X/Y} = \lfloor L^{ss}_{X/Y} \rfloor = \lfloor L_{X/Y} \rfloor$, that is, $L^{ss}_{X/Y}$ is a Weil divisor.*

Proof. Let $\pi : Y' \to Y$ be a finite Kummer covering that induces $f' : X' \to Y'$ a semistable reduction in codimension one. Of course, we can assume that $\pi^{-1}(\Sigma)$ is simple normal crossing. Then $\pi^* L^{ss}_{X/Y} = f'_* \omega_{X'/Y'} = \pi^* \lfloor L^{ss}_{X/Y} \rfloor$ by Proposition 3.7 and Lemma 4.4. So, we obtain the required equality $L^{ss}_{X/Y} = \lfloor L^{ss}_{X/Y} \rfloor$. □

§5. Applications to algebraic fiber spaces

In this section, we apply Theorems 2.10 and 2.11 for algebraic fiber spaces. There are many applications of Theorems 2.10 and 2.11, but we restrict to the following ones.

The next theorem already appeared in the introduction (see Theorem 1.2).

THEOREM 5.1. *Let $f : X \to Y$ be a surjective morphism between nonsingular projective varieties X and Y. Let $\mathcal{L}$ be an f-ample line bundle on X. Assume that there exists a simple normal crossing divisor Σ on Y such that every fiber of f over $Y_0 := Y \setminus \Sigma$ is a $K3$ surface (resp. an n-dimensional Abelian variety). Then we obtain a polarized variation of Hodge structures of weight two (resp. one). Let $p_0 : Y_0 \to \mathcal{S}$ be the period map of the weight two (resp. one) polarized Hodge structures and $p : Y \to \bar{\mathcal{S}}$ be the extension of p_0. We note that the projective variety $\bar{\mathcal{S}}$ is the Baily-Borel-Satake compactification of $\mathcal{S}$, which is embedded into a projective space by automorphic forms of weight k, and that p always exists by Borel's extension theorem. We define $a = 19k$ (resp. $a = k(n+1)$). Then $aL^{ss}_{X/Y}$ is a Weil divisor and*

$$\mathcal{O}_Y(aL^{ss}_{X/Y}) \simeq p^* \mathcal{O}_{\bar{\mathcal{S}}}(1).$$

We note that a is decided by the polarized Hodge structures of the fibers of f.

Proof. If $f : X \to Y$ is a $K3$ fibration, then we define $\mathcal{H} := \mathcal{O}_{Y_0} \otimes (R^2 f_{0*} \mathbb{Z}_{X_0})_{prim}$ and $\mathcal{F}^2 := f_{0*} \omega_{X_0/Y_0}$, where *prim* means the primitive part with respect to the fixed polarization $\mathcal{L}$.

When $f : X \to Y$ is an Abelian fibration, we put $\mathcal{H} := R^1 f_{0*} \mathbb{Z}_{X_0} \otimes \mathcal{O}_{Y_0}$ and $\mathcal{F}^1 := f_{0*} \Omega^1_{X_0/Y_0}$. Then $f_{0*} \omega_{X_0/Y_0} \simeq \bigwedge^n \mathcal{F}^1$. We note that $f_* \omega_{X/Y}$ is the canonical extension of $f_{0*} \omega_{X_0/Y_0}$ if the local monodromies around Σ are unipotent.

Let $\pi : M \to Y$ be a finite Kummer covering such that the algebraic fiber space $g : Z \to M$ induced by π is semistable in codimension one. We denote $\Sigma' = \pi^{-1}(\Sigma)$, $Y_0 = Y \setminus \Sigma$, $M_0 = M \setminus \Sigma'$, and $\pi_0 : M_0 \to Y_0$. We can assume that Z is isomorphic to $M \times_Y X$ over M_0. By pulling back the polarized variation of Hodge structures by π_0, we obtain a polarized variation of Hodge structures on M_0. When $f : X \to Y$ is a $K3$ (resp. an Abelian) fibration, the corresponding polarized variation of Hodge structures is type (WII) and $g = 19$ (resp. type (WI) and $g = n$) by the notation in 2.3. So, we use the same notation as in Section 2. We give names as follows;

$$q : M \xrightarrow{\ \pi\ } Y \xrightarrow{\ p\ } \bar{\mathcal{S}},$$

and $q_0 := q|_{M_0}$. By 2.21, we obtain isomorphisms $\rho_{M_0} : q_0^*\mathcal{O}_{\mathcal{S}}(1) \simeq (g_{0*}\omega_{Z_0/M_0})^{\otimes a}$ and $\rho_{Y_0} : p_0^*\mathcal{O}_{\mathcal{S}}(1) \simeq (f_{0*}\omega_{X_0/Y_0})^{\otimes a}$, which are compatible, that is, $\rho_{M_0} = \pi_0^*\rho_{Y_0}$. By 2.22, the above isomorphism ρ_{M_0} extends to $\rho_M : q^*\mathcal{O}_{\bar{\mathcal{S}}}(1) \simeq (g_*\omega_{Z/M})^{\otimes a}$. So, this isomorphism is Galois invariant. Taking the Galois invariant part, we obtain $p^*\mathcal{O}_{\bar{\mathcal{S}}}(1) \simeq \mathcal{O}_Y(\lfloor aL^{ss}_{X/Y} \rfloor)$. We note that $\pi \circ p = q$ and $\pi^* L^{ss}_{X/Y} = g_*\omega_{Z/M}$. Therefore, $aL^{ss}_{X/Y}$ is a Weil divisor and $p^*\mathcal{O}_{\bar{\mathcal{S}}}(1) \simeq \mathcal{O}_Y(aL^{ss}_{X/Y})$. □

Remark 5.2. In Theorem 5.1, we further assume that f is semistable in codimension one. Then the left hand side is $(f_*\omega_{X/Y})^{\otimes a}$ (cf. Proposition 3.6). So we obtain $(f_*\omega_{X/Y})^{\otimes a} \simeq p^*\mathcal{O}_{\bar{\mathcal{S}}}(1)$.

THEOREM 5.3. *Let $f : X \to Y$ be a surjective morphism between nonsingular projective varieties X and Y with connected fibers. Assume that there exists a simple normal crossing divisor Σ on Y such that every fiber of f over $Y_0 := Y \setminus \Sigma$ is a smooth curve of genus $g \geq 1$. If f is semistable in codimension one, then we obtain*

$$(\det f_*\omega_{X/Y})^{\otimes a} \simeq p^*\mathcal{O}_{\bar{\mathcal{S}}}(1),$$

where $p_0 : Y_0 \to \mathcal{S}$ is the period map of weight one polarized Hodge structures and $p : Y \to \bar{\mathcal{S}}$ is the extension of p_0. Note that $\mathcal{S} = \mathbb{H}_g/\operatorname{Sp}(2g, \mathbb{Z})$ and $\bar{\mathcal{S}}$ is the Satake compactification of $\mathcal{S}$, which is embedded into a projective space by automorphic forms of weight k, and that $a = k(g+1)$.

Proof. We use the same notation as in the proof of Theorem 2.11. Let $H := R^1 f_{0*}\mathbb{Z}_{X_0}$ and $\mathcal{H} := H \otimes \mathcal{O}_{Y_0}$. We put $\mathcal{F}^1 := f_{0*}\omega_{X_0/Y_0} \subset \mathcal{H}$. We consider the period map of polarized Hodge structures of weight one. In this case, it is well-known that $D = \mathbb{H}_g$, $\Gamma = \operatorname{Sp}(2g, \mathbb{Z})$, and $\mathcal{S} = H_g/\operatorname{Sp}(2g, \mathbb{Z})$, where g is the genus of general fibers. Thus, we obtain $p^*\mathcal{O}_{\bar{\mathcal{S}}}(1) \simeq \left(\bigwedge^g \mathcal{F}^1(\overline{\mathcal{H}})\right)^{\otimes a} \simeq (\det f_*\omega_{X/Y})^{\otimes a}$ by Theorem 2.11. □

Remark 5.4. Since we have the global Torelli theorem for curves, Theorem 5.3 implies that the Conjecture $Q_{n,n-1}$ is true (see [Mo, (7.2) Theorem]). For the conjectures $Q_{n,m}$, $C^+_{n,m}$, and $C_{n,m}$, see [Mo, (7.1), (7.3)] and Conjecture 7.1 below. Therefore, we obtain $C^+_{n,n-1}$ (cf. [V1]).

Remark 5.5. If $f : X \to Y$ is an elliptic fibration in Theorem 5.1, it is well-known that $p = J$, $\bar{\mathcal{S}} = \mathbb{P}^1$, and a can be taken to be 12, where J is the j-function (see [U1], [Ft, Section 2], [Ka3, Theorem 20], and [Ny3, Chapter 3]).

Theorem 5.1 for semistable families of Abelian varieties and Theorem 5.3 are more or less known to specialists, see [Ar, Theorem 1.1], [U2, Theorem 2.1], and [BPV, III **17**].

The next theorem directly follows from Theorem 2.10. It may be useful for the study of symplectic manifolds. We use the same notation as in Section 2.

THEOREM 5.6. *Let $f : X \to Y$ be an algebraic fiber space. Assume that there exists a simple normal crossing divisor Σ on Y such that $f_0 : X_0 \to Y_0$ is smooth and $h^{2,0}(X_y) = h^{0,2}(X_y) = 1$ and $h^{1,1}(X_y) \geq 4$ for every $y \in Y_0$. Let $\mathcal{L}$ be an f-ample line bundle on X. We define $\mathcal{H} = \mathcal{O}_{Y_0} \otimes (R^2 f_{0*}\mathbb{Z}_{X_0})_{prim}$ and $\mathcal{F}^2 = f_{0*}\Omega^2_{X_0/Y_0}$. Then $\mathcal{H} \to Y_0$ is a polarized variation of Hodge structures of type* (WII). *We assume that the local monodromies around Σ on $\mathcal{H}$ are unipotent. Then we obtain*

$$\mathcal{F}^2(\overline{\mathcal{H}})^{\otimes a} \simeq p^*\mathcal{O}_{\bar{\mathcal{S}}}(1).$$

The following remark is well-known.

Remark 5.7. In Theorem 5.6, we further assume that Y is a curve and f is semistable. Then

$$\mathcal{F}^2(\overline{\mathcal{H}}) = f_*\Omega^2_{X/Y}(\log V),$$

where $V = f^*\Sigma$.

§6. Remarks on the semi-ampleness of $L^{ss}_{X/Y}$

In this section, we generalize the semi-ampleness of the semistable part $L^{ss}_{X/Y}$ for more general fiber spaces.

THEOREM 6.1. *Let $f : X \to Y$ be an algebraic fiber space as in* 3.1. *Assume that the geometric generic fiber $X_{\bar{\eta}}$ has a generically finite cover that is birationally equivalent to an Abelian variety. We further assume that there exists a generically finite cover $g : W \to X$ and a simple normal crossing divisor Σ on Y as in Lemma* 4.1. *Then the semistable part $L^{ss}_{X/Y}$ is semi-ample.*

Proof. By the assumption, there exists a surjective morphism $\pi : Y' \to Y$ from a non-singular projective variety such that $\Sigma' := \pi^{-1}(\Sigma)$ is a simple normal crossing divisor and $f' : X' \to Y'$ is an algebraic fiber space induced by π with the following properties:

(i) there exists a generically finite morphism $Z \to X'$ from a non-singular projective variety Z,

(ii) there is an algebraic fiber space $Z' \to Y'$, which is birationally equivalent to $Z \to Y'$,

(iii) there is a simple normal crossing divisor $D \supset \Sigma'$ such that $Z \to Y'$ and $Z' \to Y'$ are smooth over $Y' \setminus D$,

(iv) every closed fiber of $Z' \to Y'$ over $Y' \setminus D$ is an Abelian variety.

We note that Abelian variety V is characterized by the conditions that $K_V \sim 0$ and the irregularity $q(V) = \dim V$.

Applying Theorem 5.1 to $Z' \to Y'$, we obtain that $L^{ss}_{Z'/Y'}$ is semi-ample and $L^{ss}_{Z/Y'}$ is so since $L^{ss}_{Z/Y'} = L^{ss}_{Z'/Y'}$. By Proposition 4.2, $\pi^* L^{ss}_{X/Y} = L^{ss}_{X'/Y'}$, where $f' : X' \to Y'$ is induced by π. By Lemma 4.1, $L^{ss}_{X'/Y'} = b L^{ss}_{Z/Y'}$. Therefore, the semistable part $L^{ss}_{X/Y}$ is semi-ample. ∎

Remark 6.2. By taking a canonical cover introduced in Definition 3.8, we can always take a required cover $W \to X$ and a simple normal crossing divisor Σ on Y as in Lemma 4.1 if we modify $f : X \to Y$ birationally.

By the same argument as in Theorem 6.1, we obtain the following theorem. It is stated in [Mo] with the idea of the proof.

THEOREM 6.3. (cf. [Mo, Part II (5.15.9)(ii)]) *Let $f : X \to Y$ be an algebraic fiber space such that the generic fiber is a surface with the Kodaira dimension zero. Assume that there exists a generically finite cover $g : W \to X$ and a simple normal crossing divisor Σ on Y as in Lemma* 4.1. *Then the semistable part $L^{ss}_{X/Y}$ is semi-ample.*

Sketch of the proof. By the same argument as in Theorem 6.1, we can reduce it to the case where general fibers are $K3$ or Abelian surfaces. Thus we obtain the result by Theorem 5.1. □

By the above theorem and Remark 6.2, we obtain the following corollary.

COROLLARY 6.4. *Let $f : X \to Y$ be an algebraic fiber space such that the generic fiber is a surface with the Kodaira dimension zero. After we modify $f : X \to Y$ birationally, we obtain that $L^{ss}_{X/Y}$ is semi-ample.*

6.5. If $\dim Y = 1$, then the semi-ampleness of $L^{ss}_{X/Y}$ holds without extra assumptions. The next theorem is a reformulation of Kawamata's result.

THEOREM 6.6. (cf. [Ka2]) *Let $f : X \to Y$ be an algebraic fiber space as in* 3.1. *Assume that* $\dim Y = 1$. *Then $L^{ss}_{X/Y}$ is semi-ample.*

Proof. By Lemma 4.1, Definition 3.8, and Proposition 4.2, we can assume that $b = 1$ and $f : X \to Y$ is semistable. So we have $\mathcal{O}_Y(L^{ss}_{X/Y}) \simeq f_*\omega_{X/Y}$ by Proposition 3.6. Thus, it is sufficient to prove $\kappa(L^{ss}_{X/Y}) \geq 0$. We note that $L^{ss}_{X/Y}$ is nef.

If $\deg L^{ss}_{X/Y} > 0$, then $\kappa(L^{ss}_{X/Y}) = 1$. So we assume that $\deg L^{ss}_{X/Y} = 0$. In this case, $\mathcal{O}_Y(kL^{ss}_{X/Y}) \simeq \mathcal{O}_Y$ for some positive integer k (see [Ka2, p. 69, lines 12–16]). Therefore, we obtain the required result. □

§7. Applications to Iitaka's conjecture

The following is a famous conjecture by Iitaka [Ii]. For the details, see [Mo, Sections 6, 7] (see also Remark 5.4).

CONJECTURE 7.1. (Conjecture $C_{n,m}$) *Let $f : X \to Y$ be an algebraic fiber space with* $\dim X = n$ *and* $\dim Y = m$. *Then we have*

$$\kappa(X) \geq \kappa(Y) + \kappa(X_\eta).$$

The next theorem claims that $\kappa(L^{ss}_{X/Y}) \geq 0$ implies $C_{n,m}$ is true when the Kodaira dimension of the generic fiber is zero.

THEOREM 7.2. *Let $f : X \to Y$ be an algebraic fiber space as in Lemma* 4.1. *Then $\kappa(K_{X/Y}) \geq \kappa(L^{ss}_{X/Y})$. If $\kappa(L^{ss}_{X/Y}) \geq 0$ and $\kappa(Y) \geq 0$, then $\kappa(X) \geq \max\{\kappa(Y), \kappa(L^{ss}_{X/Y})\}$. In particular, $\kappa(L^{ss}_{X/Y}) \geq 0$ implies $\kappa(X) \geq \kappa(Y)$.*

Proof. By applying the flattening theorem, we obtain a birational morphism $\pi : Y' \to Y$ from a non-singular projective variety Y' such that $\pi^{-1}(\Sigma)$ is simple normal crossing and the following diagram;

$$\begin{array}{ccc} X & \xleftarrow{\mu} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xleftarrow[\pi]{} & Y', \end{array}$$

as in Lemma 3.9. So we use the same notation as in Lemma 3.9. By Lemma 3.9 and Proposition 4.2, we obtain $\kappa(K_{X/Y}) = \kappa(K_{X'/Y}) = \kappa(K_{X'/Y}(B'_-)) \geq \kappa(K_{X'/Y'}(B'_-)) \geq \kappa(L^{ss}_{X'/Y'}) = \kappa(L^{ss}_{X/Y})$. We note that B'_- is effective and μ-exceptional. The latter part of the statement is obvious. ∎

By the above theorem, Theorem 6.1, and Remark 6.2, we obtain:

COROLLARY 7.3. (cf. [Ka3, Theorem 13]) *Let $f : X \to Y$ be an algebraic fiber space whose generic fiber has the Kodaira dimension zero. Assume that the geometric generic fiber has a generically finite cover that is birationally equivalent to an Abelian variety. Then $\kappa(X) \geq \max\{\kappa(Y), \kappa(L^{ss}_{X/Y})\}$ if $\kappa(Y) \geq 0$.*

We prove Conjecture 7.1 for special fiber spaces. The following argument heavily relies on [Ka3].

THEOREM 7.4. *Let $f : X \to Y$ be an algebraic fiber space. Assume that the Kodaira dimension of the generic fiber of f is one, that is, $\kappa(X_\eta) = 1$. Let*

$$f : X \xrightarrow{g} Z \xrightarrow{h} Y$$

be the relative Iitaka fibering. Assume the following condition;

($\diamondsuit$) *By replacing $g : X \to Z$ birationally, there exists a proper surjective morphism $\alpha : W \to X \to Z$ with connected fibers such that*

(i) *W is a non-singular projective variety, and $W \to X$ is generically finite,*

(ii) *the Kodaira dimension of the generic fiber of α is zero, that is, $\kappa(W_\eta) = 0$, and the geometric genus $p_g(W_\eta) = 1$,*

(iii) *there exists a simple normal crossing divisor D on Z, such that g and α are smooth over $Z \setminus D$,*

(iv) *the semistable part $L^{ss}_{X/Z}$ is semi-ample.*

Then the inequality $\kappa(X) \geq \kappa(Y) + \kappa(X_\eta) = \kappa(Y) + 1$ holds.

Remark 7.5. If the geometric generic fiber of g has a generically finite cover that is birationally equivalent to an Abelian variety, then the condition ($\diamondsuit$) is satisfied (see Theorem 6.1 and Remark 6.2).

Therefore, we obtain

Corollary 7.6. *Let $f : X \to Y$ be an algebraic fiber space. Assume that the Kodaira dimension of the generic fiber of f is one. Let*

$$f : X \xrightarrow{\ g\ } Z \xrightarrow{\ h\ } Y$$

be the relative Iitaka fibering. If the geometric generic fiber of g has a generically finite cover that is birationally equivalent to an Abelian variety, then $\kappa(X) \geq \kappa(Y) + 1$.

The following proposition is a consequence of the easy addition theorem (see, for example, [Ka1, Theorem 11]). For the proof, see [Ka3].

Proposition 7.7. ([Ka3, Proposition 6]) *Let X, Y, and Z be non-singular projective algebraic varieties, and let $f : X \to Y$ and $g : X \to Z$ be surjective morphisms with $g_*\mathcal{O}_X \simeq \mathcal{O}_Z$. Then for a sufficiently general point z of Z,*

$$\dim Y - \kappa(Y) \geq \dim(f(X_z)) - \kappa(f(X_z)),$$

where $f(X_z)$ is considered as an irreducible algebraic variety and κ is defined as the Kodaira dimension of its non-singular model.

Lemma 7.8. *Let $f : X \to Y$ be an algebraic fiber space that factors as*

$$X \longrightarrow Y' \xrightarrow{\ \mu\ } Y,$$

where μ is a proper birational morphism from a non-singular projective variety Y'. Then $\kappa(K_{X/Y'}) \leq \kappa(K_{X/Y})$.

Proof. It is obvious since $K_{Y'} - \mu^* K_Y$ is effective. ∎

The following lemma is a modified version of [Ka3, Lemma 19].

LEMMA 7.9. (Induction Lemma) *For the proof of Theorem* 7.4, *it is enough to prove that*

$$(\spadesuit) \qquad \kappa(X) > 0 \quad \text{if} \quad \kappa(Y) \geq 0.$$

Proof. If $\kappa(Y) = -\infty$, then there is nothing to prove. Suppose $\kappa(Y) \geq 0$. Let $\Phi : X \to V$ be the Iitaka fibering of X. By ($\spadesuit$), $\dim V > 0$ and hence $X_v \neq X$, where X_v is a sufficiently general fiber of Φ. Let

$$f : X \xrightarrow{\ g\ } Z \xrightarrow{\ h\ } Y$$

be the relative Iitaka fibering. Since $\Phi : X \to V$ is the Iitaka fibering, we can assume that Φ factors as $X \to Z \to V$ (see [Mo, (2.4)(ii)]). By Proposition 7.7,

$$\dim Y - \kappa(Y) \geq \dim(f(X_v)) - \kappa(f(X_v)),$$

where v is a sufficiently general point of V. By the Stein factorization of $X_v \to f(X_v)$, we obtain an algebraic fiber space $f_v : X_v \to Y_v$. Since $\kappa(Y) \geq 0$, we have $\kappa(f(X_v)) \geq 0$, and hence $\kappa(Y_v) \geq 0$. We note that $\kappa(X_v) = 0$. If $\dim f_v = \dim f$, then the fiber of f_v is the fiber of f. By applying ($\spadesuit$), we obtain $\kappa(X_v) > 0$, which contradicts $\kappa(X_v) = 0$. Therefore, $\dim f_v < \dim f$. Since $\dim h = 1$, $\dim f_v = \dim f - 1$. So the fiber of f_v is the fiber of $X_v \to Z_v$. Therefore, $0 = \kappa(X_v) \geq \kappa(Y_v) \geq \kappa(f(X_v)) \geq 0$ (see Theorem 7.2 and Proposition 4.2). By Proposition 7.7,

$$\begin{aligned}\dim Y - \kappa(Y) &\geq \dim X_v - \dim f_v \\ &= \dim X - \kappa(X) - (\dim f - 1) \\ &= \dim X - \kappa(X) - \dim f + 1.\end{aligned}$$

Thus, we obtain the required inequality $\kappa(X) \geq \kappa(Y) + 1$. ∎

Proof of Theorem 7.4. Let q be the genus of the general fiber of h. If $q \geq 2$, then $\kappa(X) \geq \kappa(Z) \geq \kappa(Y) + 1$ (see Corollary 7.2) and we are done. By the same way, if $q = 1$ and h have non-constant moduli, then $\kappa(X) \geq \kappa(Z) \geq \kappa(Y) + \mathrm{Var}(h) \geq \kappa(Y) + 1$ (for the definition of $\mathrm{Var}(h)$, see [V2], and see also Remark 5.4). Thus, what are remaining are the following two cases:

(i) $q = 1$, and h have the constant moduli, or

(ii) $q = 0$.

Proof of Case (i). We shall prove that $\kappa(X, K_{X/Y}) > 0$ for a suitable birational model of $f : X \to Y$. Using Lemma 7.8 and [Ka3, Theorems 8, 9], we reduce it to the case where Z is birationally equivalent to a product $Y \times E$ for an elliptic curve E. Thus we come to the following situation:

$$f : X \xrightarrow{\ g\ } Z \xrightarrow{\ \nu\ } Y \times E \xrightarrow{\ h_1\ } Y,$$

where

(a) E is an elliptic curve,

(b) f is the given fiber space, h_1 is the projection, and ν is a proper birational morphism,

(c) g is the fiber space satisfying ($\diamondsuit$), and f factors as

$$X \xrightarrow{\ \mu\ } \widetilde{X} \longrightarrow Y,$$

where μ is birational and $\widetilde{X}$ is a non-singular projective variety such that B_- is an effective μ-exceptional divisor (see Lemma 3.9). We note that

$$bK_X = g^*(bK_Z + L^{ss}_{X/Z}) + \sum_{D_i} s_{D_i} g^* D_i + B,$$

(d) there exists a simple normal crossing divisor D on Z as in (iii) of ($\diamondsuit$),

(e) the horizontal part D^h with respect to $Z \to Y$ is smooth.

By the canonical bundle formula, we have

$$g_* K^{\otimes m}_{X/Z}((m/b)B_-) \simeq \mathcal{O}_Z\Bigg(\sum_i (m/b)s_{D_i} D_i + (m/cb)A\Bigg),$$

where m is a positive integer such that $(m/b)s_{D_i}$, m/b, and m/cb are integers, and A is a general member of the free linear system $|cL^{ss}_{X/Z}|$. By restricting the canonical bundle formula to $X_y \to Z_y$, where y is a sufficiently general point of Y, we obtain an irreducible component D_1 of D^h such that $s_{D_1} \neq 0$ or an irreducible component of A^h since $\kappa(X_y) = 1$. We denote such an irreducible horizontal divisor as D_0. If D_0 is an irreducible component of A^h (resp. $D_0 = D_1$), we define $s_{D_0} := 1/c$ (resp. $s_{D_0} := s_{D_1}$).

Let $\overline{D}_0$ be the image of D_0 on $Y \times E$. Then $\kappa(Y \times E, \overline{D}_0) > 0$ by Lemma 7.10. On the other hand, every irreducible component of $\nu^*\overline{D}_0 - D_0$ is ν-exceptional and $H^0(Z, \mathcal{O}_Z((ms_{D_0}/b)(D_0 - \nu^*\overline{D}_0) \otimes K_{Z/Y}^{\otimes mk}) \neq 0$ for a sufficiently large integer k. We note that $K_{Y\times E} = h_1^* K_Y$. Combining the above, we obtain

$$H^0(Z, g_* K_{X/Y}^{\otimes mk}(k(m/b)B_-) \otimes \mathcal{O}_Z(-(ms_{D_0}/b)\nu^*\overline{D}_0)) \neq 0.$$

Therefore,

$$\kappa(X, K_{X/Y}) \geq \kappa(Z, \nu^*\overline{D}_0) = \kappa(Y \times E, \overline{D}_0) > 0.$$

We note that B_- is effective and exceptional over $\widetilde{X}$. Thus, we finish the proof of Case (i). □

The following lemma was already used in the proof of Case (i).

LEMMA 7.10. *Let Y be a non-singular projective variety and E is an elliptic curve. Let $p : Y \times E \to Y$ be the first projection and D an irreducible divisor that is mapped onto Y by p. Then the Kodaira dimension $\kappa(Y \times E, D) > 0$.*

Proof. This directly follows from the next lemma: Lemma 7.11. □

The following lemma is a variant of the theorem of cube.

LEMMA 7.11. *Under the notation and assumptions of Lemma 7.10, we have $2D \sim T_x^* D + T_{-x}^* D$, where $x \in E$ and T_x is the translation $+x : E \to E$.*

Proof. See [F1, Section 5]. It is a special case of [F1, Corollary 5.6]. □

Proof of Case (ii). By the same argument as in Case (i), we reduce it to the following situation:

$$f : X \xrightarrow{\ g\ } Z \xrightarrow{\ \nu\ } Y \times \mathbb{P}^1 \xrightarrow{\ h_1\ } Y,$$

where

(a) f is the given algebraic fiber space, h_1 is the projection, and ν is a proper birational morphism,

(b) g is the fiber space satisfying ($\diamondsuit$),

(c) there is a simple normal crossing divisor D on Z as in (iii) of ($\diamondsuit$),

(d) the horizontal part D^h with respect to $Z \to Y$ is smooth.

Let A be a general member of the free linear system $|cL^{ss}_{X/Z}|$, where $c \geq 2$. We denote the irreducible decomposition $A := \sum D'_j$. By Lemma 7.8, [Ka3, Theorems 8, 9], Lemma 3.9, and Proposition 4.2, we can assume that $\deg_Y D_i \leq 1$ and $\deg_Y D'_j \leq 1$ for every i and j. By the canonical bundle formula, we have

$$g_* K^{\otimes m}_{X/Z}((m/b)B_-) \simeq \mathcal{O}_Z\Bigl(\sum_i (m/b)s_{D_i} D_i + (m/cb)A\Bigr)$$

by Proposition 4.2, where m is a sufficiently divisible positive integer such that $(m/b)s_{D_i}$, m/b, and m/cb are integers. We further assume that

(i) $f : X \to Y$ factors as

$$f : X \xrightarrow{\ \mu\ } \widetilde{X} \longrightarrow Y$$

such that B_- is μ-exceptional (see Lemma 3.9),

(ii) $\operatorname{Supp} D^h \cup \operatorname{Supp} A^h$ is smooth.

By restricting the canonical bundle formula to $X_y \to Z_y$, where y is a sufficiently general point of Y, we have

$$\deg_{Z_y}\Bigl(\sum (m/b)s_{D_i} D_i + (m/cb)\sum D'_j\Bigr) > 2m,$$

since $\kappa(X_y) = 1$.

Since $(m/b)s_{D_i} < m$ and $m/cb < m$, we can make a sequence $(C_1, C_2), \ldots, (C_{2m+1}, C_{2m+2})$ of divisors on Z such that

(1) C_k is an irreducible component of either D^h or A^h, and $\deg_Y C_k = 1$ for $1 \leq k \leq 2m+2$,

(2) $C_{2k+1} \neq C_{2k+2}$ for $0 \leq k \leq m$, and $C_{2m-1} \neq C_{2m+1}$, $C_{2m} = C_{2m+2}$,

(3) $\sum_{k=1}^{2m+1} C_k \leq \sum (m/b)s_{D_i} D_i + (m/cb)\sum D'_j$.

We note that $\sum C_k$ is smooth. We shall show that

$$h^0(Z, K_{Z/Y} \otimes \mathcal{O}_Z(C_{2m-1} + C_{2m} + C_{2m+1})) \geq 2$$

and

$$h^0(Z, K_{Z/Y} \otimes \mathcal{O}_Z(C_{2k-1} + C_{2k})) \geq 1$$

for $1 \leq k \leq m-1$. Since

$$f_* K_{X/Y}^{\otimes m}((m/b)B_-) = h_*\big(g_* K_{X/Z}^{\otimes m}((m/b)B_-) \otimes K_{Z/Y}^{\otimes m}\big),$$

and

$$g_* K_{X/Z}^{\otimes m}((m/b)B_-) \supset \mathcal{O}_Z\Big(\sum C_k\Big),$$

it follows that

$$h^0\big(X, K_{X/Y}^{\otimes m}\big) = h^0\big(\widetilde{X}, K_{\widetilde{X}/Y}^{\otimes m}\big) = h^0\big(X, K_{X/Y}^{\otimes m}((m/b)B_-)\big) \geq 2$$

and we are done. We note that B_- is effective and exceptional over $\widetilde{X}$.

Let $Y' \to Y$ be a birational morphism from a non-singular projective variety Y' and Z' a resolution of $Z \times_Y Y'$. We obtain the following commutative diagram;

$$\begin{array}{ccc} Z' & \xrightarrow{\ \alpha\ } & Z \\ {\scriptstyle h'}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ Y' & \longrightarrow & Y. \end{array}$$

We can assume that there exists a simple normal crossing divisor Σ on Y' such that

$$\varphi_k : Z'_0 := (h')^{-1}(Y'_0) \simeq Y'_0 \times \mathbb{P}^1$$

with $\varphi_k(C'_{2k-1}|_{Z'_0}) = Y'_0 \times \{0\}$ and $\varphi_k(C'_{2k}|_{Z'_0}) = Y'_0 \times \{\infty\}$ for $1 \leq k \leq m+1$, where $Y'_0 := Y' \setminus \Sigma$ and C'_k is the proper transform of C_k. By eliminating the indeterminacy, we can further assume that there exists $\psi_k : Z' \to \mathbb{P}^1$ such that $\psi_k|_{Z'_0} = p_2 \circ \varphi_k$, where p_2 is the second projection $Y'_0 \times \mathbb{P}^1 \to \mathbb{P}^1$. We also assume that $C'_{2k-1} \cup C'_{2k} \cup ((h')^*\Sigma)_{red}$ is simple normal crossing for every k. We note that

$$K_{Z'/Y'}(C'_{2k-1} + C'_{2k}) \prec K_{Z'/Y}(C'_{2k-1} + C'_{2k}) \prec \alpha^*(K_{Z/Y}(C_{2k-1} + C_{2k})).$$

So, for our purpose, we can replace Y, Z, C_k etc. with Y', Z', C'_k. Therefore, we omit $'$ for simplicity.

The following lemma corresponds to [Ka3, Claim]. In our formulation, the proof is obvious. We note that $C_{2k-1} \cup C_{2k} \cup (h^*\Sigma)_{red}$ is simple normal crossing and $Z \setminus (C_{2k-1} \cup C_{2k} \cup (h^*\Sigma)_{red}) \simeq Y_0 \times \mathbb{C}^\times$.

Lemma 7.12. *Under the same notation and assumptions, we have*

$$\bigwedge \psi_k^*\Big(\frac{\mathrm{d}z}{z}\Big) \in \mathrm{Hom}(h^*(K_Y+\Sigma), K_Z + C_{2k-1} + C_{2k} + (h^*\Sigma)_{red})$$
$$= H^0(Z, K_{Z/Y}(C_{2k-1}+C_{2k}+(h^*\Sigma)_{red} - h^*\Sigma))$$
$$\subset H^0(Z, K_{Z/Y}(C_{2k-1}+C_{2k})).$$

for $1 \le k \le m+1$*, where* z *denotes a suitable inhomogeneous coordinate of* $\mathbb{P}^1$*.*

Therefore, $h^0(Z, K_{Z/Y}(C_{2k-1}+C_{2k})) \ge 1$ for $1 \le k \le m-1$ and $h^0(Z, K_{Z/Y}(C_{2m-1}+C_{2m}+C_{2m+1})) \ge 2$. Thus, we finish the proof of Case (ii). □

Thus, we obtain Theorem 7.4. □

References

[Am] F. Ambro, *Shokurov's boundary property*, preprint (2002).

[Ar] S. Ju. Arakelov, *Families of algebraic curves with fixed degeneracies*, Math. USSR-Izv., **5** (1971), 1277–1302.

[AMRT] A. Ash, D. Mumford, M. Rapoport and Y. Tai, Smooth compactification of locally symmetric varieties, Lie Groups: History, Frontiers and Applications, Vol. IV. Math. Sci. Press, Brookline, Mass., 1975, iv+335 pp.

[Ba] W. L. Baily, Jr., Introductory lectures on automorphic forms, Kan Memorial Lectures, No. 2. Publications of the Mathematical Society of Japan, No. 12. Iwanami Shoten, Publishers, Tokyo; Princeton University Press, Princeton, N.J., 1973, xv+262 pp.

[BB] W. L. Baily, Jr. and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. (2), **84** (1966), 442–528.

[BPV] W. Barth, C. Peters, and A. Van de Ven, Compact Complex Surfaces, Springer, 1984.

[Bo] A. Borel, *Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem*, J. Differential Geometry, **6** (1972), 543–560.

[Ca] E. Cattani, *Mixed Hodge structures, compactifications and monodromy weight filtration*, Topics in transcendental algebraic geometry, (Princeton, N.J., 1981/1982), 75–100, Ann. of Math. Stud., **106**, Princeton Univ. Press, Princeton, NJ, 1984.

[F1] O. Fujino, *Algebraic fiber spaces whose general fibers are of maximal Albanese dimension*, Nagoya Math. J., **172** (2003), 111–127.

[F2] O. Fujino, *Remarks on algebraic fiber spaces*, preprint (2002).

[FM] O. Fujino, and S. Mori, *A canonical bundle formula*, J. Differential Geom., **56** (2000), no. 1, 167–188.

[Ft] T. Fujita, *Zariski decomposition and canonical rings of elliptic threefolds*, J. Math. Soc. Japan, **38** (1986), 20–37.

[GS] P. Griffiths, and W. Schmid, *Recent developments in Hodge theory: a discussion of techniques and results*, Discrete subgroups of Lie groups and applications to moduli, (Internat. Colloq., Bombay, 1973), pp. 31–127, Oxford Univ. Press, Bombay, 1975.

[Gr] V. Gritsenko, *Modular forms and moduli spaces of abelian and $K3$ surfaces*, Algebra i Analiz, **6** (1994), no. 6, 65–102, (Russian); translation in St. Petersburg Math. J., **6** (1995), no. 6, 1179–1208.

[Ii] S. Iitaka, *Genera and classification of algebraic varieties. 1*, Sûgaku, **24** (1972), 14–27, (Japanese).

[Ka1] Y. Kawamata, *Characterization of abelian varieties*, Compositio Math., **43** (1981), no. 2, 253–276.

[Ka2] Y. Kawamata, *Kodaira dimension of algebraic fiber spaces over curves*, Invent. Math., **66** (1982), 57–71.

[Ka3] Y. Kawamata, *Kodaira dimension of certain algebraic fiber spaces*, J. Fac. Sci. Univ. Tokyo Sect.IA Math., **30** (1983), no. 1, 1–24.

[Kod] K. Kodaira, *On the structure of compact complex analytic surfaces, I*, Amer. J. Math., **86** (1964), 751–798.

[Kol] J. Kollár, *Higher direct images of dualizing sheaves. II*, Ann. of Math. (2), **124** (1986), no. 1, 171–202.

[Kon] S. Kondo, *On the Kodaira dimension of the moduli space of $K3$ surfaces*, Compositio Math., **89** (1993), no. 3, 251–299.

[Ku] V. Kulikov, *Degenerations of $K3$ surfaces and Enriques surfaces*, Math. USSR-Izv., **11** (1977), no. 5, 957–989.

[Mo] S. Mori, *Classification of higher-dimensional varieties*, Proc. Symp. Pure Math., **46** (1987), 269–331.

[Mw] A. Moriwaki, *Torsion freeness of higher direct images of canonical bundles*, Math. Ann., **276** (1987), no. 3, 385–398.

[Mu] D. Mumford, Abelian varieties, Oxford Univ. Press, Oxford, 1970.

[Ny1] N. Nakayama, *Hodge filtrations and the higher direct images of canonical sheaves*, Invent. Math., **85** (1986), no. 1, 217–221.

[Ny2] N. Nakayama, *Invariance of the Plurigenera of Algebraic Varieties*, preprint, RIMS-**1191** (1998).

[Ny3] N. Nakayama, *Local structure of an elliptic fibration*, Higher dimensional birational geometry, (Kyoto, 1997), 185–295, Adv. Stud. Pure Math., **35**, Math. Soc. Japan, Tokyo, 2002.

[Nm1] Y. Namikawa, *Toroidal degeneration of abelian varieties. II*, Math. Ann., **245** (1979), no. 2, 117–150.

[Nm2] Y. Namikawa, Toroidal compactification of Siegel spaces, Lecture Notes in Mathematics, **812**, Springer, Berlin, 1980.

[Od] T. Oda, *On modular forms associated with indefinite quadratic forms of signature* $(2, n-2)$, Math. Ann., **231** (1977/78), no. 2, 97–144.

[Sa] I. Satake, Algebraic structures of symmetric domains, Kan Memorial Lectures,

4. Iwanami Shoten, Tokyo; Princeton University Press, Princeton, N.J., 1980. xvi+321 pp.

[Sc] F. Scattone, *On the compactification of moduli spaces for algebraic $K3$ surfaces*, Mem. Amer. Math. Soc., **70** (1987), no. 374, x+86 pp.

[Sd] W. Schmid, *Variation of Hodge structure: the singularities of the period mapping*, Invent. Math., **22** (1973), 211–319.

[Se] J-P. Serre, A course in arithmetic, Translated from the French. Graduate Texts in Mathematics, No. **7**. Springer-Verlag, New York-Heidelberg, 1973, viii+115 pp.

[U1] K. Ueno, *Classification of algebraic varieties. I*, Compositio Math., **27** (1973), 277–342.

[U2] K. Ueno, *On algebraic fibre spaces of abelian varieties*, Math. Ann., **237** (1978), no. 1, 1–22.

[V1] E. Viehweg, *Canonical divisors and the additivity of the Kodaira dimension for morphisms of relative dimension one*, Compositio Math., **35** (1977), no. 2, 197–223.

[V2] E. Viehweg, *Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces*, Algebraic varieties and analytic varieties, (Tokyo, 1981), 329–353, Adv. Stud. Pure Math., **1**, North-Holland, Amsterdam-New York, 1983.

Graduate School of Mathematics
Nagoya University
Chikusa-ku
Nagoya, 464-8602
Japan
fujino@math.nagoya-u.ac.jp

Current address:
Institute for Advanced Study
Einstein Drive
Princeton, NJ 08540
USA
fujino@math.ias.edu