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L^p EXTENSION OF HOLOMORPHIC FUNCTIONS FROM SUBMANIFOLDS TO STRICTLY PSEUDOCONVEX DOMAINS WITH NON-SMOOTH BOUNDARY

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Abstract. Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ (with not necessarily smooth boundary) and let X be a submanifold in a neighborhood of $\overline{D}$. Then any L^p $(1 \leq p < \infty)$ holomorphic function in $X \cap D$ can be extended to an L^p holomorphic function in D.

§1. Introduction

Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary and let X be a submanifold in a neighborhood of $\overline{D}$ which intersects ∂D transversally. Then Henkin [4] proved that any bounded holomorphic function f in $X \cap D$ can be extended to a bounded holomorphic function F in D. Moreover, he proved that if f is holomorphic in $X \cap D$ and continuous on $\overline{X \cap D}$, then F is holomorphic in D and continuous on $\overline{D}$. Henkin-Leiterer [5] obtained the above results in the case when D is a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with non-smooth boundary, without assuming that the submanifold X and ∂D intersect transversally. On the other hand, Beatrous [1] and Cumenge [3] obtained L^p extensions of holomorphic functions from a submanifold $X \cap D$ of a bounded strictly pseudoconvex domain D in $\mathbb{C}^n$ with smooth boundary under the hypothesis that the submanifold X and ∂D intersect transversally. Using a quite different method, Ohsawa-Takegoshi [6] have done the remarkable discovery concerning L^2 extensions. They obtained the L^2 extension of holomorphic functions from the intersection of a complex hyperplane and a bounded pseudoconvex domain which involves weight functions. In their theorem

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the transversality is not assumed. When $p > 2$, Cho [2] gave a counter-example in some pseudoconvex domain such that the L^p extension does not hold. In this paper, we show that any L^p $(1 \le p < \infty)$ holomorphic function in $X \cap D$ can be extended to an L^p holomorphic function in D in the case when D is a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with non-smooth boundary, without assuming that the submanifold X and ∂D intersect transversally. The proof is based on the estimates of the integral formula for holomorphic functions in $X \cap D$ which was used to prove the bounded and continuous extension of holomorphic functions by Henkin-Leiterer [5]. We also use the estimate of the volume form by means of local coordinates in a neighborhood of a singular points of $X \cap \partial D$ obtained by Schmalz [7].

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§2. Preliminaries

Let $D \Subset \mathbb{C}^n$ be a strictly pseudoconvex open set and let $\theta \Subset \mathbb{C}^n$ be a neighborhood of ∂D, and let ρ be a strictly plurisubharmonic C^2 function in a neighborhood of $\bar{\theta}$ such that

$$D \cap \theta = \{z \in \theta : \rho(z) < 0\}.$$

Let $N(\rho) = \{z \in \bar{\theta} : \rho(z) = 0\}$, and assume that $N(\rho) \Subset \theta$. By Henkin-Leiterer [4], we can choose numbers $\varepsilon, \beta > 0$ and C^1 functions a_{jk} on $\bar{\theta}$ such that the following estimates hold:

$$\operatorname{dist}(N(\rho), \partial\theta) > 2\varepsilon,$$

$$\inf_{\zeta \in \bar{\theta}} \sum_{j,k=1}^{n} \frac{\partial^2 \rho(\zeta)}{\partial \zeta_j \partial \bar{\zeta}_k} \xi_j \bar{\xi}_k > 3\beta |\xi|^2 \quad \text{for all } 0 \ne \xi \in \mathbb{C}^n,$$

$$\sup_{\zeta \in \bar{\theta}} \left| \frac{\partial^2 \rho(\zeta)}{\partial \zeta_j \partial \zeta_k} - a_{jk}(\zeta) \right| < \frac{\beta}{n^2},$$

$$\left| \frac{\partial^2 \rho(\zeta)}{\partial x_j \partial x_k} - \frac{\partial^2 \rho(z)}{\partial x_j \partial x_k} \right| < \frac{\beta}{2n^2} \quad \text{for } \zeta, z \in \bar{\theta} \text{ with } |\zeta - z| \le 2\varepsilon,$$

where $\zeta_j = x_j + i x_{j+n}$. We define

$$F(z, \zeta) = 2 \sum_{j=1}^{n} \frac{\partial \rho(\zeta)}{\partial \zeta_j} (\zeta_j - z_j) - \sum_{j,k=1}^{n} a_{jk}(\zeta)(\zeta_j - z_j)(\zeta_k - z_k).$$

Then, by Henkin-Leiterer [5] there exist $\varepsilon > 0$ and $c > 0$ such that

$$\operatorname{Re} F(z,\zeta) \geq \rho(\zeta) - \rho(z) + c|\zeta - z|^2 \quad (\zeta, z \in \overline{\theta}, |\zeta - z| \leq 2\varepsilon).$$

Moreover, Henkin-Leiterer [5] proved the following:

THEOREM 1. *There exist a neighborhood $U \Subset \theta$ of $N(\rho)$ and C^1 functions $\Phi(z,\zeta)$, $\widetilde{\Phi}(z,\zeta)$, $M(z,\zeta)$ and $\widetilde{M}(z,\zeta)$ for $\zeta \in U$ and $z \in U \cup D$ such that the following conditions are fulfilled:*

(i) *$\Phi(z,\zeta)$ and $\widetilde{\Phi}(z,\zeta)$ depends holomorphically on $z \in U \cup D$.*
(ii) *$\Phi(z,\zeta) \neq 0$ and $\widetilde{\Phi}(z,\zeta) \neq 0$ for $\zeta \in U$, $z \in U \cup D$ with $|\zeta - z| \geq \varepsilon$.*
(iii) *$M(z,\zeta) \neq 0$ and $\widetilde{M}(z,\zeta) \neq 0$ for $\zeta \in U$, $z \in U \cup D$.*
(iv) *$\Phi(z,\zeta) = F(z,\zeta)M(z,\zeta)$ and $\widetilde{\Phi}(z,\zeta) = (F(z,\zeta) - 2\rho(\zeta))\widetilde{M}(z,\zeta)$ for $\zeta \in U$, $z \in U \cup D$ with $|\zeta - z| \leq \varepsilon$.*
(v) *Let V_1, V_0 be neighborhoods of $N(\rho)$ such that $V_0 \cup D$ is a strictly pseudoconvex open set and $V_1 \Subset V_0 \Subset U$. Then there exist the C^1 map $w(z,\zeta) = (w_1(z,\zeta), \ldots, w_n(z,\zeta))$ for $\zeta \in V_0$, $z \in V_0 \cup D$ with the following properties:*

(a)

$$\langle w(z,\zeta), \zeta - z\rangle = \Phi(z,\zeta) \quad (\zeta \in V_0, z \in V_0 \cup D).$$

(b) *We choose a neighborhood V_2 of $N(\rho)$ such that $V_2 \Subset V_1$ and a C^∞ function χ on $\mathbb{C}^n$ such that*

$$\chi = 0 \quad \textit{on} \quad \mathbb{C}^n \backslash V_1 \quad \textit{and} \quad \chi = 1 \quad \textit{on} \quad V_2.$$

Then there exist constants $\alpha > 0$ and $c < \infty$ such that

$$|\widetilde{\Phi}(z,\zeta)| \geq \alpha\big(|\rho(\zeta)| + |\rho(z)| + |\operatorname{Im} F(z,\zeta)| + |\zeta - z|^2\big) \quad \textit{for} \;\; z, \zeta \in V_2 \cap D.$$

$$|w(z,\zeta)| \leq c(\|d\rho(\zeta)\| + |\zeta - z|) \quad \textit{for} \;\; \zeta, z \in V_2.$$

$$\left|\frac{\partial\widetilde{\Phi}(z,\zeta)}{\partial\bar{\zeta}_j}\right| \leq c\left(\left|\frac{\partial\rho(\zeta)}{\partial\bar{\zeta}_j}\right| + |\zeta - z| + |\rho(\zeta)|\right) \quad \textit{for} \;\; \zeta, z \in V_2, \; j = 1, \ldots, n.$$

§3. L^p extension

We define

$$\zeta' = (\zeta_1, \ldots, \zeta_{n-1}), \quad (w(z,\zeta))' = (w_1(z,\zeta), \ldots, w_{n-1}(z,\zeta)),$$

$$\bar{\partial}_{\zeta'} = \sum_{j=1}^{n-1} \frac{\partial}{\partial \bar{\zeta}_j} d\bar{\zeta}_j, \quad \omega_{\zeta'}(\zeta) = d\zeta_1 \wedge \cdots \wedge d\zeta_{n-1},$$

$$\overline{\omega}_{\zeta'}\left(\frac{\chi(\zeta)(w(z,\zeta))'}{\widetilde{\Phi}(z,\zeta)}\right) = \bigwedge_{j=1}^{n-1} \bar{\partial}_{\zeta'}\left(\frac{\chi(\zeta) w_j(z,\zeta)}{\widetilde{\Phi}(z,\zeta)}\right).$$

Let $X = \{z \in \mathbb{C}^n : z_n = 0\}$. We denote by dV and dV' the volume form on $\mathbb{C}^n$ and $\mathbb{C}^{n-1}$, respectively. For an L^p holomorphic function f in $D \cap X$ ($p \geq 1$) and $z \in D$, we define

$$Ef(z) = \frac{(n-1)!}{(2\pi i)^{n-1}} \int_{D\cap X} f(\zeta)\overline{\omega}_{\zeta'}\left(\frac{\chi(\zeta)(w(z,\zeta))'}{\widetilde{\Phi}(z,\zeta)}\right) \wedge \omega_{\zeta'}(\zeta). \tag{3.1}$$

Then Ef is holomorphic in D and satisfies $Ef|_{D\cap X} = f$.

Using Schmalz [7], we have the following lemma:

LEMMA 1. *Let $t(z,\zeta) = \operatorname{Im}\langle w(z,\zeta), \zeta - z\rangle$. We set $\zeta_j = \xi_j + i\xi_{j+n}$, $z_j = \eta_j + i\eta_{j+n}$ and $E_\delta(z) = \{\zeta \in D : |\zeta - z| < \delta \|d\rho(z)\|\}$ for all $\delta > 0$. Then there are constants $c < \infty$, $\gamma > 0$, and numbers $\mu, \nu \in \{1, \ldots, 2n\}$ such that, $\{\rho, t(z,\zeta), \xi_1, \ldots, \hat{\mu}, \hat{\nu}, \ldots, \xi_{2n}\}$ (ξ_μ and ξ_ν have to be omitted) forms coordinates system in $E_\gamma(z)$ ($\{\rho, t(z,\zeta), \eta_1, \ldots, \hat{\mu}, \hat{\nu}, \ldots, \eta_{2n}\}$ forms coordinates system in $E_\gamma(\zeta)$, respectively) and we have the estimates*

$$dV \leq \frac{c}{\|d\rho(z)\|^2} |d\rho(\zeta) \wedge d_\zeta t(z,\zeta) \wedge \ldots, \hat{\mu}, \hat{\nu}, \cdots \wedge d\xi_{2n}| \quad \text{on } E_\gamma(z)$$

$$dV \leq \frac{c}{\|d\rho(\zeta)\|^2} |d\rho(z) \wedge d_z t(z,\zeta) \wedge \ldots, \hat{\mu}, \hat{\nu}, \cdots \wedge d\eta_{2n}| \quad \text{on } E_\gamma(\zeta),$$

where dV is the Euclidean volume form on $\mathbb{C}^n$.

Using Lemma 1, we prove the following theorem:

THEOREM 2. *Let X be a closed complex submanifold of some neighborhood of $\overline{D}$. Let f be an L^p holomorphic function in $D \cap X$ ($p \geq 1$). Then there exists an L^p holomorphic function F in D such that $F|_{D\cap X} = f$.*

Proof. We may assume $X = \{z_n = 0\}$. We set $\widetilde{U} = D \cap U$. The integral of the right hand side of (3.1) consists of the following two types integrals:

$$I_1(z) = \int_{X\cap\widetilde{U}} f(\zeta)\frac{G(z,\zeta)}{\widetilde{\Phi}(z,\zeta)^{n-1}}\,dV'(\zeta),$$

$$I_2(z) = \int_{X\cap\widetilde{U}} f(\zeta)G(z,\zeta)\frac{w_j(z,\zeta)\frac{\partial}{\partial\bar{\zeta}_\nu}\widetilde{\Phi}(z,\zeta)}{\widetilde{\Phi}(z,\zeta)^{n}}\,dV'(\zeta),$$

where $G(z,\zeta)$ is a smooth function in $\overline{D}\times\overline{D}$. At first we prove the theorem in the case when $p=1$. Using Fubini's theorem, we have

$$\begin{aligned}\int_D |I_1(z)|\,dV(z) &\lesssim \int_{X\cap\widetilde{U}} |f(\zeta)|\left\{\int_D \frac{1}{|\widetilde{\Phi}(z,\zeta)|^{n-1}}\,dV(z)\right\}dV'(\zeta)\\ &\lesssim \int_{X\cap\widetilde{U}} |f(\zeta)|\left\{\int_{|\zeta-z|\le M} \frac{1}{(|\zeta-z|^2)^{n-1}}\,dV(z)\right\}dV'(\zeta)\\ &\lesssim \int_{X\cap\widetilde{U}} |f(\zeta)|\,dV'(\zeta).\end{aligned}$$

Using the inequality

$$|w_j(z,\zeta)|\left|\frac{\partial\widetilde{\Phi}(z,\zeta)}{\partial\bar{\zeta}_\nu}\right| \lesssim \big(\|d\rho(\zeta)\|^2 + |\zeta-z| + |\rho(\zeta)|\big),$$

we have

$$\begin{aligned}&\int_D |I_2(z)|\,dV(z)\\ &\quad\lesssim \int_{X\cap\widetilde{U}} |f(\zeta)|\left(\int_D \frac{\|d\rho(\zeta)\|^2 + |\zeta-z| + |\rho(\zeta)|}{|\widetilde{\Phi}(z,\zeta)|^n}\,dV(z)\right)dV'(\zeta).\end{aligned}$$

In view of Lemma 1, if we set $t'=(t_3,\dots,t_{2n})$, we obtain

$$\begin{aligned}&\int_D \frac{\|d\rho(\zeta)\|^2}{|\widetilde{\Phi}(z,\zeta)|^n}\,dV(z)\\ &\quad= \int_{z\in E_\gamma(\zeta)} \frac{\|d\rho(\zeta)\|^2}{|\widetilde{\Phi}(z,\zeta)|^n}\,dV(z) + \int_{z\notin E_\gamma(\zeta)} \frac{\|d\rho(\zeta)\|^2}{|\widetilde{\Phi}(z,\zeta)|^n}\,dV(z)\\ &\quad\lesssim \int_{|t|\le M} \frac{dt_1dt_2dt'}{(|t_1|+|t_2|+|t'|^2)^n} + \int_{z\notin E_\gamma(\zeta)} \frac{|\zeta-z|^2}{|\widetilde{\Phi}(z,\zeta)|^n}\,dV(z)\\ &\quad\lesssim \int_0^M \frac{r^{2n-3}}{(r^2)^{n-2}}\,dr \lesssim 1.\end{aligned}$$

The other cases are similar. Thus we have

$$\int_D |I_2(z)|\,dV(z) \lesssim \int_{X\cap\widetilde{U}} |f(\zeta)|\,dV'(\zeta),$$

which completes the proof when $p = 1$. Next we assume $1 < p < \infty$. Let q be a positive number such that $p^{-1} + q^{-1} = 1$. We choose $\varepsilon > 0$ so small that $2\varepsilon p < 1$ and $2\varepsilon q < 1$. Using Hölder's inequality, we have

$$\begin{aligned}|I_1(z)|^p &\lesssim \Big(\int_{X\cap\widetilde{U}} \frac{|f(\zeta)|^p}{|\widetilde{\Phi}(z,\zeta)|^{n-1+\varepsilon p}}\,dV'(\zeta)\Big)\Big(\int_{X\cap\widetilde{U}} \frac{dV'(\zeta)}{|\widetilde{\Phi}(z,\zeta)|^{n-1-\varepsilon q}}\Big)^{p/q}\\ &\lesssim \int_{X\cap\widetilde{U}} \frac{|f(\zeta)|^p}{|\widetilde{\Phi}(z,\zeta)|^{n-1+\varepsilon p}}\,dV'(\zeta).\end{aligned}$$

Thus we have

$$\begin{aligned}\int_D |I_1(z)|^p\,dV(z) &\lesssim \int_{X\cap\widetilde{U}} |f(\zeta)|^p\Big(\int_D \frac{dV(z)}{|\widetilde{\Phi}(z,\zeta)|^{n-1+\varepsilon p}}\Big)\,dV'(\zeta)\\ &\lesssim \int_{X\cap\widetilde{U}} |f(\zeta)|^p\,dV'(\zeta).\end{aligned}$$

Next we estimate $I_2(z)$. It is sufficient to prove that the following $I_2^1(z)$, $I_2^2(z)$ and $I_2^3(z)$ belong to $L^p(D)$:

$$\begin{aligned}I_2^1(z) &= \int_{X\cap\widetilde{U}} \frac{|f(\zeta)|\|d\rho(\zeta)\|^2}{|\widetilde{\Phi}(z,\zeta)|^n}\,dV'(\zeta),\\ I_2^2(z) &= \int_{X\cap\widetilde{U}} \frac{|f(\zeta)|\|d\rho(\zeta)\||\zeta - z|}{|\widetilde{\Phi}(z,\zeta)|^n}\,dV'(\zeta),\\ I_2^3(z) &= \int_{X\cap\widetilde{U}} \frac{|f(\zeta)|\|d\rho(\zeta)\||\rho(\zeta)|}{|\widetilde{\Phi}(z,\zeta)|^n}\,dV'(\zeta).\end{aligned}$$

We prove that $I_2^1(z)$ belongs to $L^p(D)$. The other cases are similar. Using Hölder's inequality

$$\begin{aligned}I_2^1(z)^p \le \Big(\int_{X\cap\widetilde{U}} |f(\zeta)|^p \frac{\|d\rho(\zeta)\|^2}{|\widetilde{\Phi}(z,\zeta)|^{n+\varepsilon p}}\,dV'(\zeta)\Big)&\\ \times\Big(\int_{X\cap\widetilde{U}} \frac{\|d\rho(\zeta)\|^2}{|\widetilde{\Phi}(z,\zeta)|^{n-\varepsilon q}}\,dV'(\zeta)\Big)^{p/q}&.\end{aligned}$$

We set $\zeta' = (\zeta_1, \dots, \zeta_{n-1})$, $z' = (z_1, \dots, z_{n-1})$. Then we have

$$\int_{X\cap\widetilde{U}} \frac{\|d\rho(\zeta)\|^2}{|\widetilde{\Phi}(z,\zeta)|^{n-\varepsilon q}}\, dV'(\zeta) = \int_{\zeta'\in E_\gamma(z')} \frac{\|d\rho(\zeta)\|^2}{|\widetilde{\Phi}(z,\zeta)|^{n-\varepsilon q}}\, dV'(\zeta) + \int_{\zeta'\notin E_\gamma(z')} \frac{\|d\rho(\zeta)\|^2}{|\widetilde{\Phi}(z,\zeta)|^{n-\varepsilon q}}\, dV'(\zeta).$$

In view of Lemma 1, if we set $t' = (t_3, \dots, t_{2n-2})$, then there exists a positive constant M such that

$$\int_{\zeta'\in E_\gamma(z')} \frac{\|d\rho(\zeta)\|^2}{|\widetilde{\Phi}(z,\zeta)|^{n-\varepsilon q}}\, dV'(\zeta) \lesssim \int_{|t|\le M} \frac{dt_1 dt_2 dt'}{(|t_1| + |t_2| + |t'|^2)^{n-\varepsilon q}} \lesssim \int_0^M \frac{dr}{r^{1-2\varepsilon q}} \lesssim 1.$$

$$\int_{\zeta'\notin E_\gamma(z')} \frac{\|d\rho(\zeta)\|^2}{|\widetilde{\Phi}(z,\zeta)|^{n-\varepsilon q}}\, dV'(\zeta) \lesssim \int_{X\cap\tilde{U}} \frac{|\zeta' - z'|^2}{|\widetilde{\Phi}(z,\zeta)|^{n-\varepsilon q}}\, dV'(\zeta) \lesssim \int_0^M \frac{dr}{r^{1-2\varepsilon q}} \lesssim 1.$$

By Fubini's theorem, we obtain

$$\int_D I_2^1(z)^p\, dV(z) \lesssim \int_{X\cap\widetilde{U}} |f(\zeta)|^p \left(\int_D \frac{\|d\rho(\zeta)\|^2}{|\widetilde{\Phi}(z,\zeta)|^{n+\varepsilon p}}\, dV(z) \right) dV'(\zeta).$$

Using the inequality

$$\|d\rho(\zeta)\| \lesssim \|d\rho(z)\| + |\zeta - z|,$$

it is sufficient to estimate the following two integrals $J_1(\zeta)$ and $J_2(\zeta)$:

$$J_1(\zeta) = \int_D \frac{\|d\rho(z)\|^2}{|\widetilde{\Phi}(z,\zeta)|^{n+\varepsilon p}}\, dV(z),$$

$$J_2(\zeta) = \int_D \frac{|\zeta - z|^2}{|\widetilde{\Phi}(z,\zeta)|^{n+\varepsilon p}}\, dV(z).$$

We estimate $J_1(\zeta)$. The other case is similar. In view of Lemma 1, we have

$$\begin{aligned} J_1(\zeta) &= \int_{z\in E_\gamma(\zeta)} \frac{\|d\rho(z)\|^2}{|\widetilde{\Phi}(z,\zeta)|^{n+\varepsilon p}}\, dV(z) + \int_{z\notin E_\gamma(\zeta)} \frac{\|d\rho(z)\|^2}{|\widetilde{\Phi}(z,\zeta)|^{n+\varepsilon p}}\, dV(z) \\ &\lesssim \int_{|t|\le M} \frac{dt_1 dt_2 dt'}{(|t_1| + |t_2| + |t'|^2)^{n+\varepsilon p}} + \int_D \frac{dV(z)}{(|\zeta - z|^2)^{n-1+\varepsilon p}} \\ &\lesssim \int_0^M r^{1-2\varepsilon p}\, dr \lesssim 1. \end{aligned}$$

Thus we have proved that

$$\int_D I_2^1(z)^p \, dV(z) \lesssim \int_{X\cap\widetilde{U}} |f(\zeta)|^p \, dV'(\zeta).$$

This completes the proof of Theorem 2.

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