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Nagoya Math. J.
Vol. 172 (2003), 59–102

ASYMPTOTIC EXPANSIONS OF DOUBLE ZETA-FUNCTIONS OF BARNES, OF SHINTANI, AND EISENSTEIN SERIES

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Abstract. The present paper contains three main results. The first is asymptotic expansions of Barnes double zeta-functions, and as a corollary, asymptotic expansions of holomorphic Eisenstein series follow. The second is asymptotic expansions of Shintani double zeta-functions, and the third is the analytic continuation of n-variable multiple zeta-functions (or generalized Euler-Zagier sums). The basic technique of proving those results is the method of using the Mellin-Barnes type of integrals.

§1. Introduction and statement of results

The double zeta-function of Barnes, introduced by Barnes [5], is defined by the meromorphic continuation of the series

$$\zeta_2(v;\beta,w)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\beta+m+nw)^{-v}, \tag{1.1}$$

where $\beta>0$ and w is a non-zero complex number with $|\arg w|<\pi$. Each term of the right-hand side should be understood as

$$\exp(-v\log(\beta+m+nw)),$$

where the logarithm takes the principal branch. The series (1.1) is convergent absolutely for $\Re v>2$, and its continuation is holomorphic with respect to v except for the poles at $v=1$ and $v=2$.

Let

$$\zeta(v)=\sum_{n=1}^{\infty}n^{-v},\quad \zeta(v,\beta)=\sum_{n=0}^{\infty}(n+\beta)^{-v}$$

Received June 12, 2000.
2000 Mathematics Subject Classification: 11M41.

be the Riemann zeta and the Hurwitz zeta-function (with the parameter $\beta > 0$), respectively, and put

$$\binom{v}{n} = \begin{cases} \dfrac{v(v-1)\cdots(v-n+1)}{n!} & \text{if } n \text{ is a positive integer,} \\ 1 & \text{if } n = 0. \end{cases}$$

Denote by $\mathbf{Z}$ and $\mathbf{C}$ the ring of rational integers and the complex number field, respectively. We fix a number θ_0 satisfying $0 < \theta_0 < \pi$ and put

$$\mathcal{W}_\infty = \{w \in \mathbf{C} \mid |w| \geq 1, \ |\arg w| \leq \theta_0\}$$

and

$$\mathcal{W}_0 = \{w \in \mathbf{C} \mid |w| \leq 1, \ w \neq 0, \ |\arg w| \leq \theta_0\}.$$

The first purpose of the present paper is to prove the following

THEOREM 1. *For any positive integer N, we have*

$$\begin{aligned} \zeta_2(v;\beta,w) &= \zeta(v,\beta) + \frac{\zeta(v-1)}{v-1} w^{1-v} \\ &\quad + \sum_{k=0}^{N-1} \binom{-v}{k} \zeta(-k,\beta)\zeta(v+k) w^{-v-k} \\ &\quad + O(|w|^{-\Re v - N}) \end{aligned} \tag{1.2}$$

in the region $\Re v > -N+1$ and $w \in \mathcal{W}_\infty$, and also

$$\begin{aligned} \zeta_2(v;\beta,w) &= \zeta(v,\beta) + \frac{\zeta(v-1,\beta)}{v-1} w^{-1} \\ &\quad + \sum_{k=0}^{N-1} \binom{-v}{k} \zeta(v+k,\beta)\zeta(-k) w^{k} \\ &\quad + O(|w|^{N}) \end{aligned} \tag{1.3}$$

in the region $\Re v > -N+1$ and $w \in \mathcal{W}_0$, where the implied constants in (1.2) *and* (1.3) *depend only on N, v, β and θ_0.*

The formulas (1.2) and (1.3) give the asymptotic expansions of $\zeta_2(v;\beta,w)$ with respect to w, when $|w| \to +\infty$ and $|w| \to 0$, respectively.

Remark 1. Some factor in the above formulas has a singularity when v is an integer, $-N+2 \leq v \leq 2$. When $v = 2$ or $v = 1$, the function $\zeta_2(v;\beta,w)$ indeed has a pole. The singularities appearing at the points $v = 0, -1, -2, \ldots, -N+2$ are cancelled with the zeros coming from the binomial coefficients, and the above formulas are valid in this sense. But actually, the values of $\zeta_2(v;\beta,w)$ at $v = 0, -1, -2, \ldots, -N+2$ can be written in a closed form. See Theorem 5 of [14].

Remark 2. In the proof we will see that, if β is restricted to $0 < \beta \leq 1$, then the implied constant in (1.2) is independent of β. This point is important in the proof of Corollary 2. On the other hand, even in the case $0 < \beta \leq 1$, the implied constant in (1.3) is not independent of β. An alternative asymptotic formula, whose error estimate is uniform in β, will be given in Section 6.

The expansion (1.3) is entirely new; while (1.2) was, in the special case $w > 0$, already proved in the author's former article [14](see also [15]). Moreover it is noted in [14] that it is possible to generalize the result to the case of complex w with $\Re w > 0$. However, it seems that the method given in [14][15] cannot be applied to the case $\Re w \leq 0$.

In [14], we introduced the generalized double zeta-function

$$\zeta_2((u,v);\beta,w) = \sum_{m=0}^{\infty}(\beta+m)^{-u}\sum_{n=0}^{\infty}(\beta+m+nw)^{-v}. \tag{1.4}$$

(Actually in [14] the summation with respect to n is from 1 to infinity, but here we modify as above.) We studied the properties of $\zeta_2((u,v);\beta,w)$ in [14] by a method similar to that developed in Katsurada and Matsumoto [10][11], and obtained the asymptotic expansion of $\zeta_2((u,v);\beta,w)$ when w is positive real and $w \to +\infty$. The papers [10][11] of Katsurada and Matsumoto are devoted to the study of certain mean values of Dirichlet L-functions and Hurwitz zeta-functions. Later, Katsurada [7][8] discovered a new method of using the Mellin-Barnes type of integrals, and by which he gave simpler proofs of the results in [10][11], in a more generalized form.

In the present paper, we will use Katsurada's new method, and consequently we will prove (1.2) and (1.3) for any complex w with $|\arg w| < \pi$. This is one important advantage of our present method; in fact, this is the reason why we can deduce from Theorem 1 the following asymptotic expansions of holomorphic Eisenstein series.

Let $\mathcal{H}$ be the upper half-plane, and $w \in \mathcal{H}$. Define

$$G_v(w) = \sum_{m\in\mathbf{Z}} \sum_{n\in\mathbf{Z}}{}' (m+nw)^{-v}, \tag{1.5}$$

where

$$(m+nw)^{-v} = \exp(-v\log(m+nw))$$

with $-\pi \le \arg(m+nw) < \pi$, and $\sum\sum'$ indicates that the term $m=n=0$ is excluded from the summation. The series (1.5) is convergent absolutely for $\Re v > 2$, and it can be easily seen that the series can be expressed by $\zeta_2(v;1,\pm w)$ and $\zeta(v)$ (see (7.3) below). This especially implies that $G_v(w)$ can be continued meromorphically to the whole complex v-plane. Define

$$\mathcal{W}_\infty^* = \{w \in \mathcal{H} \mid |w| \ge 1,\ \pi-\theta_0 \le \arg w \le \theta_0\}$$

and

$$\mathcal{W}_0^* = \{w \in \mathcal{H} \mid |w| \le 1,\ w \ne 0,\ \pi-\theta_0 \le \arg w \le \theta_0\}$$

for $\pi/2 < \theta_0 < \pi$. Applying Theorem 1, we obtain

Corollary 1. *Assume $\pi/2 < \theta_0 < \pi$. For any positive integer N, we have*

$$G_v(w) = (1+e^{\pi i v})\zeta(v) + O(|w|^{-\Re v - N}) \tag{1.6}$$

for $\Re v > -N+1$, $w \in \mathcal{W}_\infty^*$, and also

(1.7)
$$\begin{aligned} G_v(w) &= (1+w^{-v})(1+e^{\pi i v})\zeta(v) + (e^{\pi i v} - e^{-\pi i v})\frac{\zeta(v-1)}{v-1}w^{-1} \\ &\quad - \left(1 + \frac{e^{\pi i v}+e^{-\pi i v}}{2}\right)\zeta(v) \\ &\quad + \sum_{\substack{1\le k\le N-1 \\ k:\mathrm{odd}}} (e^{\pi i v} - e^{-\pi i v})\binom{-v}{k}\zeta(v+k)\zeta(-k)w^k + O(|w|^N) \end{aligned}$$

for $\Re v > -N+1$, $w \in \mathcal{W}_0^*$, where the implied constants in (1.6) *and* (1.7) *depend only on N, v and θ_0.*

When $v = 2l$ is an even integer, (1.7) reduces to

$$G_{2l}(w) = 2w^{-2l}\zeta(2l) + O(|w|^N). \tag{1.8}$$

In particular, $G_{2l}(w)$ is a modular form for $l \geq 2$, and in this case (1.6)(with $v = 2l$) and (1.8) are immediate consequences of the well-known Fourier expansion of $G_{2l}(w)$, and they agree with the modular relation $G_{2l}(-w^{-1}) = w^{2l}G_{2l}(w)$. When v is not an even integer, then $G_v(w)$ is not modular. The existence of the terms of order w^k $(-1 \leq k \leq N-1)$ on the right-hand side of (1.7) expresses how far is the property of $G_v(w)$ from the modularity.

From Section 2 to Section 6, we will develop the proof of Theorem 1. Section 7 will be devoted to the proof of Corollary 1.

The double gamma-function $\Gamma_2(\beta,(1,w))$ is defined by

$$\log\Big(\frac{\Gamma_2(\beta,(1,w))}{\rho_2(1,w)}\Big) = \zeta_2'(0;\beta,w),$$

where 'prime' denotes the differentiation with respect to v and

$$-\log\rho_2(1,w) = \lim_{\beta\to 0}\{\zeta_2'(0;\beta,w) + \log\beta\}.$$

From Theorem 1 we can deduce asymptotic expansions of $\log\Gamma_2(\beta,(1,w))$ with respect to w. Let $\psi(v) = (\Gamma'/\Gamma)(v)$ and γ be the Euler constant. We have

Corollary 2. *For any positive integer $N \geq 2$, we have*

$$\begin{aligned}\log\Gamma_2&(\beta,(1,w))\\ &= -\frac{1}{2}\beta\log w + \log\Gamma(\beta) + \frac{1}{2}\beta\log 2\pi\\ &\quad + (\zeta(-1,\beta)-\zeta(-1))w^{-1}\log w - (\zeta(-1,\beta)-\zeta(-1))\gamma w^{-1}\\ &\quad + \sum_{k=2}^{N-1}\frac{(-1)^k}{k}(\zeta(-k,\beta)-\zeta(-k))\zeta(k)w^{-k} + O(|w|^{-N})\end{aligned} \tag{1.9}$$

for $w \in \mathcal{W}_\infty$ with the implied constant depending only on N, β and θ_0, and

$$\begin{aligned}\log\Gamma_2&(\beta,(1,w))\\ &= \log\Gamma(\beta w^{-1}+1) + \frac{1}{2}\log\Gamma(\beta+1) - \log\beta + \beta w^{-1}\log w\\ &\quad + \left\{\zeta(-1)+\zeta'(-1)-\zeta_1(-1,\beta)-\zeta_1'(-1,\beta)\right\}w^{-1}\end{aligned} \tag{1.10}$$

$$- \frac{1}{12}(\gamma + \psi(\beta+1))w$$
$$- \sum_{k=2}^{N-1} \frac{(-1)^k}{k}(\zeta(k) - \zeta_1(k,\beta))\zeta(-k)w^k + O(|w|^N)$$

for $w \in \mathcal{W}_0$ with the implied constant depending only on N and θ_0.

In the present paper we do not give the detailed proof of Corollary 2, though it is not difficult for the readers to reconstruct it. The details and related remarks will be discussed in a forthcoming article.

Our second major purpose of the present paper is to prove the asymptotic expansions of Shintani double zeta-functions.

In the course of the proof of Theorem 1, we will introduce the double infinite series

$$\zeta_2((u,v);(\alpha,\beta),w) = \sum_{m=0}^{\infty}(\alpha+m)^{-u}\sum_{n=0}^{\infty}(\beta+m+nw)^{-v}, \tag{1.11}$$

which is a further generalization of (1.4); here, u and v are complex, $\beta \geq \alpha > 0$, and w is complex with $|\arg w| < \pi$. Later we will show that the series (1.11) is convergent absolutely for $\Re u > -\delta$ and $\Re v > 2+\delta$, where δ is any positive number, and it can be continued meromorphically to wider regions. We will apply Katsurada's method [7][8] of Mellin-Barnes type of integrals to $\zeta_2((u,v);(\alpha,\beta),w)$, and will prove the asymptotic expansions of it (Propositions 1, 2 and 3 and also (9.2) and (9.9) below).

The generalized form $\zeta_2((u,v);(\alpha,\beta),w)$ is actually not necessary if our aim is restricted to the proof of Theorem 1. However, the analytic properties of $\zeta_2((u,v);(\alpha,\beta),w)$ are essentially used in our proof of the following Theorems 2 and 3, which gives the asymptotic expansions of Shintani double zeta-functions.

Let $a > 0$, $b > 0$, $w_1 \geq 0$, $w_2 \geq 0$, and we assume that at least one of w_1 and w_2 is not zero. Put $A = (a,b)$ and $W = (w_1,w_2)$. Shintani [17][18] introduced the double zeta-function of the form

$$\zeta_{SH,2}(v;A,W) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\{(a+m+(b+n)w_1)(a+m+(b+n)w_2)\}^{-v}, \tag{1.12}$$

and used it in his study on Kronecker limit formulas for real quadratic fields. (Actually in [17] Shintani considered a more general situation; see also [19].) Here we introduce the following two-variable version of Shintani double zeta-functions:

$$\begin{aligned}(1.13)\quad & \zeta_{SH,2}((u,v);A,W)\\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a+m+(b+n)w_1)^{-u}(a+m+(b+n)w_2)^{-v}.\end{aligned}$$

Without loss of generality we may assume $w_1 \leq w_2$. Moreover, when $w_1 = w_2$, then clearly

$$\zeta_{SH,2}((u,v);A,(w_1,w_1)) = \zeta_2(u+v;a+bw_1,w_1), \tag{1.14}$$

hence everything is reduced to the theory of Barnes double zeta-functions. Therefore hereafter we assume $w_1 < w_2$.

The series (1.13) is convergent absolutely and uniformly in any compact subset of the region $\Re(u+v) > 2$ and $\Re v > 0$, because in this region

$$\begin{aligned}&|(a+m+(b+n)w_1)^{-u}(a+m+(b+n)w_2)^{-v}|\\ &\qquad\qquad \leq (a+m+(b+n)w_1)^{-\Re(u+v)}.\end{aligned}$$

We will apply the method of Mellin-Barnes type of integrals to (1.13), and continue it meromorphically to wider regions. The basic formula is (8.2) which will be proved in Section 8. Our main results are the asymptotic expansions with respect to the smaller variable w_1. Hence we assume $w_1 \neq 0$.

Throughout this paper, ε denotes an arbitrarily small positive number, not necessarily the same at each occurrence. By $(\mathbf{Z}/2)_{\leq 1}$ we denote the set of all integers and half-integers ≤ 1. We will prove the following

THEOREM 2. *Let M, N be positive integers satisfying $M \leq N$. Then we have*

$$\begin{aligned}(1.15)\quad & \zeta_{SH,2}(v;A,W)\\ &= \sum_{j=0}^{M-1}\binom{-v}{j}(w_2-w_1)^j w_1^{-2v-j}\Bigg\{-\frac{1}{1-2v-j}\zeta(2v-1,b)w_1\\ &\qquad + \sum_{k=0}^{N-1}\binom{-2v-j}{k}\zeta(2v+k,b)\zeta(-k,a)w_1^{-k} + O(w_1^{-N})\Bigg\}\\ &\quad + O\big((w_2-w_1)^{M-\varepsilon}w_1^{-2\Re v-M+1+\varepsilon}\big)\end{aligned}$$

for $\Re v > (1-M)/2+\varepsilon$, $v \notin (\mathbf{Z}/2)_{\leq 1}$ and $w_1 \geq 1$.

THEOREM 3. *Let M be a positive integer, $-M+\varepsilon < \Re v < M+1-\varepsilon$, $-\Re v+\varepsilon \notin \mathbf{Z}$, $v \notin \mathbf{Z} \cup (\mathbf{Z}/2)_{\leq 1}$, and N a positive integer satisfying*

$$N > \max\{\Re v + M - 1,\ -\Re v + M + 1\}.$$

Then we have

(1.16)

$$\begin{aligned}
&\zeta_{SH,2}(v;A,W)\\
&\quad= \frac{\Gamma(1-v)\Gamma(2v-1)}{\Gamma(v)}\zeta(2v-1,b)(w_2-w_1)^{1-2v}\\
&\qquad - \sum_{j=0}^{M-1}\binom{-v}{j}(w_2-w_1)^{-v-j}w_1^{-v+j}\\
&\qquad\qquad \times\left\{\frac{\Gamma(1-v-j)\Gamma(2v-1)}{\Gamma(v-j)}\zeta(2v-1,a)w_1^{2v-1}\right.\\
&\qquad\qquad\qquad + \sum_{k=0}^{N-1}\binom{-v+j}{k}\zeta(v+j-k,b)\zeta(v-j+k,a)w_1^{v-j+k}\\
&\qquad\qquad\qquad \left. + O(w_1^{\Re v-j+N})\right\}\\
&\qquad + \sum_{k=0}^{N-1} w_1^k J_{M,k}(v;A,W) + O\big((w_2-w_1)^{-\Re v-M+\varepsilon}w_1^{\Re v+M-1-\varepsilon}\big)
\end{aligned}$$

for $0 < w_1 \leq 1$, where

(1.17)

$$\begin{aligned}
J_{M,k}(v;A,W) = \frac{1}{2\pi i}\int_{-\Re v-M+\varepsilon-i\infty}^{-\Re v-M+\varepsilon+i\infty} &\frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}\binom{-2v-z}{k}\\
&\times \zeta(-z-k,b)\zeta(2v+z+k,a)(w_2-w_1)^z dz.
\end{aligned}$$

Theorems 2 and 3 may be regarded as the asymptotic expansions of the Shintani double zeta-function $\zeta_{SH,2}(v;A,W)$ with respect to w_1, when $w_1 \to \infty$ and $w_1 \to 0$, respectively. The proofs of them will be developed in Sections 8 to 11. An interesting feature of Theorem 3 is the appearance of the sum including $J_{M,k}(v;A,W)$; there is no corresponding sum in Theorem 2.

In the above formulas (1.15) and (1.16), the larger variable w_2 appears only in the form $w_2 - w_1$. Therefore, to consider the behaviour with respect to w_2 is almost the same thing as to consider the behaviour with respect to $w_2 - w_1$. The above expansions include powers of $w_2 - w_1$ at several places. In particular, $(w_2 - w_1)^z$ are included in the definition (1.17) of $J_{M,k}(v; A, W)$. How to treat those factors, and how to find the asymptotic behaviour with respect to w_2 or $w_2 - w_1$, will be discussed at the end of Section 11.

In the final Section 12, we will discuss another important aspect of our present method. So far we have only studied the double zeta-functions, but more general multiple zeta-functions of the form

$$\sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} (\alpha + m_1 w_1 + \cdots + m_n w_n)^{-v} \tag{1.18}$$

was already introduced for any $n \geq 1$ by Barnes [6]. Let $v_1, \ldots, v_n$, $w_1, \ldots, w_n$ be complex numbers with $|\arg w_j| < \pi$, $w_j \neq 0$ $(1 \leq j \leq n)$, and $\alpha_1, \ldots, \alpha_n$ be positive numbers. Here we introduce the following n-variable generalization of (1.18), which is at the same time a generalization of (1.11):

$$\begin{aligned} \text{(1.19)} \quad & \zeta_n((v_1, \ldots, v_n); (\alpha_1, \ldots, \alpha_n), (w_1, \ldots, w_n)) \\ &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} (\alpha_1 + m_1 w_1)^{-v_1} (\alpha_2 + m_1 w_1 + m_2 w_2)^{-v_2} \\ &\qquad \times \cdots \times (\alpha_n + m_1 w_1 + m_2 w_2 + \cdots + m_n w_n)^{-v_n}. \end{aligned}$$

Under suitable conditions on $\arg w_j$, This series is convergent absolutely if $\Re v_j > 1$ for $1 \leq j \leq n$. Putting $w_j = 1$ for all j, we have

$$\begin{aligned} \text{(1.20)} \quad & \zeta_n((v_1, \ldots, v_n); (\alpha_1, \ldots, \alpha_n), (1, \ldots, 1)) \\ &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} (\alpha_1 + m_1)^{-v_1} (\alpha_2 + m_1 + m_2)^{-v_2} \\ &\qquad \times \cdots \times (\alpha_n + m_1 + m_2 + \cdots + m_n)^{-v_n}, \end{aligned}$$

which we abbreviate as $\zeta_n((v_1, \ldots, v_n); (\alpha_1, \ldots, \alpha_n))$. In (1.20) we further put $\alpha_j = j$ $(1 \leq j \leq n)$. Then

$$\begin{aligned} \text{(1.21)} \quad & \zeta_n((v_1, \ldots, v_n); (1, 2, \ldots, n)) \\ &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} (1 + m_1)^{-v_1} (2 + m_1 + m_2)^{-v_2} \end{aligned}$$

$$\times \cdots \times (n + m_1 + m_2 + \cdots + m_n)^{-v_n}$$

$$= \sum_{1 \le m_1 < m_2 < \cdots < m_n} m_1^{-v_1} m_2^{-v_2} \dots m_n^{-v_n},$$

which is the well-known Euler-Zagier sum (see Zagier [23]). Hence our class of multiple zeta-functions (1.19) includes both Barnes multiple zeta-functions (1.18) and the Euler-Zagier sum (1.21) as special cases. Therefore the study of the behaviour of (1.19) is highly desirable. The full account of this problem will require (at least) one more separate paper, hence here we only consider the special case (1.20), and will prove the following

THEOREM 4. *If $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, then the multiple series* (1.20) *can be continued meromorphically to the whole $\mathbf{C}^n$ as a function of $v_1, \dots, v_n$.*

This result was first proved very recently by Akiyama and Ishikawa [2], by using the Euler-Maclaurin summation formula. Some history concerning this problem will be mentioned at the beginning of Section 12. In the present paper we will prove the above Theorem 4 by the method of Mellin-Barnes type of integrals. Moreover, many information on the analytic behaviour of $\zeta_n(v_1, \dots, v_n; \alpha_1, \dots, \alpha_n)$ can also be obtained in the course of the proof (see Proposition 6 in Section 12). Therefore, our method will provide a new basis of further researches on multiple zeta-functions. Some initial motivations of the present paper can be found in the concluding remarks at the end of the paper.

The author expresses his sincere gratitude to Professor Masanori Katsurada for important suggestions and stimulating discussions, and to Professor Makoto Ishibashi and the referee for useful comments.

§2. The basic integral formula for Barnes double zeta-functions

Let δ be any positive number. First of all we show

LEMMA 1. *The series* (1.11) *is convergent absolutely and uniformly in any compact subset of the region $\Re u > -\delta$, $\Re v > 2 + \delta$.*

Proof. We only consider the case $0 \le \arg w < \pi$, since the case $-\pi < \arg w \le 0$ can be treated similarly. It is easily seen that

$$|\beta + m + nw| \ge C(w)(\beta + m)$$

for any non-negative n, where

$$C(w)=\begin{cases}1 & \text{if } 0\le \arg w\le \pi/2,\\ \sin(\pi-\arg w) & \text{if } \pi/2<\arg w<\pi.\end{cases}$$

Hence

$$\left|\frac{\beta+m+nw}{\alpha+m}\right|\ge C(w)$$

because $\beta\ge\alpha$. Therefore,

$$\begin{aligned}(2.1)\qquad &\sum_{m=0}^{\infty}|(\alpha+m)^{-u}|\sum_{n=0}^{\infty}|(\beta+m+nw)^{-v}|\\
&\le e^{\pi|\Im v|}\sum_{m=0}^{\infty}(\alpha+m)^{-\Re u}\sum_{n=0}^{\infty}|\beta+m+nw|^{-\Re v}\\
&\le e^{\pi|\Im v|}\Bigg\{\alpha^{-\Re u}\sum_{n=0}^{\infty}|\beta+nw|^{-\Re v}\\
&\qquad+\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left|\frac{\alpha+m}{\beta+m+nw}\right|^{\delta}|\beta+m+nw|^{-\Re v+\delta}\Bigg\}\\
&\le e^{\pi|\Im v|}\Bigg\{\alpha^{-\Re u}\sum_{n=0}^{\infty}|\beta+nw|^{-\Re v}\\
&\qquad+C(w)^{-\delta}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}|\beta+m+nw|^{-\Re v+\delta}\Bigg\}.\end{aligned}$$

The convergence of the above double series can be discussed by the same method as the well-known argument on Eisenstein series. In fact, for any positive integer j, let $\Gamma(j)$ be the parallelogram with the vertices $\pm j\pm jw$, and

$$l(j)=\min\{|z|\mid z\in\Gamma(j)\}.$$

On $\Gamma(j)$ there are exactly $2j$ points of the form $m+nw$ with integers $m\ge 1$, $n\ge 0$. Also if $m+nw\in\Gamma(j)$, then $|\beta+m+nw|\ge l(j)$. Hence we have

$$\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}|\beta+m+nw|^{-\Re v+\delta}\le 2\sum_{j=1}^{\infty}jl(j)^{-\Re v+\delta}.$$

Since $l(j)=jl(1)$, we find that the above series is convergent when $\Re v>2+\delta$, which completes the proof of Lemma 1. ∎

In view of Lemma 1, at first we assume $\Re u > -\delta$ and $\Re v > 2+\delta$. Our starting point is the same as the starting point of Katsurada's argument in [7][8], that is the formula

$$\Gamma(v)(1+\lambda)^{-v} = \frac{1}{2\pi i}\int_{(c)} \Gamma(v+z)\Gamma(-z)\lambda^z dz \tag{2.2}$$

(Whittaker-Watson [22, Section 14.51, p.289, Corollary]), where v and λ are complex with $\Re v > 0$, $|\arg\lambda| < \pi$, $\lambda \neq 0$, and the path is the vertical line from $c - i\infty$ to $c + i\infty$. In [22] this formula is stated with $c = 0$ (with a suitable modification of the path near the point $z = 0$), but it is clear that the formula is also valid for $-\Re v < c < 0$. We put $\lambda = ((\beta-\alpha)+nw)/(m+\alpha)$ in (2.2) and divide the both sides by $\Gamma(v)(\alpha+m)^{u+v}$ to obtain

$$\begin{aligned}(\alpha+m)^{-u}&(\beta+m+nw)^{-v}\\ &= \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}(\alpha+m)^{-u-v-z}((\beta-\alpha)+nw)^z dz\end{aligned} \tag{2.3}$$

for any $m \geq 0$ and $n \geq 1$. Therefore, if we assume

$$\max\{-\Re v,\ 1-\Re(u+v)\} < c < -1, \tag{2.4}$$

and put

$$\xi((u,v);(\alpha,\beta)) = \sum_{m=0}^{\infty}(\alpha+m)^{-u}(\beta+m)^{-v}, \tag{2.5}$$

then from (1.11) and (2.3) we have

$$\begin{aligned}\zeta_2&((u,v);(\alpha,\beta),w)\\ &= \xi((u,v);(\alpha,\beta)) + \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}\\ &\qquad \times \sum_{m=0}^{\infty}(\alpha+m)^{-u-v-z}\sum_{n=1}^{\infty}((\beta-\alpha)+nw)^z dz\\ &= \xi((u,v);(\alpha,\beta)) + \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}\\ &\qquad \times \zeta(u+v+z,\alpha)\zeta_1\Big(-z, \frac{\beta-\alpha}{w}\Big)w^z dz\end{aligned} \tag{2.6}$$

for $\Re u > -\delta$ and $\Re v > 2+\delta$, where

$$\zeta_1(z,\beta) = \zeta(z,\beta) - \beta^{-z}.$$

To show the last equality in (2.6) we have used the fact

$$\arg((\beta-\alpha)+nw) = \arg w + \arg\Big(n + \frac{\beta-\alpha}{w}\Big). \tag{2.7}$$

This can be checked easily; for example, if $\arg w = \theta > 0$, then $0 < \arg((\beta-\alpha)+nw) \le \theta$ since $\beta \ge \alpha$. Also, since $\arg((\beta-\alpha)/w) = -\theta$ (or 0 if $\beta=\alpha$), we see that $-\theta < \arg(n+(\beta-\alpha)/w) \le 0$. Hence (2.7) follows.

§3. Shifting the path to the left

Hereafter, until the end of Section 6, we assume $|\arg w| \le \theta_0$.

Let N be a positive integer satisfying

$$N > \Re u - 1 + \varepsilon, \tag{3.1}$$

and put $c_N = -\Re v - N + \varepsilon$. Our first purpose in this section is to estimate the order of the integrand on the right-hand side of (2.6) in the strip $c_N \le \Re z \le c$.

Since $\Re(-z) \ge -c > 1$ in this region, we have

$$\begin{aligned}\Big|\zeta_1\Big(-z,\frac{\beta-\alpha}{w}\Big)w^z\Big| &\le \sum_{n=1}^{\infty}|(nw+\beta-\alpha)^z| \\ &= \sum_{n=1}^{\infty}|nw+\beta-\alpha|^x\exp(-y\arg(nw+\beta-\alpha)),\end{aligned}$$

where $x=\Re z$ and $y=\Im z$. If $|w|\ge 2(\beta-\alpha)$, then $|nw+\beta-\alpha| \ge n|w|/2$, and

$$|\arg(nw+\beta-\alpha)| \le |\arg(nw)| = |\arg w|.$$

Hence

$$\begin{aligned}\Big|\zeta_1\Big(-z,\frac{\beta-\alpha}{w}\Big)w^z\Big| &\le \sum_{n=1}^{\infty}\Big(\frac{n}{2}\Big)^x|w|^x\exp(|y\arg w|) \\ &\ll |w|^x\exp(|y\arg w|)\end{aligned} \tag{3.2}$$

for $w\in\mathcal{W}(\beta-\alpha)$, where

$$\mathcal{W}(\beta-\alpha) = \{w \mid |w| \ge 2(\beta-\alpha),\ |\arg w| \le \theta_0\}.$$

Next we show

Lemma 2. *Let x_1, x_2 be two real numbers with $x_1 < x_2$. In the strip $x_1 \le x \le x_2$, except for the points near $z = 1$, we have*

$$\zeta(z,\beta) = O\big((|y|+1)^{A(x)}\big) \tag{3.3}$$

for any $\beta > 0$, where $A(x)$ is a constant depending only on x, and the implied constant depends on β, x_1 and x_2. When $x_2 < 0$, we can take $A(x) = 1/2 - x$. Moreover, when $x_2 < 0$ and $0 < \beta \le 1$, the implied constant is independent of β.

Proof. In the case $0 < \beta \le 1$, the estimate (3.3) is shown in Whittaker-Watson [22, Sections 13.5 and 13.51]. One of the main tools of the proof given there is the formula

$$\begin{aligned}\zeta(z,\beta) = \frac{2\Gamma(1-z)}{(2\pi)^{1-z}}\bigg\{&\sin\Big(\frac{\pi z}{2}\Big)\sum_{m=1}^{\infty}\frac{\cos(2m\pi\beta)}{m^{1-z}} \\ &+ \cos\Big(\frac{\pi z}{2}\Big)\sum_{m=1}^{\infty}\frac{\sin(2m\pi\beta)}{m^{1-z}}\bigg\}\end{aligned} \tag{3.4}$$

(see Titchmarsh [21, (2.17.3)] or Whittaker-Watson [22, Section 13.15, p.269]), which is valid if $0 < \beta \le 1$ and $x < 0$. In particular, from (3.4) and Stirling's formula for the gamma-function it immediately follows that

$$\zeta(z,\beta) = O\big((|y|+1)^{\frac{1}{2}-x}\big)$$

for $x \le x_2 < 0$, and the implied constant here is independent of β. If $\beta > 1$, then we write $\beta = L + \beta_1$, where L is a positive integer and $0 < \beta_1 \le 1$. Then, in the region $x > 1$ we have

$$\zeta(z,\beta) = \zeta(z,\beta_1) - \sum_{j=0}^{L-1}(\beta_1 + j)^{-z}.$$

But this relation is valid for any z by analytic continuation. Applying the proved estimate to $\zeta(z,\beta_1)$, and noting

$$\Big|\sum_{j=0}^{L-1}(\beta_1 + j)^{-z}\Big| \le \sum_{j=0}^{L-1}(\beta_1 + j)^{-x}$$

which only depends on β, x_1 and x_2, we obtain the assertions of Lemma 2 for $\beta > 1$. ∎

Now we estimate the integrand on the right-hand side of (2.6) in the strip $c_N \le \Re z \le c$. By using (3.2), Lemma 2 and Stirling's formula, we get

$$
\begin{aligned}
(3.5)\qquad &\frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}\zeta(u+v+z,\alpha)\zeta_1\Big(-z,\frac{\beta-\alpha}{w}\Big)w^z\\
&\ll e^{-\frac{1}{2}\pi|y+\Im v|}(|y+\Im v|+1)^{\Re v+x-\frac{1}{2}}e^{-\frac{1}{2}\pi|y|}(|y|+1)^{-x-\frac{1}{2}}\\
&\quad\times(|y+\Im(u+v)|+1)^{A(x+\Re(u+v))}|w|^x\exp(\,|y\arg w|\,)\\
&\ll(|y|+1)^{\Re v-1+A(x+\Re(u+v))}\\
&\quad\times\exp\Big(-\frac{1}{2}\pi|y+\Im v|-\frac{1}{2}\pi|y|+|\arg w|\,|y|\Big)|w|^x
\end{aligned}
$$

for $w \in \mathcal{W}(\beta-\alpha)$, except for the singular points on the left-hand side. The implied constants in (3.5) depend on c_N, c, u, v, θ_0 and α. Note that if $\Re(u+v+z)<0$ and $0<\alpha\le 1$, then the implied constant is independent of α. Since $|\arg w|\le\theta_0<\pi$, the right-hand side of (3.5) tends to zero when $|y|\to\infty$. Therefore we may shift the path of integration on the right-hand side of (2.6) to $\Re z = c_N$.

Denote by $\rho(a)$ the residue of the integrand on the right-hand side of (2.6) at the pole $z=a$. We find that, if u is not a positive integer, then all the poles at $z=1-u-v$ and $z=-v-k$ (where k is any non-negative integer) are simple, and the residues are

$$
(3.6)\qquad \rho(1-u-v)=\frac{\Gamma(1-u)\Gamma(u+v-1)}{\Gamma(v)}\zeta_1\Big(u+v-1,\frac{\beta-\alpha}{w}\Big)w^{1-u-v}
$$

and

$$
(3.7)\qquad \rho(-v-k)=\binom{-v}{k}\zeta(u-k,\alpha)\zeta_1\Big(v+k,\frac{\beta-\alpha}{w}\Big)w^{-v-k},
$$

respectively. Therefore, shifting the path as indicated above, we obtain

$$
\begin{aligned}
(3.8)\qquad &\zeta_2((u,v);(\alpha,\beta),w)\\
&=\xi((u,v);(\alpha,\beta))\\
&\quad+\frac{\Gamma(1-u)\Gamma(u+v-1)}{\Gamma(v)}\zeta_1\Big(u+v-1,\frac{\beta-\alpha}{w}\Big)w^{1-u-v}\\
&\quad+\sum_{k=0}^{N-1}\binom{-v}{k}\zeta(u-k,\alpha)\zeta_1\Big(v+k,\frac{\beta-\alpha}{w}\Big)w^{-v-k}\\
&\quad+R_N((u,v);(\alpha,\beta),w),
\end{aligned}
$$

if $w \in \mathcal{W}(\beta-\alpha)$, $\Re u > -\delta$, $\Re v > 2+\delta$ and u is not a positive integer, where

$$(3.9)\qquad R_N((u,v);(\alpha,\beta),w) = \frac{1}{2\pi i}\int_{(c_N)} \frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)} \times \zeta(u+v+z,\alpha)\zeta_1\Big(-z,\frac{\beta-\alpha}{w}\Big)w^z dz.$$

§4. The function $\xi((u,v);(\alpha,\beta))$

In this section we study the meromorphic continuation of the function $\xi((u,v);\ (\alpha,\beta))$ defined by (2.5). At first assume $\Re u > -\delta$, $\Re v > 2+\delta$.

If $\alpha=\beta$, then

$$(4.1)\qquad \xi((u,v);(\alpha,\beta)) = \zeta(u+v,\alpha),$$

hence the continuation is obvious. Therefore hereafter in this section we assume $\alpha<\beta$. Then the formula (2.3) is valid for $n=0$, because we can put $\lambda=(\beta-\alpha)/(\alpha+m)$ in (2.2). Hence, under the assumption (2.4), we obtain

$$(4.2)\qquad \begin{aligned} &\xi((u,v);(\alpha,\beta)) \\ &= \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)} \sum_{m=0}^{\infty}(\alpha+m)^{-u-v-z}(\beta-\alpha)^z dz \\ &= \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}\zeta(u+v+z,\alpha)(\beta-\alpha)^z dz. \end{aligned}$$

Let N be the same as in Section 3, and we shift the path of integration on the right-hand side of (4.2) to the line $\Re z = c_N = -\Re v - N + \varepsilon$. Using Stirling's formula we can easily see that this shifting is possible. Assume that u is not a positive integer. Then the integrand has simple poles at $z=1-u-v$ and $z=-v-k$, where k is any non-negative integer. The residue at $z=1-u-v$ is

$$\frac{\Gamma(1-u)\Gamma(u+v-1)}{\Gamma(v)}(\beta-\alpha)^{1-u-v},$$

and the residue at $z=-v-k$ is

$$\binom{-v}{k}(\beta-\alpha)^{-v-k}\zeta(u-k,\alpha).$$

Therefore

$$\begin{aligned}\xi((u,v);(\alpha,\beta)) = {} & \frac{\Gamma(1-u)\Gamma(u+v-1)}{\Gamma(v)}(\beta-\alpha)^{1-u-v} \\ & + \sum_{k=0}^{N-1}\binom{-v}{k}(\beta-\alpha)^{-v-k}\zeta(u-k,\alpha) \\ & + \xi_N((u,v);(\alpha,\beta)),\end{aligned} \tag{4.3}$$

where

$$\xi_N((u,v);(\alpha,\beta)) = \frac{1}{2\pi i}\int_{(c_N)}\frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}\zeta(u+v+z,\alpha)(\beta-\alpha)^z dz. \tag{4.4}$$

Putting $z = z' - N - v$ in (4.4), we have

$$\begin{aligned}\xi_N((u,v);(\alpha,\beta)) = {} & \frac{1}{2\pi i}\int_{(\varepsilon)}\frac{\Gamma(z'-N)\Gamma(-z'+N+v)}{\Gamma(v)} \\ & \times \zeta(u+z'-N,\alpha)(\beta-\alpha)^{z'-N-v}dz'.\end{aligned} \tag{4.5}$$

The poles of the above integrand are at $z' = -u+N+1$, $z' = v+N+k$ ($k = 0,1,2,\dots$), and $z' = N-l$ ($l = 0,1,2,\dots$). Therefore, the integral (4.5) can be continued holomorphically to

$$\mathcal{C}(N;\varepsilon) = \{(u,v) \mid \Re u < N+1-\varepsilon, \Re v > -N+\varepsilon\},$$

because the above poles do not lie on the path of integration if $(u,v) \in \mathcal{C}(N;\varepsilon)$. This gives, via (4.3), the analytic continuation of $\xi((u,v);(\alpha,\beta))$ to the region

$$\mathcal{C}^*(N;\varepsilon) = \{(u,v) \in \mathcal{C}(N;\varepsilon);\ u \text{ is not a positive integer}\}.$$

§5. Completion of the proof of (1.2)

Putting $z = z' - N - v$ in (3.9), we have

$$\begin{aligned}R_N((u,v);(\alpha,\beta),w) = {} & \frac{1}{2\pi i}\int_{(\varepsilon)}\frac{\Gamma(-N+z')\Gamma(v+N-z')}{\Gamma(v)} \\ & \times \zeta(u-N+z',\alpha)\zeta_1\Big(v+N-z',\frac{\beta-\alpha}{w}\Big)w^{-v-N+z'}dz'.\end{aligned} \tag{5.1}$$

The poles of the integrand are at $z' = -u+N+1$, $z' = v+N+k$ ($k = -1,0,1,2,\dots$), and $z' = N-l$ ($l = 0,1,2,\dots$). Note that N satisfies

(3.1). Similarly to the case of $\xi((u,v);(\alpha,\beta))$ in the last section, we see that the integral (5.1) can be continued holomorphically to

$$\mathcal{D}(N;\varepsilon)=\{(u,v)\mid \Re u<N+1-\varepsilon,\ \Re v>-N+1+\varepsilon\}.$$

Let

$$\mathcal{D}^*(N;\varepsilon)=\{(u,v)\in\mathcal{D}(N;\varepsilon);\ u \text{ is not a positive integer}\}.$$

Since $\mathcal{D}^*(N;\varepsilon)\subset\mathcal{C}^*(N;\varepsilon)$, the result in Section 4 implies that $\xi((u,v);(\alpha,\beta))$ can also be continued to $\mathcal{D}^*(N;\varepsilon)$. Therefore, by (3.8), we now obtain the analytic continuation of $\zeta_2((u,v);(\alpha,\beta),w)$ to the region $\mathcal{D}^*(N;\varepsilon)$.

Next we estimate $R_N((u,v);(\alpha,\beta),w)$ in this region. Since the estimate (3.5) is also valid for $\Re z=c_N$ and $(u,v)\in\mathcal{D}^*(N;\varepsilon)$, from the expression (3.9) we obtain

$$R_N((u,v);(\alpha,\beta),w)=O(|w|^{-\Re v-N+\varepsilon}), \tag{5.2}$$

and if $\Re u<N-\varepsilon$ and $0<\alpha\le 1$, then the implied constant is independent of α. Therefore, we now find that the expansion (3.8) is valid for any $(u,v)\in\mathcal{D}^*(N;\varepsilon)$ and $w\in\mathcal{W}(\beta-\alpha)$, with the error estimate (5.2).

Consider (3.8) with $N+1$ instead of N, and compare it with the original (3.8). We find

$$\begin{aligned}R_N((u,v);(\alpha,\beta),w)=\binom{-v}{N}\zeta(u-N,\alpha)\zeta_1\Big(v+N,\frac{\beta-\alpha}{w}\Big)w^{-v-N}\\+R_{N+1}((u,v);(\alpha,\beta),w).\end{aligned} \tag{5.3}$$

The right-hand side is defined in $\mathcal{D}^*(N+1;\varepsilon)$, hence (5.3) gives the analytic continuation of R_N to $\mathcal{D}^*(N+1;\varepsilon)$. In particular, now R_N is defined in

$$\begin{aligned}\mathcal{D}^*(N)&=\mathcal{D}^*(N;0)\\&=\left\{(u,v)\ \middle|\ \begin{array}{l}\Re u<N+1,\Re v>-N+1,\\ u \text{ is not a positive integer}\end{array}\right\}.\end{aligned}$$

Since $\Re(v+N)>1$ in this region, similarly to (3.2) we see that

$$\zeta_1\Big(v+N,\frac{\beta-\alpha}{w}\Big)w^{-v-N}\ll|w|^{-\Re v-N}\exp(|\Im v|\theta_0)$$

for $w\in\mathcal{W}(\beta-\alpha)$. Therefore, if $|w|\ge 1$, from (5.3) we have

$$R_N((u,v);(\alpha,\beta),w)\ll|w|^{-\Re v-N}+|w|^{-\Re v-(N+1)+\varepsilon} \tag{5.4}$$

$$\ll |w|^{-\Re v - N}.$$

The implied constant depends only on u, v, N, θ_0 and α; if $\Re u < N$ and $0 < \alpha \leq 1$, then it is independent of α. We summarize the above as the following

PROPOSITION 1. *The asymptotic expansion* (3.8) *holds for any positive integer* N, *any* $w \in \mathcal{W}(\beta - \alpha) \cap \{|w| \geq 1\}$ *and any* $(u, v) \in \mathcal{D}^*(N)$, *and the error term* $R_N((u, v); (\alpha, \beta), w)$ *satisfies the estimate* (5.4).

In particular, putting $u = 0$ and $\alpha = \beta$ in the above proposition, we obtain (1.2).

The right-hand side of (3.8) includes the factors of the form $\zeta_1(v, (\beta - \alpha)/w)$. However, it is easy to expand these factors with respect to w, because the formula

$$\zeta_1(v, \beta) = \zeta(v) + \sum_{k=1}^{\infty} \binom{-v}{k} \zeta(v + k) \beta^k \tag{5.5}$$

was proved by Mikolás [16]. Therefore, Proposition 1 essentially gives the asymptotic expansion of $\zeta_2((u, v); (\alpha, \beta), w)$ with respect to w when $|w|$ tends to infinity.

§6. Shifting the path to the right (Proof of (1.3))

Now we return to the integral expression (2.6), and consider the shifting of the path to the right. It is Professor Masanori Katsurada who first suggested the possibility of finding the asymptotic behaviour of Barnes double zeta-functions when w tends to zero, by shifting the path to the right (in a private conversation with the author).

Let N be a positive integer, and we shift the path to $\Re z = N - \varepsilon$. To show the validity of this shifting, we should estimate the integrand in the strip $c \leq \Re z \leq N - \varepsilon$. The quantity $\zeta_1(-z, (\beta - \alpha)/w)$ is not easily handled in general, hence we restrict ourselves to the special case

$$\beta - \alpha = aw, \quad a \geq 0. \tag{6.1}$$

For our purpose in the present paper, it is sufficient to consider this special situation. Under the assumption (6.1), we have

$$\zeta_1\Big(-z, \frac{\beta - \alpha}{w}\Big) = \zeta_1(-z, a)$$

$$= \begin{cases} \zeta(-z) & \text{if } a = 0, \\ \zeta(-z,a) - a^z & \text{if } a > 0. \end{cases}$$

Therefore, using Lemma 2 and Stirling's formula, we obtain an estimate of the integrand which is similar to (3.5), and it implies the validity of the shifting.

The poles of the integrand on the right-hand side of (2.6) in the half-plane $\Re z > c$ are located at $z = k$ ($k = -1, 0, 1, 2, \ldots$), and the residue at $z = k$ is

$$-\binom{-v}{k}\zeta(u+v+k,\alpha)\zeta_1\Big(-k,\frac{\beta-\alpha}{w}\Big)w^k$$

for $k \geq 0$, while the residue at $z = -1$ is

$$\frac{1}{1-v}\zeta(u+v-1,\alpha)w^{-1}.$$

Therefore, after shifting the path, we obtain

$$\begin{aligned} (6.2)\qquad &\zeta_2((u,v);(\alpha,\beta),w) \\ &= \xi((u,v);(\alpha,\beta)) - \frac{1}{1-v}\zeta(u+v-1,\alpha)w^{-1} \\ &\quad + \sum_{k=0}^{N-1}\binom{-v}{k}\zeta(u+v+k,\alpha)\zeta_1\Big(-k,\frac{\beta-\alpha}{w}\Big)w^k \\ &\quad + S_N((u,v);(\alpha,\beta),w), \end{aligned}$$

where

$$\begin{aligned} (6.3)\qquad &S_N((u,v);(\alpha,\beta),w) \\ &= \frac{1}{2\pi i}\int_{(N-\varepsilon)}\frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}\zeta(u+v+z,\alpha)\zeta_1\Big(-z,\frac{\beta-\alpha}{w}\Big)w^z dz. \end{aligned}$$

The integral (6.3) can be holomorphically continued to

$$\mathcal{F}(N;\varepsilon) = \{(u,v) \mid \Re v > -N+\varepsilon,\ \Re(u+v) > 1-N+\varepsilon\},$$

hence (6.2) gives the analytic continuation of $\zeta_2((u,v);(\alpha,\beta),w)$ to $\mathcal{F}(N;\varepsilon)$. The estimate

$$(6.4)\qquad S_N((u,v);(\alpha,\beta),w) = O(|w|^{N-\varepsilon})$$

can be shown similarly to (5.2). The implied constant depends on α, while if $0 \le a \le 1$, then it is independent of a. Moreover, if $|w| \le 1$, then similarly to the argument in the last section, we obtain that $S_N((u,v);(\alpha,\beta),w)$ can be further continued to $\mathcal{F}(N) = \mathcal{F}(N;0)$, and

$$S_N((u,v);(\alpha,\beta),w) = O(|w|^N) \tag{6.5}$$

holds in this region. Hence we now arrive at the following

Proposition 2. *Under the assumption* (6.1)*, the asymptotic expansion* (6.2) *holds for any positive integer N, any $w \in \mathcal{W}_0$ and any $(u,v) \in \mathcal{F}(N)$, and the error term $S_N((u,v);(\alpha,\beta),w)$ satisfies the estimate* (6.5)*.*

The formula (1.3) is the special case $u = 0$, $\alpha = \beta$ (hence $a = 0$) of this proposition.

Under the asssumption (6.1), the factor $\zeta_1(-k,(\beta-\alpha)/w)$ does not include w, hence the formula (6.2) gives the asymptotic expansion of $\zeta_2((u,v);(\alpha,\beta),w)$ with respect to w when $|w|$ tends to zero.

Sometimes it is useful to obtain an error estimate which is uniform in α. For this purpose, we separate from (2.6) the terms corresponding to $m = 0$. Those terms contribute to $\zeta_2((u,v);(\alpha,\beta),w)$ as

$$\sum_{n=1}^{\infty} \alpha^{-u}(\beta+nw)^{-v} = \alpha^{-u}w^{-v}\zeta_1\Big(v,\frac{\beta}{w}\Big),$$

hence we can modify (2.6) to

$$\begin{aligned} \zeta_2&((u,v);(\alpha,\beta),w) \\ &= \xi((u,v);(\alpha,\beta)) + \alpha^{-u}w^{-v}\zeta_1\Big(v,\frac{\beta}{w}\Big) \\ &\quad + \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}\zeta_1(u+v+z,\alpha)\zeta_1\Big(-z,\frac{\beta-\alpha}{w}\Big)w^z dz. \end{aligned} \tag{6.6}$$

Shifting the path to $\Re z = N - \varepsilon$ as in the proof of Proposition 2, we obtain

(6.7)

$$\begin{aligned} &\zeta_2((u,v);(\alpha,\beta),w) \\ &\quad = \xi((u,v);(\alpha,\beta)) + \alpha^{-u}w^{-v}\zeta_1\Big(v,\frac{\beta}{w}\Big) - \frac{1}{1-v}\zeta_1(u+v-1,\alpha)w^{-1} \end{aligned}$$

$$+\sum_{k=0}^{N-1}\binom{-v}{k}\zeta_1(u+v+k,\alpha)\zeta_1\Big(-k,\frac{\beta-\alpha}{w}\Big)w^k$$
$$+S_{1,N}((u,v);(\alpha,\beta),w),$$

under the assumption (6.1), where

$$\begin{aligned}(6.8)\quad & S_{1,N}((u,v);(\alpha,\beta),w)\\ &=\frac{1}{2\pi i}\int_{(N-\varepsilon)}\frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}\zeta_1(u+v+z,\alpha)\zeta_1\Big(-z,\frac{\beta-\alpha}{w}\Big)w^z dz.\end{aligned}$$

Again similarly as above, $S_{1,N}((u,v);(\alpha,\beta),w)$ can be continued to $\mathcal{F}(N)$ and satisfies the estimate

$$S_{1,N}((u,v);(\alpha,\beta),w)=O(|w|^N) \tag{6.9}$$

in this region. Therefore we obtain

Proposition 3. *Under the assumption* (6.1), *the asymptotic expansion* (6.7) *holds for any positive integer N, any $w\in\mathcal{W}_0$ and any $(u,v)\in\mathcal{F}(N)$, and the error term $S_{1,N}((u,v);(\alpha,\beta),w)$ satisfies the estimate* (6.9).

The important point here is that $S_{1,N}$ includes the factor $\zeta_1(u+v+z,\alpha)$ instead of $\zeta(u+v+z,\alpha)$, hence the implied constant in (6.9) does not depend on α.

§7. Eisenstein series

In this section we prove Corollary 1. We begin with the expression (1.5), for $w\in\mathcal{H}$ and $\Re v>2$. Divide the sum (1.5) as follows:

$$\begin{aligned}G_v(w)=&\sum_{m=1}^{\infty}\Big\{\sum_{n=1}^{\infty}(m+nw)^{-v}+m^{-v}+\sum_{n=1}^{\infty}(m-nw)^{-v}\Big\}\\ &+\sum_{n=1}^{\infty}(nw)^{-v}+\sum_{n=1}^{\infty}(-nw)^{-v}\\ &+\sum_{m=1}^{\infty}\Big\{\sum_{n=1}^{\infty}(-m+nw)^{-v}+(-m)^{-v}+\sum_{n=1}^{\infty}(-m-nw)^{-v}\Big\}.\end{aligned}$$

Note that

$$(-m\pm nw)^{-v}=e^{\mp\pi iv}(m\mp nw)^{-v},$$

$(-nw)^{-v} = e^{\pi iv}(nw)^{-v}$ and $(-m)^{-v} = e^{\pi iv}m^{-v}$ for any positive m and n. Hence we obtain

(7.1)
$$G_v(w) = (1+e^{\pi iv})\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m+nw)^{-v} + (1+e^{-\pi iv})\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m-nw)^{-v} + (1+e^{\pi iv})(1+w^{-v})\zeta(v).$$

On the other hand, we have

$$\zeta_2(v;1,w) = \zeta(v) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m+nw)^{-v} \tag{7.2}$$

for any $w \in \mathbf{C}$ satisfying $|\arg w| < \pi$, $w \neq 0$, if $\Re v > 2$. From (7.1) and (7.2) we obtain

(7.3)
$$G_v(w) = (1+e^{\pi iv})\{\zeta_2(v;1,w)-\zeta(v)\} + (1+e^{-\pi iv})\{\zeta_2(v;1,-w)-\zeta(v)\} + (1+e^{\pi iv})(1+w^{-v})\zeta(v)$$

for $w \in \mathcal{H}$. This relation gives the analytic continuation of $G_v(w)$ to the whole complex v-plane.

Now we assume that $w \in \mathcal{W}_\infty^*$. Then both w and $-w$ are in the region $\mathcal{W}_\infty$, hence we can apply (1.2) to the right-hand side of (7.3). Since

$$(-w)^{-v-k} = (e^{-\pi i}w)^{-v-k} = (-1)^k e^{\pi iv} w^{-v-k}$$

for $w \in \mathcal{H}$, we obtain

$$\begin{aligned} G_v(w) &= \{(1+e^{\pi iv}) - e^{\pi iv}(1+e^{-\pi iv})\}\frac{\zeta(v-1)}{v-1}w^{1-v} \\ &\quad + \sum_{k=0}^{N-1}\{(1+e^{\pi iv}) + (-1)^k e^{\pi iv}(1+e^{-\pi iv})\} \\ &\qquad \times \binom{-v}{k}\zeta(-k)\zeta(v+k)w^{-v-k} \\ &\quad + (1+e^{\pi iv})(1+w^{-v})\zeta(v) + O(|w|^{-\Re v-N}) \\ &= \sum_{\substack{0\le k\le N-1 \\ k:\text{even}}} 2(1+e^{\pi iv})\binom{-v}{k}\zeta(-k)\zeta(v+k)w^{-v-k} \end{aligned}$$

$$+ (1+e^{\pi iv})(1+w^{-v})\zeta(v) + O(|w|^{-\Re v-N}).$$

Since the term corresponding to $k=0$ in the above sum is

$$2(1+e^{\pi iv})\zeta(0)\zeta(v)w^{-v} = -(1+e^{\pi iv})\zeta(v)w^{-v},$$

and $\zeta(-k)=0$ for positive even k, we obtain (1.6). Similarly, applying (1.3) to (7.3), we obtain (1.7). Corollary 1 is proved.

Remark . The analytic continuation of Eisenstein series (1.5) was already discussed, by a quite different method, in Lewittes [12] [13].

§8. The basic integral expression for Shintani double zeta-functions

Now we proceed to the study of Shintani double zeta-functions. The purpose is to prove asymptotic expansions of the two-variable double zeta-function $\zeta_{SH,2}((u,v);A,W)$. Recall that $A=(a,b)$, $W=(w_1,w_2)$, $a>0$, $b>0$, $w_2>w_1>0$.

We already mentioned in Section 1 that the series (1.13) is convergent absolutely when $\Re(u+v)>2$ and $\Re v>0$. Under these conditions we can find a number c' satisfying

$$\max\{2-\Re(u+v), -\Re v\} < c' < 0. \tag{8.1}$$

Putting $c=c'$ and

$$\lambda = \frac{(b+n)(w_2-w_1)}{a+m+(b+n)w_1}$$

in (2.2), and dividing the both sides by $\Gamma(v)(a+m+(b+n)w_1)^{u+v}$, we obtain

$$\begin{aligned}&(a+m+(b+n)w_1)^{-u}(a+m+(b+n)w_2)^{-v}\\ &\quad=\frac{1}{2\pi i}\int_{(c')}\frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}(b+n)^z(w_2-w_1)^z\\ &\qquad\qquad\times(a+m+(b+n)w_1)^{-u-v-z}dz,\end{aligned}$$

hence

$$\begin{aligned}&\zeta_{SH,2}((u,v);A,W)\\ &\quad=\frac{1}{2\pi i}\int_{(c')}\frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}(w_2-w_1)^z w_1^{-u-v-z}\end{aligned} \tag{8.2}$$

$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(b+n)^z\Big(b+n+\frac{a+m}{w_1}\Big)^{-u-v-z}dz$$
$$= \frac{1}{2\pi i}\int_{(c')}\frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}(w_2-w_1)^z w_1^{-u-v-z}$$
$$\times\, \zeta_2((-z,u+v+z);(b,b+aw_1^{-1}),w_1^{-1})dz$$

in the region $\Re(u+v)>2$, $\Re v>0$. This expression is fundamental in our method, and this is the reason why we introduced the generalized form $\zeta_2((u,v);(\alpha,\beta),w)$ in this paper.

§9. Properties of $\zeta_2((-z,u+v+z);(b,b+aw_1^{-1}),w_1^{-1})$

For our purpose it is necesssary to study the analytic properties of the function $\zeta_2((-z,u+v+z);(b,b+aw_1^{-1}),w_1^{-1})$ as a function in z.

When $\beta>\alpha$, the formula (2.3) also holds for $n=0$. Hence, instead of (2.6), we have

$$\zeta_2((u,v);(\alpha,\beta),w) = \frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(v+z')\Gamma(-z')}{\Gamma(v)}\zeta(u+v+z',\alpha)\zeta\Big(-z',\frac{\beta-\alpha}{w}\Big)w^{z'}dz' \tag{9.1}$$

for $\Re u>-\delta$, $\Re v>2+\delta$. Shifting the path to the right as in Section 6, we get

$$\begin{aligned}\zeta_2((u,v);(\alpha,\beta),w) &= -\frac{1}{1-v}\zeta(u+v-1,\alpha)w^{-1}\\ &+\sum_{k=0}^{N-1}\binom{-v}{k}\zeta(u+v+k,\alpha)\zeta\Big(-k,\frac{\beta-\alpha}{w}\Big)w^k\\ &+S_{0,N}((u,v);(\alpha,\beta),w),\end{aligned} \tag{9.2}$$

under the assumption (6.1), where $S_{0,N}((u,v);(\alpha,\beta),w)$ is defined similarly to $S_N((u,v);(\alpha,\beta),w)$ in (6.3) but with replacing $\zeta_1(-z,(\beta-\alpha)/w)$ by $\zeta(-z,(\beta-\alpha)/w)$, and (9.2) is valid in the region $\mathcal{F}(N;\varepsilon)$ by analytic continuation. In particular, we have

$$\begin{aligned}\zeta_2((-z,u+v+z);(b,b+aw_1^{-1}),w_1^{-1}) &= -\frac{1}{1-u-v-z}\zeta(u+v-1,b)w_1\end{aligned} \tag{9.3}$$

$$+\sum_{k=0}^{N-1}\binom{-u-v-z}{k}\zeta(u+v+k,b)\zeta(-k,a)w_1^{-k}$$
$$+S_{0,N}((-z,u+v+z);(b,b+aw_1^{-1}),w_1^{-1})$$

under the conditions

$$\Re(u+v)>1-N+\varepsilon \tag{9.4}$$

and

$$x=\Re z>-\Re(u+v)-N+\varepsilon. \tag{9.5}$$

Moreover we assume $u+v\neq l$, where l is any integer satisfying $2-N\leq l\leq 2$, because if $u+v=l$ then the right-hand side of (9.3) is singular. Since

$$\begin{aligned}&S_{0,N}((-z,u+v+z);(b,b+aw_1^{-1}),w_1^{-1})\\&\quad=\frac{1}{2\pi i}\int_{(N-\varepsilon)}\frac{\Gamma(u+v+z+z')\Gamma(-z')}{\Gamma(u+v+z)}\zeta(u+v+z',b)\zeta(-z',a)w_1^{-z'}dz'\end{aligned}$$

is holomorphic in z under the condition (9.5), we can see from (9.3) that the only pole of the function $\zeta_2((-z,u+v+z);(b,b+aw_1^{-1}),w_1^{-1})$, as a function in z, in the region (9.5) is at $z=1-u-v$.

Next we estimate $S_{0,N}$. Since $\Re(u+v+z')>1$ by (9.4), we have $\zeta(u+v+z',b)=O(1)$. Therefore, using Stirling's formula and Lemma 2, we have

$$\begin{aligned}&S_{0,N}(-z,u+v+z;b,b+aw_1^{-1},w_1^{-1})\\&\quad\ll w_1^{-N+\varepsilon}\int_{-\infty}^{\infty}\exp\left\{-\frac{\pi}{2}(|y'|+|y+y'+\Im(u+v)|-|y+\Im(u+v)|)\right\}\\&\qquad\qquad\times(|y+y'+\Im(u+v)|+1)^{\Re(u+v)+x+N-\varepsilon-\frac{1}{2}}\\&\qquad\qquad\times(|y+\Im(u+v)|+1)^{-\Re(u+v)-x+\frac{1}{2}}dy',\end{aligned}$$

where $y=\Im z$. Here we put $y+y'=\eta$ to get that the above is

$$\begin{aligned}&=w_1^{-N+\varepsilon}e^{\frac{\pi}{2}|y+\Im(u+v)|}(\,|y+\Im(u+v)|+1)^{-\Re(u+v)-x+\frac{1}{2}}\\&\quad\times\int_{-\infty}^{\infty}\exp\left\{-\frac{\pi}{2}(|\eta-y|+|\eta+\Im(u+v)|\,)\right\}\\&\qquad\times(\,|\eta+\Im(u+v)|+1)^{\Re(u+v)+x+N-\varepsilon-\frac{1}{2}}d\eta.\end{aligned}$$

Since $\exp(-(\pi/2)|\eta - y|) \leq 1$, the above integral is bounded uniformly in y, hence

$$\begin{aligned}(9.6)\qquad & S_{0,N}((-z, u+v+z); (b, b+aw_1^{-1}), w_1^{-1}) \\ &\quad = O\Big(w_1^{-N+\varepsilon} e^{\frac{\pi}{2}|y+\Im(u+v)|}(\,|y+\Im(u+v)|+1)^{-\Re(u+v)-x+\frac{1}{2}}\Big)\end{aligned}$$

under the conditions (9.4) and (9.5), where the implied constant is independent of y. Combining this estimate with (9.3), we obtain

$$\begin{aligned}(9.7)\quad \zeta_2((-z, u+v+z); (b, b+aw_1^{-1}), w_1^{-1}) &\ll \sum_{k=-1}^{N-1}\left(\frac{|y|+1}{w_1}\right)^k \\ &\quad + w_1^{-N+\varepsilon} e^{\frac{\pi}{2}|y+\Im(u+v)|}(|y+\Im(u+v)|+1)^{-\Re(u+v)-x+\frac{1}{2}},\end{aligned}$$

hence

(9.8)

$$\begin{aligned}&\frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}(w_2-w_1)^z w_1^{-u-v-z}\zeta_2((-z, u+v+z); (b, b+aw_1^{-1}), w_1^{-1}) \\ &\quad \ll \exp\Big\{-\frac{\pi}{2}(|y+\Im v|+|y|)\Big\}(|y+\Im v|+1)^{x+\Re v-\frac{1}{2}} \\ &\qquad \times (|y|+1)^{-x-\frac{1}{2}}(w_2-w_1)^x w_1^{-\Re(u+v)-x}\left\{\sum_{k=-1}^{N-1}\left(\frac{|y|+1}{w_1}\right)^k\right. \\ &\qquad \left. + w_1^{-N+\varepsilon} e^{\frac{\pi}{2}|y+\Im(u+v)|}(|y+\Im(u+v)|+1)^{-\Re(u+v)-x+\frac{1}{2}}\right\},\end{aligned}$$

both (9.7) and (9.8) are valid under the conditions (9.4), (9.5) and $u + v \notin \mathbf{Z}_{\leq 2}$ (where $\mathbf{Z}_{\leq 2}$ denotes the set of all integers ≤ 2), and the implied constants are independent of y.

Next we consider the situation when we shift the path on the right-hand side of (9.1) to the left. Similarly to the argument in Sections 3 and 5, we obtain that

$$\begin{aligned}(9.9)\qquad & \zeta_2((u, v); (\alpha, \beta), w) \\ &\quad = \frac{\Gamma(1-u)\Gamma(u+v-1)}{\Gamma(v)}\zeta\Big(u+v-1, \frac{\beta-\alpha}{w}\Big)w^{1-u-v} \\ &\qquad + \sum_{k=0}^{N-1}\binom{-v}{k}\zeta(u-k, \alpha)\zeta\Big(v+k, \frac{\beta-\alpha}{w}\Big)w^{-v-k}\end{aligned}$$

$$+ R_{0,N}((u,v);(\alpha,\beta),w)$$

holds in the region $\mathcal{D}^*(N;\varepsilon)$, where $R_{0,N}((u,v);(\alpha,\beta),w)$ is defined similarly to $R_N((u,v);(\alpha,\beta),w)$ but with replacing $\zeta_1(-z,(\beta-\alpha)/w)$ by $\zeta(-z,(\beta-\alpha)/w)$. In particular we have

$$\begin{aligned}(9.10)\quad & \zeta_2((-z,u+v+z);(b,b+aw_1^{-1}),w_1^{-1}) \\ &= \frac{\Gamma(1+z)\Gamma(u+v-1)}{\Gamma(u+v+z)}\zeta(u+v-1,a)w_1^{u+v-1} \\ &+ \sum_{k=0}^{N-1}\binom{-u-v-z}{k}\zeta(-z-k,b)\zeta(u+v+z+k,a)w_1^{u+v+z+k} \\ &+ R_{0,N}((-z,u+v+z);(b,b+aw_1^{-1}),w_1^{-1})\end{aligned}$$

for any z which is not a negative integer and satisfying

$$(9.11)\qquad x > \max\{-N-1+\varepsilon, -\Re(u+v)-N+1+\varepsilon\}.$$

Similarly to the case of (9.3), we assume $u+v \notin \mathbf{Z}_{\leq 2}$.

From the expression (9.10) we see that the possibility of the poles of $\zeta_2((-z,u+v+z);(b,b+aw_1^{-1}),w_1^{-1})$ in the region (9.11) exists at $z=-j$ and $z=2-u-v-j$, where j is a positive integer. However, the poles at $z=2-u-v-j$ for any $j\geq 2$ are cancelled with the zeros coming from the binomial coefficients. Also, there are two factors $\Gamma(1+z)$ and $\zeta(-z-(j-1),b)$ which have a pole at $z=-j$ $(j\geq 1)$, and it can be seen that the sum of the residues of those factors is equal to zero. Hence the real pole is only at $z=1-u-v$, which agrees with our previous conclusion.

Now we estimate

$$\begin{aligned}& R_{0,N}((-z,u+v+z);(b,b+aw_1^{-1}),w_1^{-1}) \\ &= \frac{1}{2\pi i}\int_{(c'_N)}\frac{\Gamma(u+v+z+z')\Gamma(-z')}{\Gamma(u+v+z)}\zeta(u+v+z',b)\zeta(-z',a)w_1^{-z'}dz',\end{aligned}$$

where $c'_N = -\Re(u+v)-x-N+\varepsilon$. Using Stirling's formula and Lemma 2, we have

(9.12)

$$\begin{aligned}& R_{0,N}((-z,u+v+z);(b,b+aw_1^{-1}),w_1^{-1}) \\ &\ll w_1^{\Re(u+v)+x+N-\varepsilon}e^{\frac{\pi}{2}|y+\Im(u+v)|}(|y+\Im(u+v)|+1)^{-\Re(u+v)-x+\frac{1}{2}}\end{aligned}$$

$$\times \int_{-\infty}^{\infty} \exp\Big\{-\frac{\pi}{2}(|y+y'+\Im(u+v)|+|y'|)\Big\}$$
$$\times (|y+y'+\Im(u+v)|+1)^{-N-\frac{1}{2}+\varepsilon}(|y'|+1)^{\Re(u+v)+x+N-\frac{1}{2}-\varepsilon}$$
$$\times (|y'+\Im(u+v)|+1)^{A(-x-N+\varepsilon)}dy',$$

where $y=\Im z$. Since

$$\exp\Big(-\frac{\pi}{2}(|y+y'+\Im(u+v)|)\Big)(|y+y'+\Im(u+v)|+1)^{-N-\frac{1}{2}+\varepsilon}\le 1,$$

the integrand on the right-hand side of (9.12) is bounded uniformly in y. Hence we obtain

(9.13)

$$R_{0,N}((-z,u+v+z);(b,b+aw_1^{-1}),w_1^{-1})$$
$$= O\big(w_1^{\Re(u+v)+x+N-\varepsilon}e^{\frac{\pi}{2}|y+\Im(u+v)|}(|y+\Im(u+v)|+1)^{-\Re(u+v)-x+\frac{1}{2}}\big)$$

under the condition (9.11), with the implied constant independent of y.

§10. Proof of Theorem 2

In this section we prove the asymptotic expansion for $\zeta_{SH,2}((u,v);A,W)$ when w_1 tends to infinity. Our starting point is the expansion (8.2), which is valid for $\Re(u+v)>2$, $\Re v>0$.

Let M be a positive integer. We shift the path of integration on the right-hand side of (8.2) to $\Re z=M-\varepsilon$. From (8.1) we see that (9.5) holds for $\Re z\ge c'$, hence we can use the estimate (9.8) in the strip $c'\le\Re z\le M-\varepsilon$. The right-hand side of (9.8) tends to zero when $|y|\to\infty$, hence we may shift the path as indicated above. Since c' satisfies (8.1), the only pole $z=1-u-v$ of the function $\zeta_2((-z,u+v+z);(b,b+aw_1^{-1}),w_1^{-1})$ is irrelavant to this shifting. The residue at $z=j$ $(j=0,1,2,\dots)$ is

$$-\binom{-v}{j}(w_2-w_1)^j w_1^{-u-v-j}\zeta_2((-j,u+v+j);(b,b+aw_1^{-1}),w_1^{-1}).$$

Therefore we obtain

(10.1)

$$\zeta_{SH,2}((u,v);A,W)$$
$$=\sum_{j=0}^{M-1}\binom{-v}{j}(w_2-w_1)^j w_1^{-u-v-j}\zeta_2((-j,u+v+j);(b,b+aw_1^{-1}),w_1^{-1})$$

$$+ S_M^*((u,v);A,W),$$

where

$$(10.2)\quad S_M^*((u,v);A,W) = \frac{1}{2\pi i}\int_{(M-\varepsilon)} \frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}(w_2-w_1)^z \times w_1^{-u-v-z}\zeta_2((-z,u+v+z);(b,b+aw_1^{-1}),w_1^{-1})dz.$$

Now we assume $N \geq M$. Then, if $(u,v)\in\mathcal{F}(M;\varepsilon)$, we see that

$$\Re(u+v) > 1-M+\varepsilon \geq 1-N+\varepsilon,$$

so (9.4) holds. Also (9.5) obviously holds for $x = M-\varepsilon$. Hence we can apply (9.8) to the right-hand side of (10.2), if $(u,v)\in\mathcal{F}(M;\varepsilon)$ with $u+v\notin\mathbf{Z}_{\leq 2}$, and get the absolute convergence of the integral in this region. The poles of the integrand are located only at $z=-v-l$ $(l=0,1,2,\dots)$, $z=l$ $(l=0,1,2,\dots)$, and $z=1-u-v$. Hence by (10.2) we find that $S_M^*((u,v);A,W)$ can be continued meromorphically to

$$\mathcal{F}^{**}(M;\varepsilon) = \{(u,v)\in\mathcal{F}(M;\varepsilon);\ u+v\notin\mathbf{Z}_{\leq 2}\}.$$

Therefore (10.1) gives the analytic continuation of the function $\zeta_{SH,2}((u,v);A,W)$ to $\mathcal{F}^{**}(M;\varepsilon)$.

By using (9.8), we get

$$(10.3)\quad S_M^*((u,v);A,W) \ll (w_2-w_1)^{M-\varepsilon}w_1^{-\Re(u+v)-M+\varepsilon}\left(\sum_{k=-1}^{N-1} w_1^{-k} + w_1^{-N+\varepsilon}\right) \ll (w_2-w_1)^{M-\varepsilon}w_1^{-\Re(u+v)-M+1+\varepsilon}$$

if $(u,v)\in\mathcal{F}^{**}(M;\varepsilon)$ and $w_1\geq 1$. Also, since (9.5) holds if $z=j\geq 0$, we can substitute (9.3) with $z=j$ into (10.1). The error term $S_{0,N}((-j,u+v+j);(b,b+aw_1^{-1}),w_1^{-1})$ is estimated as $O(w_1^{-N+\varepsilon})$ by (9.6), but the term $+\varepsilon$ in the exponent can be removed, similarly to the case of (6.5). The above arguments now lead to the following

Proposition 4. *Let M, N be positive integers satisfying $M\leq N$. Then we have*

(10.4)

$$\zeta_{SH,2}((u,v);A,W)$$

$$= \sum_{j=0}^{M-1} \binom{-v}{j} (w_2 - w_1)^j w_1^{-u-v-j} \Bigg\{ -\frac{1}{1-u-v-j} \zeta(u+v-1,b) w_1 + \sum_{k=0}^{N-1} \binom{-u-v-j}{k} \zeta(u+v+k,b) \zeta(-k,a) w_1^{-k} + O(w_1^{-N}) \Bigg\} + S_M^*((u,v);A,W)$$

for $(u,v) \in \mathcal{F}^{**}(M;\varepsilon)$, $w_1 \geq 1$, *and the error term* $S_M^*((u,v);A,W)$ *satisfies the estimate* (10.3).

Theorem 2 is the special case $u = v$ of this proposition.

Remark . If we use (6.2) or (6.7) instead of (9.2)(9.3), we can show a slightly different from of asymptotic expansion.

§11. Proof of Theorem 3

We return to the expression (8.2), which is valid for $\Re(u+v) > 2$, $\Re v > 0$, and this time we shift the path of integration to the left.

Let M be a positive integer satisfying

$$M > \Re u - 2 + \varepsilon, \tag{11.1}$$

and we consider the shifting to $\Re z = c_M = -\Re v - M + \varepsilon$. The condition (11.1) assures that $c_M < c'$. If N_1 is sufficiently large, then $c_M > -\Re(u + v) - N_1 + \varepsilon$, hence we can use (9.8) with $N = N_1$ in the strip $c_M \leq \Re z \leq c'$ to see that the above shifting is possible. Assume that u is not a positive integer. The relevant poles are at $z = -v - j$ $(j = 0, 1, 2, \ldots, M-1)$ with the residue

$$-\binom{-v}{j} (w_2 - w_1)^{-v-j} w_1^{-u+j} \zeta_2((v+j, u-j); (b, b + aw_1^{-1}), w_1^{-1})$$

and at $z = 1 - u - v$ with the residue

$$\frac{\Gamma(1-u)\Gamma(u+v-1)}{\Gamma(v)} \zeta(u+v-1,b)(w_2 - w_1)^{1-u-v}.$$

Therefore we obtain

(11.2) $\quad \zeta_{SH,2}((u,v);A,W)$

$$
\begin{aligned}
&= \frac{\Gamma(1-u)\Gamma(u+v-1)}{\Gamma(v)}\zeta(u+v-1,b)(w_2-w_1)^{1-u-v} \\
&\quad - \sum_{j=0}^{M-1}\binom{-v}{j}(w_2-w_1)^{-v-j}w_1^{-u+j} \\
&\qquad \times \zeta_2((v+j,u-j);(b,b+aw_1^{-1}),w_1^{-1}) \\
&\quad + R_M^*((u,v);A,W),
\end{aligned}
$$

where

$$
\begin{aligned}
(11.3)\quad R_M^*((u,v);A,W) &= \frac{1}{2\pi i}\int_{(c_M)}\frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}(w_2-w_1)^z \\
&\quad \times w_1^{-u-v-z}\zeta_2((-z,u+v+z);(b,b+aw_1^{-1}),w_1^{-1})dz.
\end{aligned}
$$

Moreover, similarly to the case of $S_M^*((u,v);A,W)$ in the last section, we can show by using (11.3) and (9.8) (with $N=N_1$, sufficiently large) that $R_M^*((u,v);A,W)$ may be continued to the region

$$
\mathcal{C}^{**}(M;\varepsilon)=\{(u,v)\in\mathcal{C}^*(M;\varepsilon);\ u+v\notin\mathbf{Z}_{\le 2}\}.
$$

Therefore (11.2) gives the analytic continuation of $\zeta_{SH,2}((u,v);A,W)$ to $\mathcal{C}^{**}(M,\varepsilon)$.

Now, let $(u,v)\in\mathcal{C}^{**}(M,\varepsilon)$, and we use the expression (9.10). Let N be a positive integer such that

$$
(11.4)\qquad N>\max\{\Re v+M-1,\ -\Re u+M+1\}.
$$

Then $x=\Re z=c_M$ sastisfies (9.11). Moreover we assume that $-\Re v+\varepsilon\notin\mathbf{Z}$ (hence $c_M\notin\mathbf{Z}$). Then we may substitute (9.10) into (11.3) to obtain

$$
\begin{aligned}
(11.5)\qquad R_M^*((u,v);A,W) &= P_M((u,v);A,W) \\
&\quad + \sum_{k=0}^{N-1}w_1^kJ_{M,k}((u,v);A,W)+Q_{M,N}((u,v);A,W),
\end{aligned}
$$

where

$$
\begin{aligned}
(11.6)\qquad P_M((u,v);A,W) &= \frac{1}{2\pi i}\Gamma(u+v-1)\zeta(u+v-1,a) \\
&\quad \times\int_{(c_M)}\frac{\Gamma(v+z)\Gamma(-z)\Gamma(1+z)}{\Gamma(v)\Gamma(u+v+z)}(w_2-w_1)^zw_1^{-1-z}dz,
\end{aligned}
$$

$$J_{M,k}((u,v);A,W)=\frac{1}{2\pi i}\int_{(c_M)}\frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}\binom{-u-v-z}{k}\times\zeta(-z-k,b)\zeta(u+v+z+k,a)(w_2-w_1)^z dz, \tag{11.7}$$

and

$$Q_{M,N}((u,v);A,W)=\frac{1}{2\pi i}\int_{(c_M)}\frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}\times R_{0,N}((-z,u+v+z);(b,b+aw_1^{-1}),w_1^{-1})(w_2-w_1)^z w_1^{-u-v-z}dz. \tag{11.8}$$

Since $u+v\notin \mathbf{Z}_{\le 2}$, it is clear that

$$P_M((u,v);A,W)=O((w_2-w_1)^{-\Re v-M+\varepsilon}w_1^{\Re v+M-1-\varepsilon}). \tag{11.9}$$

Also, using (9.13) which is uniform in t, we get

$$Q_{M,N}((u,v);A,W)=O((w_2-w_1)^{-\Re v-M+\varepsilon}w_1^{N-\varepsilon}). \tag{11.10}$$

When $w_1\le 1$, (11.10) is absorbed into (11.9) because of (11.4). Hence from (11.5) we have

$$R_M^*((u,v);A,W)=\sum_{k=0}^{N-1}w_1^k J_{M,k}((u,v);A,W)+O((w_2-w_1)^{-\Re v-M+\varepsilon}w_1^{\Re v+M-1-\varepsilon}). \tag{11.11}$$

Lastly, since N satisfies (11.4), we see that (9.11) holds for $z=-v-j$ $(j=0,1,2,\dots,M-1)$. Hence we can apply (9.10) and (9.13) to the terms $\zeta_2((v+j,u-j);(b,b+aw_1^{-1}),w_1^{-1})$ on the right-hand side of (11.2), if $v+j$ is not a positive integer. The error coming from (9.13) is $O(w_1^{\Re u-j+N-\varepsilon})$, but the term $-\varepsilon$ in the exponent can be removed as before. Therefore, combining with (11.11), we now complete the proof of the following

PROPOSITION 5. *Let M be a positive integer, $(u,v)\in\mathcal{C}^{**}(M;\varepsilon)$. We assume that v is not an integer $\ge -M+2$, and $-\Re v+\varepsilon$ is not an integer. Let $N=N(u,v,M)$ a positive integer satisfying* (11.4). *Then we have*

$$\zeta_{SH,2}((u,v);A,W)=\frac{\Gamma(1-u)\Gamma(u+v-1)}{\Gamma(v)}\zeta(u+v-1,b)(w_2-w_1)^{1-u-v} \tag{11.12}$$

$$
\begin{aligned}
&-\sum_{j=0}^{M-1}\binom{-v}{j}(w_2-w_1)^{-v-j}w_1^{-u+j}\\
&\quad\times\Biggl\{\frac{\Gamma(1-v-j)\Gamma(u+v-1)}{\Gamma(u-j)}\zeta(u+v-1,a)w_1^{u+v-1}\\
&\qquad+\sum_{k=0}^{N-1}\binom{-u+j}{k}\zeta(v+j-k,b)\zeta(u-j+k,a)w_1^{u-j+k}\\
&\qquad+O(w_1^{\Re u-j+N})\Biggr\}\\
&+\sum_{k=0}^{N-1}w_1^k J_{M,k}((u,v);A,W)\\
&+O((w_2-w_1)^{-\Re v-M+\varepsilon}w_1^{\Re v+M-1-\varepsilon})
\end{aligned}
$$

for $0<w_1\leq 1$.

In particular, putting $u=v$, we obtain the assertion of Theorem 3. Similarly to the remark at the end of the last section, we can prove a slightly different formula if we use (3.8) instead of (9.9)(9.10).

At several places on the right-hand side of (10.4) and (11.12), there appear powers of w_2-w_1. We have only assumed $w_2>w_1$, hence various cases are possible on the behaviour of w_2-w_1. If w_2 is not so much larger than w_1, for instance if $w_2=Cw_1$ with a constant $C>1$, then we can replace $(w_2-w_1)^z w_1^{-u-v-z}$ on the right-hand side of (8.2) by $(C-1)^z w_1^{-u-v}$, and simply substitute this into (9.3) or (9.10) to obtain the nice asymptotic expansion. If w_2 is much larger than w_1, then the formula

$$(w_2-w_1)^c=w_2^c\Biggl\{1+\sum_{j=1}^{\infty}\Bigl(\frac{w_1}{w_2}\Bigr)^j\Biggr\}^c$$

is useful.

The behaviour of the integral $J_{M,k}((u,v);A,W)$ with respect to w_2-w_1 can be easily seen by using the shifting argument again. Here we assume $u\notin\mathbf{Z}$, $v\notin\mathbf{Z}$. Then all the relevant poles are simple. Let L be a positive integer. If w_2-w_1 is large, then we shift the path to $\Re z=c_M-L$ to obtain

(11.13)

$$J_{M,k}((u,v);A,W)$$

$$
\begin{aligned}
&= \sum_{l=M}^{M+L-1} \binom{-v}{l}\binom{-u+l}{k}\zeta(v+l-k,b)\zeta(u-l+k,a)(w_2-w_1)^{-v-l} \\
&\quad - \delta_1(k)\frac{k!\Gamma(v-k-1)}{\Gamma(v)}\binom{1-u-v-k}{k}\zeta(u+v-1,a)(w_2-w_1)^{-1-k} \\
&\quad + O((w_2-w_1)^{-\Re v-M-L+\varepsilon}),
\end{aligned}
$$

where $\delta_1(k)=1$ if

$$\Re v+M-1-\varepsilon<k<\Re v+M+L-1-\varepsilon,$$

and $=0$ if not. If w_2-w_1 is small, then we shift the path to $\Re z=L-\varepsilon$ to obtain

(11.14)

$$
\begin{aligned}
&J_{M,k}((u,v);A,W) \\
&= -\sum_{l=0}^{M-1} \binom{-v}{l}\binom{-u+l}{k}\zeta(v+l-k,b)\zeta(u-l+k,a)(w_2-w_1)^{-v-l} \\
&\quad + \sum_{l=0}^{L-1} \binom{-v}{l}\binom{-u-v-l}{k}\zeta(-l-k,b)\zeta(u+v+l+k,a)(w_2-w_1)^{l} \\
&\quad - \delta_0\frac{\Gamma(1-u)\Gamma(u+v-1)}{\Gamma(v)}\zeta(u+v-1,b)(w_2-w_1)^{1-u-v} \\
&\quad + \delta_2(k)\frac{k!\Gamma(v-k-1)}{\Gamma(v)}\binom{1-u-v+k}{k}\zeta(u+v-1,a)(w_2-w_1)^{-1-k} \\
&\quad + O((w_2-w_1)^{L-\varepsilon}),
\end{aligned}
$$

where $\delta_0=1$ or 0 according as $k=0$ or not, and $\delta_2(k)=1$ or 0 according as $k<\Re v+M-1-\varepsilon$ or not.

In many cases, these asymptotic expansion formulas (11.13) and (11.14) provide sufficient information on the behaviour of $J_{M,k}((u,v);A,W)$.

§12. The analytic continuation of generalized Euler-Zagier sums

In this final section we prove Theorem 4 for the generalized Euler-Zagier sum (1.20), that is

$$\zeta_n((v_1,\dots,v_n);(\alpha_1,\dots,\alpha_n),(1,\dots,1))$$

$$= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} (\alpha_1 + m_1)^{-v_1} (\alpha_2 + m_1 + m_2)^{-v_2} \times \cdots \times (\alpha_n + m_1 + m_2 + \cdots + m_n)^{-v_n}.$$

After the paper [23] of Zagier, the Euler-Zagier sum (1.21) has been investigated extensively, but the study of analytic continuation of (1.21) has begun very recently except for the case $n = 2$.

When $n = 2$, the sums (1.20) and (1.21) are reduced to $\zeta_2((u,v);(\alpha,\beta),1)$ and $\zeta_2((u,v);(1,2),1)$, respectively. The analytic continuation of the latter was already done by Atkinson [4]. The continuation of $\zeta_2((u,v);(\alpha,\alpha),1)$ was included in Katsurada and Matsumoto [11]. Katsurada [9] studied some properties of $\zeta_2((u,v);(\alpha,\beta),1)$ (actually a more generalized one including exponential factors) by his method of using Mellin-Barnes type of integrals, in the domain of absolute convergence.

On the other hand, for general n-variable case, the first work concerning analytic continuation is Arakawa and Kaneko [3], who treated the continuation with respect to v_n. Next, by using the Euler-Maclaurin summation formula, Akiyama, Egami and Tanigawa [1] proved that (1.21) can be continued to $\mathbf{C}^n$ as a function of n-variables $v_1, \ldots, v_n$. Zhao [24] independently proved the same fact by a quite different method. Then, using the same technique as in [1], Akiyama and Ishikawa [2] has shown the continuation of (1.20), as mentioned in Section 1.

Now we prove the following proposition, which obviously includes the assertion of Theorem 4.

PROPOSITION 6. *If $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, then the n-variable multiple series $\zeta_n((v_1, \ldots, v_n);(\alpha_1, \ldots, \alpha_n))$, defined by* (1.20)*, can be continued meromorphically to the whole $\mathbf{C}^n$-space. In particular, the function*

$$\zeta_n((v_1, \ldots, v_{n-1}, v_n + z);(\alpha_1, \ldots, \alpha_n)), \tag{12.1}$$

as a function in z, has the following properties:

(i) *The only possible poles of* (12.1) *are at $1 - v_n$, $2 - (v_{n-1} + v_n) - k$, $3 - (v_{n-2} + v_{n-1} + v_n) - k, \ldots, n - (v_1 + v_2 + \cdots + v_n) - k$, where k is any non-negative integer.*

(ii) *Except for the above poles, the estimate*

$$\zeta_n((v_1, \ldots, v_{n-1}, v_n + z);(\alpha_1, \ldots, \alpha_n)) = O\Big(\exp\Big\{\frac{\pi}{2}(n-1)\varepsilon|y + \Im v_n|\Big\}(|y + \Im v_n| + 1)^{B(n, x+\Re v_n)}\Big) \tag{12.2}$$

holds for any $\varepsilon > 0$, *where* $x = \Re z$, $B(1, x) = A(x)$ (*which is defined in the statement of Lemma* 2) *and* $B(n, x) = -x + (1/2)$ *for* $n \geq 2$.

Proof. We prove this proposition by induction. When $n = 1$, $\zeta_1(v_1 + z; \alpha_1)$ is nothing but the Hurwitz zeta-function $\zeta(v_1 + z; \alpha_1)$, hence all the assertions are well-known (cf. Lemma 2). Now we assume that the proposition is true for ζ_{n-1} ($n \geq 2$). Since $\alpha_{n-1} < \alpha_n$, we may put $v = v_n$ and

$$\lambda = \frac{\alpha_n - \alpha_{n-1} + m_n}{\alpha_{n-1} + m_1 + m_2 + \cdots + m_{n-1}}$$

in (2.2) for any $m_n \geq 0$. We obtain

$$\begin{aligned}
&\Gamma(v_n)(\alpha_{n-1} + m_1 + m_2 + \cdots + m_{n-1})^{v_n}(\alpha_n + m_1 + m_2 + \cdots + m_n)^{-v_n} \\
&\quad = \frac{1}{2\pi i}\int_{(c)} \Gamma(v_n + z')\Gamma(-z')\Big(\frac{\alpha_n - \alpha_{n-1} + m_n}{\alpha_{n-1} + m_1 + m_2 + \cdots + m_{n-1}}\Big)^{z'} dz',
\end{aligned}$$

hence

(12.3)

$$\begin{aligned}
&\zeta_n((v_1, \ldots, v_n); (\alpha_1, \ldots, \alpha_n)) \\
&\quad = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(v_n + z')\Gamma(-z')}{\Gamma(v_n)} \sum_{m_1, \ldots, m_{n-1}=0}^{\infty} (\alpha_1 + m_1)^{-v_1} \\
&\qquad\qquad \times \cdots \times (\alpha_{n-2} + m_1 + m_2 + \cdots + m_{n-2})^{-v_{n-2}} \\
&\qquad\qquad \times (\alpha_{n-1} + m_1 + m_2 + \cdots + m_{n-1})^{-v_{n-1}-v_n-z'} \\
&\qquad\qquad \times \sum_{m_n=0}^{\infty} (\alpha_n - \alpha_{n-1} + m_n)^{z'} dz' \\
&\quad = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(v_n + z')\Gamma(-z')}{\Gamma(v_n)} \zeta(-z', \alpha_n - \alpha_{n-1}) \\
&\qquad\qquad \times \zeta_{n-1}((v_1, \ldots, v_{n-2}, v_{n-1} + v_n + z'); (\alpha_1, \ldots, \alpha_{n-1}))dz'
\end{aligned}$$

if $\Re v_j > 1$ $(1 \leq j \leq n)$, where c satisfies

$$\max\{1 - \Re(v_{n-1} + v_n), -\Re v_n\} < c < -1. \tag{12.4}$$

Let M be a positive integer, and we shift the path of integration on the right-hand side of (12.3) to $\Re z' = M - \varepsilon$. This shifting is possible because (12.2) holds for ζ_{n-1} by the assumption (ii) of induction. We denote by $\mathcal{P}_n(v_1,\ldots,v_n)$ the set of all points listed in the assertion (i) of the proposition. Then, the set of all possible poles of the integrand on the right-hand side of (12.3), as a function in z', can be written as

$$\mathcal{P}_{n-1}(v_1,\ldots,v_{n-2},v_{n-1}+v_n)\cup\mathcal{Q}_1(v_n)\cup\{-1,0,1,2,\ldots\} \tag{12.5}$$

by the induction assumption (i), where

$$\mathcal{Q}_1(v_n)=\{z'=-v_n-j\mid j=0,1,2,\ldots\}.$$

Besides (12.4), we may further assume

$$k-\Re(v_{n-k}+v_{n-k+1}+\cdots+v_n)<c\quad(2\leq k\leq n-1), \tag{12.6}$$

because now $\Re v_j>1$ $(1\leq j\leq n)$. Then all the poles in

$$\mathcal{P}_{n-1}(v_1,\ldots,v_{n-2},v_{n-1}+v_n)\cup\mathcal{Q}_1(v_n)$$

are located on the left of the path $\Re z'=c$. Hence, when we shift the path as indicated above, the relevant poles are at $z'=-1$ coming from $\zeta(-z',\alpha_n-\alpha_{n-1})$ and $z'=j$ $(0\leq j\leq M-1)$ coming from $\Gamma(-z')$. Counting the residues of those poles, we obtain

(12.7)

$$\begin{aligned}
&\zeta_n((v_1,\ldots,v_n);(\alpha_1,\ldots,\alpha_n))\\
&\quad=\frac{1}{v_n-1}\zeta_{n-1}((v_1,\ldots,v_{n-2},v_{n-1}+v_n-1);(\alpha_1,\ldots,\alpha_{n-1}))\\
&\qquad+\sum_{j=0}^{M-1}\binom{-v_n}{j}\zeta(-j,\alpha_n-\alpha_{n-1})\\
&\qquad\qquad\times\zeta_{n-1}((v_1,\ldots,v_{n-2},v_{n-1}+v_n+j);(\alpha_1,\ldots,\alpha_{n-1}))\\
&\qquad+S_{n,M}((v_1,\ldots,v_n);(\alpha_1,\ldots,\alpha_n)),
\end{aligned}$$

where

(12.8)

$$\begin{aligned}
&S_{n,M}((v_1,\ldots,v_n);(\alpha_1,\ldots,\alpha_n))\\
&\quad=\frac{1}{2\pi i}\int_{(M-\varepsilon)}\frac{\Gamma(v_n+z')\Gamma(-z')}{\Gamma(v_n)}\zeta(-z',\alpha_n-\alpha_{n-1})
\end{aligned}$$

$$\times \zeta_{n-1}((v_1, \ldots, v_{n-2}, v_{n-1} + v_n + z'); (\alpha_1, \ldots, \alpha_{n-1}))dz'.$$

Since the set of all poles of the integrand on the right-hand side of (12.8) is the same as (12.5), we find that $S_{n,M}$ can be continued holomorphically to the region $\mathcal{G}_n(M;\varepsilon)$, which consists of all $(v_1, \ldots, v_n)$ satisfying

$$\Re(v_{n-k} + v_{n-k+1} + \cdots + v_n) > k - M + \varepsilon \tag{12.9}$$

for all $k = 0, 1, 2, \ldots, n-1$. Since M is arbitrary, this implies, via (12.7), the analytic continuation of $\zeta_n((v_1, \ldots, v_n); (\alpha_1, \ldots, \alpha_n))$ to the whole $\mathbf{C}^n$.

The remaining task is to prove the assertions (i) and (ii) for the function $\zeta_n((v_1, \ldots, v_{n-1}, v_n + z); (\alpha_1, \ldots, \alpha_n))$. From (12.7) we have

(12.10)

$$\begin{aligned}
&\zeta_n((v_1, \ldots, v_{n-1}, v_n + z); (\alpha_1, \ldots, \alpha_n)) \\
&\quad = \frac{1}{v_n + z - 1}\zeta_{n-1}((v_1, \ldots, v_{n-2}, v_{n-1} + v_n + z - 1); (\alpha_1, \ldots, \alpha_{n-1})) \\
&\qquad + \sum_{j=0}^{M-1} \binom{-v_n - z}{j} \zeta(-j, \alpha_n - \alpha_{n-1}) \\
&\qquad\qquad \times \zeta_{n-1}((v_1, \ldots, v_{n-2}, v_{n-1} + v_n + z + j); (\alpha_1, \ldots, \alpha_{n-1})) \\
&\qquad + S_{n,M}((v_1, \ldots, v_{n-1}, v_n + z); (\alpha_1, \ldots, \alpha_n)),
\end{aligned}$$

which is valid for

$$\Re s > \max_{0 \le k \le n-1} \{k - M - \Re(v_{n-k} + v_{n-k+1} + \cdots + v_n) + \varepsilon\}. \tag{12.11}$$

Since $S_{n,M}((v_1, \ldots, v_{n-1}, v_n + z); (\alpha_1, \ldots, \alpha_n))$ is holomorphic in z in the region (12.11), the set of all poles of the right-hand side of (12.10) is

$$\{1 - v_n\} \cup \bigcup_{-1 \le j \le M-1} \mathcal{P}_{n-1}(v_1, \ldots, v_{n-2}, v_{n-1} + v_n + j).$$

Applying the induction assumption (i) we find that the above set coincides with

$$\mathcal{P}_n(v_1, \ldots, v_n) \setminus \{2 - (v_{n-1} + v_n) - k \mid k = M+1, M+2, \ldots\},$$

which tends to $\mathcal{P}_n(v_1, \ldots, v_n)$ itself when $M \to \infty$. Hence the assertion (i) is proved.

Next, let z be complex, satisfying (12.11), and $z \notin \mathcal{P}_n(v_1, \ldots, v_n)$. Applying the induction assumption (ii) and Stirling's formula to (12.8) (with replacing v_n by $v_n + z$), we get

(12.12)

$$\begin{aligned}
&S_{n,M}((v_1, \ldots, v_{n-1}, v_n + z); (\alpha_1, \ldots, \alpha_n)) \\
&\quad \ll e^{\frac{\pi}{2}|y+\Im v_n|}(|y+\Im v_n|+1)^{-x-\Re v_n+\frac{1}{2}} \\
&\qquad \times \int_{-\infty}^{\infty} \exp\Big\{\frac{\pi}{2}\big((n-2)\varepsilon|y+y'+\Im(v_{n-1}+v_n)| \\
&\qquad\qquad\qquad\qquad - |y+y'+\Im v_n| - |y'|\big)\Big\} \\
&\qquad\qquad \times (|y+y'+\Im v_n|+1)^{x+M+\Re v_n-\frac{1}{2}-\varepsilon} \\
&\qquad\qquad \times (|y+y'+\Im(v_{n-1}+v_n)|+1)^{B(n-1,x+M+\Re(v_{n-1}+v_n)-\varepsilon)}dy',
\end{aligned}$$

where $x = \Re z$ and $y = \Im z$. Putting $y + y' = \eta$, and noting

$$\begin{aligned}
&\exp\Big\{-\frac{\pi}{2}(|\eta+\Im v_n|+|\eta-y|)\Big\} \\
&\quad = \exp\Big\{-\frac{\pi}{2}(1-(n-1)\varepsilon)(|\eta+\Im v_n|+|\eta-y|)\Big\} \\
&\qquad \times \exp\Big\{-\frac{\pi}{2}(n-1)\varepsilon(|\eta+\Im v_n|+|\eta-y|)\Big\} \\
&\quad \le \exp\Big\{-\frac{\pi}{2}(1-(n-1)\varepsilon)|y+\Im v_n|\Big\}\exp\Big\{-\frac{\pi}{2}(n-1)\varepsilon|\eta+\Im v_n|\Big\}
\end{aligned}$$

(here we have used $|\eta+\Im v_n| + |\eta - y| \ge |y + \Im v_n|$ for the first factor, and $\exp(-(\pi/2)(n-1)\varepsilon|\eta-y|) \le 1$ for the second factor), and

$$\begin{aligned}
&\exp\Big\{\frac{\pi}{2}(n-2)\varepsilon|\eta+\Im(v_{n-1}+v_n)|\Big\} \\
&\quad \le \exp\Big\{\frac{\pi}{2}(n-2)\varepsilon(|\eta+\Im v_n|+|\Im v_{n-1}|)\Big\} \\
&\quad \ll \exp\Big\{\frac{\pi}{2}(n-2)\varepsilon|\eta+\Im v_n|\Big\},
\end{aligned}$$

we obtain that the integral on the right-hand side of (12.12) is

$$\begin{aligned}
&\ll e^{-\frac{\pi}{2}(1-(n-1)\varepsilon)|y+\Im v_n|}\int_{-\infty}^{\infty} e^{-\frac{\pi}{2}\varepsilon|\eta+\Im v_n|} \\
&\qquad \times (|\eta|+1)^{x+M+\Re v_n-\frac{1}{2}+B(n-1,x+M+\Re(v_{n-1}+v_n)-\varepsilon)-\varepsilon}d\eta \\
&\ll e^{-\frac{\pi}{2}(1-(n-1)\varepsilon)|y+\Im v_n|}
\end{aligned}$$

uniformly in y. Hence from (12.12) we have

$$S_{n,M}((v_1,\ldots,v_{n-1},v_n+z);(\alpha_1,\ldots,\alpha_n)) \ll e^{\frac{\pi}{2}(n-1)\varepsilon|y+\Im v_n|}(|y+\Im v_n|+1)^{-x-\Re v_n+\frac{1}{2}}. \tag{12.13}$$

We substitute this estimate into (12.10). The factors

$$\zeta_{n-1}((v_1,\ldots,v_{n-2},v_{n-1}+v_n+z+j);(\alpha_1,\ldots,\alpha_{n-1})) \quad (-1\le j\le M-1)$$

on the right-hand side of (12.10) can be estimated by the induction assumption (ii); since

$$\mathcal{P}_{n-1}(v_1,\ldots,v_{n-2},v_{n-1}+v_n+j)\subset\mathcal{P}_n(v_1,\ldots,v_n),$$

we can use (12.2) here. We obtain the desired estimate for

$$\zeta_n((v_1,\ldots,v_{n-1},v_n+z);(\alpha_1,\ldots,\alpha_n))$$

in the region (12.11). Since M is arbitrary, we now complete the proof of the assertion (ii), hence Proposition 6. □

Concluding remarks. (August 2002) The contents of this paper were already reported in the author's talk at the Kyoto Conference on analytic number theory (RIMS Kyoto Univ., Nov/Dec 1999). After the submission of this paper, many new results were proved in this direction of research. We mention some of them, which will clarify the initial motivation of this paper as well as the connection with number theory, and will give some prospect of further developments and remaining problems.

In [14] [15], we discussed the asymptotic behaviour of double gamma-functions by a different method, and as an application, we proved a formula on the special value at $s=1$ of certain Hecke L-functions attached to real quadratic fields. The method of the present paper supplies a simpler treatment on this matter. After the statement of Corollary 2, we mentioned "a forthcoming article", which was already published ([26]). In [26] we prove Corollary 2 (of the present paper) and some related results, including a formula on the special value at $s=1$ of certain zeta-functions, related to real quadratic fields, also introduced by Hecke. The asymptotic expansions of double zeta-functions (Theorem 1) are the basis of all of those results.

By generalizing the method in Section 12 of the present paper, asymptotic expansions of more general multiple zeta-functions were already discussed in [29] [30], and from which it is probably possible to deduce some formula on special values of zeta-functions related to general totally real number fields. Similar information can also be obtained by the asymptotic behaviour of Shintani multiple zeta-functions, which will be proved by generalizing Theorems 2 and 3 in the present paper.

Other applications of (some modifications of) the method and the results of the present paper can be found in [25] and [33]. In [25], asymptotic expansion formulas for (any positive even) power discrete mean values of Hurwitz zeta-functions have been proved; before this work, only the case of the mean square had been known. In [33], a simple proof of the explicit formula of the determinant of the Laplacian of high-dimensional spheres has been shown in a generalized form.

Multiple L-functions, defined by twisting (1.20) by Dirichlet characters, were studied by [2]. As an application, H. Ishikawa obtained certain estimates of multiple character sums. More general L-functions twisted by, for example, Fourier coefficients of modular forms can also be considered. The first step of this direction was already carried out by [32]. Then, analogously to Ishikawa's work, estimates on certain multiple sums of Fourier coefficients will be obtained.

The method of the present paper can be applied to other types of multiple zeta-functions, such as the Mordell-Tornheim type or the Apostol-Vu type ([27]). Indeed, by the method of using Mellin-Barnes integrals, we can give a unified viewpoint of a large family of multiple zeta-functions ([31]).

Finally, our argument of the analytic continuation might be important when one considers possible functional equations of multiple zeta-functions. The importance of arithmetic properties of the values of multiple zeta-functions at positive integer arguments is now well known; therefore functional equations, if exist, will be very interesting. So far a kind of functional equation has been shown only in the double zeta case ([28]), and so it is highly desirable to study the more general case.

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