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LOCAL ZETA FUNCTIONS AND NEWTON POLYHEDRA

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Abstract. To a polynomial f over a non-archimedean local field K and a character χ of the group of units of the valuation ring of K one associates Igusa's local zeta function $Z(s, f, \chi)$. In this paper, we study the local zeta function $Z(s, f, \chi)$ associated to a non-degenerate polynomial f, by using an approach based on the p-adic stationary phase formula and Néron p-desingularization. We give a small set of candidates for the poles of $Z(s, f, \chi)$ in terms of the Newton polyhedron $\Gamma(f)$ of f. We also show that for almost all χ, the local zeta function $Z(s, f, \chi)$ is a polynomial in q^{-s} whose degree is bounded by a constant independent of χ. Our second result is a description of the largest pole of $Z(s, f, \chi_{triv})$ in terms of $\Gamma(f)$ when the distance between $\Gamma(f)$ and the origin is at most one.

§1. Introduction

Let K be a non-archimedean local field of arbitrary characteristic. Let $\mathcal{O}_K$ be the ring of integers of K and $\mathcal{P}_K$ its maximal ideal. Let π be a fixed uniformizing parameter of K, and let the residue field of K be $\mathbb{F}_q$ the field with $q = p^r$ elements. For $x \in K$, v denotes the valuation of K such that $v(\pi) = 1$, $|x|_K = q^{-v(x)}$ and $ac(x) = x\pi^{-v(x)}$. Let $f(x) \in \mathcal{O}_K[x]$, $x = (x_1, \ldots, x_n)$ be a non-constant polynomial, and $\chi : \mathcal{O}_K^\times \to \mathbb{C}^\times$ a character of $\mathcal{O}_K^\times$, the group of units of $\mathcal{O}_K$. We formally put $\chi(0) = 0$. To these data one associates Igusa's local zeta function,

$$Z(s, f, \chi) = \int_{\mathcal{O}_K^n} \chi(acf(x))|f(x)|_K^s |dx|, \quad s \in \mathbb{C},$$

for $\mathrm{Re}(s) > 0$, where $|dx|$ denotes the Haar measure on K^n, normalized such that $\mathcal{O}_K^n$ has measure 1. In the case of K having characteristic zero,

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Igusa [I2] and Denef [D1] proved that $Z(s,f,\chi)$ is a rational function of q^{-s}.

A basic problem is to determine the poles of the meromorphic continuation of $Z(s,f,\chi)$ into $\operatorname{Re}(s)<0$. The general strategy is to take a resolution $h : X \to K^n$ of f and study the resolution data $\{(N_i,n_i)\}$ in which N_i is the multiplicity of $f\circ h$ along a exceptional divisor D_i, and n_i is the multiplicity of $h^\star(dx)$ along D_i. The set of ratios $\{\frac{-n_i}{N_i}\}\cup\{-1\}$ contains the real parts of the poles of $Z(s,f,\chi)$ as observed in [I2]. However, many examples show that most of these ratios do not correspond to poles. The problem of the determination of the actual poles of $Z(s,f,\chi)$ for arbitrary n is still an open problem. The case $n=2$ was solved for irreducible f and $\chi=\chi_{triv}$ for all primes p by Meuser [Me]. The generalization to reducible f and $\chi\neq\chi_{triv}$ but for almost all primes p was solved by Veys in [Ve].

In case of non-degenerate polynomials with respect to its Newton polyhedron and $K=\mathbb{R}$, Varchenko [Va] gave a procedure to compute a set of candidates for the poles of the complex power of f, by using toroidal resolution of singularities (see also [D-S-1], [D-S-2]).

The p-adic case is entirely similar to the real case. In this case, Lichtin and Meuser [L-M] proved in the case $n=2$ that not all candidates provided by the numerical data of a toric resolution of f are actually poles of $Z(s,f,\chi)$. In [D3] Denef gave a procedure based on monomial changes of variables to determine a small set of candidates for the poles of $Z(s,f,\chi_{triv})$ in terms of the Newton polyhedron of f.

In this paper, we study the local zeta function $Z(s,f,\chi)$ associated to a globally non-degenerate polynomial f (see Definition 1.1), by using an approach based on the p-adic stationary phase formula and Néron p-desingularization. We show the stationary phase formula gives a small set of candidates for the poles of $Z(s,f,\chi)$ in terms of the Newton polyhedron $\Gamma(f)$ of f (cf. Theorem A). When $\chi=\chi_{triv}$ and $\operatorname{char}(K)=0$ this set of poles agree with that obtained in [D3]. We also show that for almost all χ, the zeta function $Z(s,f,\chi)$ is a polynomial in q^{-s} whose degree is bounded by a constant independent of χ. Our second result shows that the stationary phase formula can be used to describe the largest pole of $Z(s,f,\chi_{triv})$ in terms of $\Gamma(f)$, when the distance between $\Gamma(f)$ and the origin is at most one (cf. Theorem B). This result was previously known for $\operatorname{char}(K)=0$. This result allows one to generalize estimates for exponential sums that were obtained in [D-Sp] to the case $\operatorname{char}(K)\neq 0$ (cf. Corollary 6.1).

We set $\mathbb{R}_+=\{x\in\mathbb{R}\mid x\geqq 0\}$. Let $f(x)=\sum_l a_l x^l\in K[x]$, $x=$

$(x_1, x_2, \dots, x_n)$ be a polynomial in n variables satisfying $f(0) = 0$. The set $supp(f) = \{l \in \mathbb{N}^n \mid a_l \neq 0\}$ is called the *support* of f. *The Newton polyhedron* $\Gamma(f)$ of f is defined as the convex hull in $\mathbb{R}^n_+$ of the set

$$\bigcup_{l \in supp(f)} (l + \mathbb{R}^n_+).$$

We denote by $\langle\ ,\ \rangle$ the usual inner product of $\mathbb{R}^n$, and identify $\mathbb{R}^n$ with its dual by means of it. We set

$$\langle a_\gamma, x\rangle = m(a_\gamma),$$

for the equation of the supporting hyperplane of a facet γ (i.e. a face of codimension 1 of $\Gamma(f)$) with perpendicular vector $a_\gamma = (a_1, a_2, \dots, a_n) \in \mathbb{N}^n \smallsetminus \{0\}$, and $|a_\gamma| := \sum_i a_i$.

Definition 1.1. A polynomial $f(x) = \sum_i a_i x^i \in K[x]$ is called *globally non-degenerate with respect to its Newton polyhedron* $\Gamma(f)$, if it satisfies the following two properties:
(GND1) the origin of K^n is a singular point of $f(x)$;
(GND2) for every face $\gamma \subset \Gamma(f)$ (including $\Gamma(f)$ itself), the polynomial

$$f_\gamma(x) := \sum_{i \in \gamma} a_i x^i$$

has the property that there is no $x \in (K \smallsetminus \{0\})^n$ such that

$$f_\gamma(x) = \frac{\partial f_\gamma}{\partial x_1}(x) = \cdots = \frac{\partial f_\gamma}{\partial x_n}(x) = 0.$$

Our first result is the following.

Theorem A. *Let K be a non-archimedean local field, and let $f(x) \in \mathcal{O}_K[x]$ be a polynomial globally non-degenerate with respect to its Newton polyhedron $\Gamma(f)$. Then the Igusa local zeta function $Z(s, f, \chi)$ is a rational function of q^{-s} satisfying*:

(i) *if s is a pole of $Z(s, f, \chi)$, then*

$$s = -\frac{|a_\gamma|}{m(a_\gamma)} + \frac{2\pi i}{\log q}\frac{k}{m(a_\gamma)}, \quad k \in \mathbb{Z}$$

for some facet γ of $\Gamma(f)$ with perpendicular a_γ, and $m(a_\gamma) \neq 0$, or

$$s = -1 + \frac{2\pi i}{\log q} k, \quad k \in \mathbb{Z};$$

(ii) *if $\chi \neq \chi_{triv}$ and the order of χ does not divide any $m(a_\gamma) \neq 0$, where γ is a facet of $\Gamma(f)$, then $Z(s, f, \chi)$ is a polynomial in q^{-s}, and its degree is bounded by a constant independent of χ.*

For a polynomial $f(x) \in K[x]$ globally non-degenerate with respect to its Newton polyhedron $\Gamma(f)$, we set

$$\beta(f) := \max_{\tau_j} \Big\{ -\frac{|a_j|}{m(a_j)} \Big\},$$

where τ_j runs through all facets of $\Gamma(f)$ satisfying $m(a_j) \neq 0$. The point

$$T_0 = \big(-\beta(f)^{-1}, \ldots, -\beta(f)^{-1} \big) \in \mathbb{Q}^n$$

is the intersection point of the boundary of the Newton polyhedron $\Gamma(f)$ with the diagonal $\Delta = \{(t, \ldots, t) \mid t \in \mathbb{R}\}$ in $\mathbb{R}^n$. Let τ_0 be the face of smallest dimension of $\Gamma(f)$ containing T_0, and ρ its codimension.

If $g(x) \in \mathcal{O}_K[x]$, $x = (x_1, \ldots, x_n)$, we denote by $\overline{g(x)}$ its reduction modulo $\mathcal{P}_K$.

The second result of this paper describes the largest pole of $Z(s, f, \chi_{triv})$, when $\beta(f) \geq -1$.

Theorem B. *Let K be a non-archimedean local field, and let $f(x) \in \mathcal{O}_K[x]$ be a globally non-degenerate polynomial with respect to its Newton polyhedron $\Gamma(f)$. If $\beta(f) > -1$, then $\beta(f)$ is a pole of $Z(s, f, \chi_{triv})$ of multiplicity ρ. If $\beta(f) = -1$, then $\beta(f)$ is a pole of $Z(s, f, \chi_{triv})$ of multiplicity less than or equal to $\rho + 1$. Moreover, if every face $\gamma \supseteqq \tau_0$ satisfies* $\operatorname{Card}\big(\{z \in \mathbb{F}_q^{\times n} \mid \bar{f}_\gamma(z) = 0\}\big) > 0$, *then the multiplicity of $\beta(f)$ is exactly $\rho + 1$.*

The largest pole of $Z(s, f, \chi_{triv})$ when f is non-degenerate with respect to its Newton polyhedron $\Gamma(f)$ and $\beta(f) > -1$ follows from observations made by Varchenko in [Va] and was originally noted in the p-adic case in [L-M] (although it is misstated there as $\beta(f) \neq -1$). The case $\beta(f) = -1$ is treated in [D-H]. The case of $\beta(f) < -1$ is more difficult and is established in [D-H] with some additional conditions on τ_0 by using a difficult result on exponential sums. Thus our Theorem B gives a different proof of the cases where $\beta(f) \geqq -1$.

The organization of this paper is as follows. In Section 2, we review Igusa's stationary phase formula. The results of this section generalize our previous results in [Z-G]. Section 3 contains some basic results about Newton polyhedra. In Section 4, we prove Theorem A. In Section 5, we prove Theorem B. Section 6 contains some consequences of the main theorems. More precisely, we give estimates for exponential sums involving globally non-degenerate polynomials (cf. Corollary 6.1). In Section 7, we compute explicitly the local zeta functions of some polynomials in two variables and discuss the relation between the largest pole of $Z(s, f, \chi_{triv})$ and $\beta(f)$.

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§2. Igusa's stationary phase formula

In [I3] Igusa introduced the stationary phase formula for π-adic integrals and suggested that a closer examination of this formula might lead to a new proof of the rationality of $Z(s, f, \chi)$ in any characteristic. Following this suggestion the author proved the rationality of the local zeta function $Z(s, f, \chi_{triv})$ attached to a semiquasihomogeneous polynomial f over an arbitrary non-archimedean local field [Z-G].

Let L be a ring and $f(x) \in L[x]$, we denote by $V_f(L)$ the corresponding L-hypersurface and by $Sing_f(L)$ the L-singular locus.

We denote by $\bar{x}$ the image of an element of $\mathcal{O}_K$ under the canonical homomorphism $\mathcal{O}_K \to \mathcal{O}_K/\pi\mathcal{O}_K \cong \mathbb{F}_q$, i.e. the reduction modulo π. Given $f(x) \in \mathcal{O}_K[x]$ such that not all its coefficients are in $\pi\mathcal{O}_K$, we denote by $\overline{f(x)}$ the polynomial obtained by reducing modulo π the coefficients of $f(x)$.

We fix a lifting R of $\mathbb{F}_q$ in $\mathcal{O}_K$. By definition, the set R is mapped bijectively onto $\mathbb{F}_q$ by the canonical homomorphism $\mathcal{O}_K \to \mathcal{O}_K/\pi\mathcal{O}_K$. Let $f(x) \in \mathcal{O}_K[x]$ be a polynomial in n variables, $P_1 = (y_1, \ldots, y_n) \in \mathcal{O}_K^n$, and $m_{P_1} = (m_1, \ldots, m_n) \in \mathbb{N}^n$. We call a K^n-isomorphism $\Phi_{m_{P_1}}(x)$ a *dilatation*, if it has the form $\Phi_{m_{P_1}}(x) = (z_1, \ldots, z_n)$, $z_i = y_i + \pi^{m_i} x_i$, for each $i = 1, 2, \ldots, n$. The *dilatation* of $f(x)$ at P_1 induced by $\Phi_{m_{P_1}}(x)$ is defined as

$$f_{P_1}(x) := \pi^{-e_{P_1}} f(\Phi_{m_{P_1}}(x)), \tag{2.1}$$

where e_{P_1} is the minimum order of π in the coefficients of $f(\Phi_{m_{P_1}}(x))$. We call the K-hypersurface $V_{f_{P_1}}(K)$ *the dilatation of* $V_f(K)$ at P_1 induced by $\Phi_{m_{P_1}}(x)$; the number e_{P_1} *the arithmetic multiplicity of* $f(x)$ *at* P_1 by $\Phi_{m_{P_1}}(x)$, and the set $S(f_{P_1})$, the lifting of $Sing_{\bar{f}_{P_1}}(\mathbb{F}_q)$, *the first generation of descendants of* P_1.

Given a sequence of dilatations $(\Phi_{m_{P_k}}(x))_{k\in\mathbb{N}}$, we define inductively $e_{P_1,\dots,P_k}$ and $f_{P_1,\dots,P_k}(x)$, $S(f_{P_1,\dots,P_k})$ as follows:

$$f_{P_1,\dots,P_k}(x) := \begin{cases} f(x), & \text{if } k = 0, \\ \pi^{-e_{P_1,\dots,P_k}} f_{P_1,\dots,P_{k-1}}(\Phi_{m_{P_k}}(x)), & \text{if } k \geqq 1, \end{cases} \tag{2.2}$$

where $P_k \in S(f_{P_1,\dots,P_{k-1}})$, and $e_{P_1,\dots,P_k}$ is the minimum order of π in the coefficients of $f_{P_1,\dots,P_{k-1}}(\Phi_{m_{P_k}}(x))$. For $k \ge 1$, the set $S(f_{P_1,\dots,P_k}) := \bigcup_{P_k} S(f_{P_1,\dots,P_{k-1},P_k})$ is called *the* k^{th}*-generation of descendants of* P_1. By definition the 0^{th}-generation of descendants of P_1 is $\{P_1\}$.
Now, we review Igusa's stationary phase formula, from the point of view of the dilatations. For that, we fix the m_{P_k}'s equal to $(1,\dots,1) \in \mathbb{N}^n$ in (2.1).

Let $\overline{D}$ be a subset of $\mathbb{F}_q^n$ and D its preimage under the canonical homomorphism $\mathcal{O}_K \to \mathcal{O}_K/\pi\mathcal{O}_K \cong \mathbb{F}_q$. Let $S(f,D)$ denote the subset of R^n (the set of representatives of $\mathbb{F}_q^n$ in $\mathcal{O}_K^n$) mapped bijectively to the set $Sing_{\bar{f}}(\mathbb{F}_q) \cap \overline{D}$. We use the simplified notation $S(f)$ in the case of $D = \mathcal{O}_K^n$. Also we define:

$$\nu(\bar{f}, D, \chi) := \begin{cases} q^{-n}\,\mathrm{Card}\{\overline{P} \in \overline{D} \mid \overline{P} \notin V_{\bar{f}}(\mathbb{F}_q)\}, & \text{if } \chi = \chi_{triv}, \\ q^{-nc_\chi} \sum_{\{P\in D \mid \overline{P}\notin V_{\bar{f}}(\mathbb{F}_q)\} \bmod \mathcal{P}_K^{c_\chi}} \chi(ac(f(P))), & \text{if } \chi \neq \chi_{triv}, \end{cases}$$

where c_χ is the conductor of χ, and

$$\sigma(\bar{f}, D, \chi) := \begin{cases} q^{-n}\,\mathrm{Card}\{\overline{P} \in \overline{D} \mid \overline{P} \text{ is a smooth point of } V_{\bar{f}}(\mathbb{F}_q)\}, & \text{if } \chi = \chi_{triv}, \\ 0, & \text{if } \chi \neq \chi_{triv}. \end{cases}$$

If $D = \mathcal{O}_K^n$, we use the simplified notation $\nu(\bar{f},\chi)$, $\sigma(\bar{f},\chi)$. We denote by $Z(D,s,f,\chi)$ the integral $\int_D \chi(ac(f(x)))|f(x)|_K^s |dx|$. With all this, we are able to establish Igusa's stationary phase formula for π-adic integrals ([I3, p. 177]):

Igusa's Stationary Phase Formula.

$$(2.3)\quad Z(D,s,f,\chi)=\nu(\bar{f},D,\chi)+\sigma(\bar{f},D,\chi)\frac{(1-q^{-1})q^{-s}}{(1-q^{-1-s})} + \sum_{P\in S(f,D)} q^{-n-e_P s}\int_{\mathcal{O}_K^n}\chi(ac(f_P(x)))|f_P(x)|_K^s|dx|,$$

where $\mathrm{Re}(s)>0$. The proof given by Igusa in [I3], for the case $\chi=\chi_{triv}$, generalizes literally to arbitrary characters.

In [Z-G] the author introduced the following index of singularity at a point $P\in\mathcal{O}_K^n$, satisfying $P\notin Sing_f(\mathcal{O}_K)$.

Definition 2.1. Let $f(x)\in\mathcal{O}_K[x]$ be a polynomial and $P=(a_1,\dots,a_n)\in\mathcal{O}_K^n$, such that $P\notin Sing_f(\mathcal{O}_K)$. We define

$$L(f,P):=\mathrm{Inf}\Big(v(f(P)),v\Big(\frac{\partial f}{\partial x_1}(P)\Big),\dots,v\Big(\frac{\partial f}{\partial x_n}(P)\Big)\Big).$$

It follows from the definition that $L(f,P)=0$ if and only if the polynomial

$$\overline{f(x)}=\alpha_0+\sum_j\alpha_j(x_j-\overline{a_j})+(\text{degree}\geq 2)\in\mathbb{F}_q[x],$$

satisfies $\alpha_j\in\mathbb{F}_q^*$ for some $j=0,1,2,\dots,n$.

The index $L(f,P)$ appears naturally associated to Igusa's stationary phase, as it was already noted in [Z-G]. In addition, this index plays an important role in the construction of the Néron π-adic desingularization of the special fiber of smooth schemes over $Spec(\mathcal{O}_K)$ (see [A], [N]).

If $A\subseteq\mathcal{O}_K^n$, we denote by A^c the complement of A with respect to $\mathcal{O}_K^n$.

Proposition 2.2. *Let $D\subseteq\mathcal{O}_K^n$ be an open and compact subset, and let $f(x)\in\mathcal{O}_K[x]$ be a polynomial such that $Sing_f(K)\cap D=\emptyset$. Then there exists a constant $C(f,D)\in\mathbb{N}$, depending only on f and D, such that*

$$(2.4)\qquad L(f,P)\leqq C(f,D),\quad \textit{for all } P\in D.$$

Proof. By contradiction, we suppose that $L(f,P)$ is not bounded on D. Thus there exists a sequence $(Q_i)_{i\in\mathbb{N}}$ of points of D satisfying $\lim L(f,Q_i)\to\infty$, when $i\to\infty$. This sequence has a limit point $Q_*\in D$. Since $Sing_f(K)$ is a closed set, we have that $Q_*\in Sing_f(K)\cap D=\emptyset$, contradiction. ∎

From now on, we shall suppose that $C(f,D)$ is minimal for condition (2.4).

We recall that a subset A of K^n is open and compact if and only if there is $m \geqq 0$ such that A is the finite union of classes modulo π^m. In particular the preimage of any subset of $\mathbb{F}_q^n$ under the canonical homomorphism $\mathcal{O}_K \to \mathcal{O}_K/\pi\mathcal{O}_K$ is an open and compact subset.

The following lemma is a generalization of Proposition 2.3 of [Z-G].

LEMMA 2.3. *Let $D \subseteq \mathcal{O}_K^n$ be the preimage under the canonical homomorphism $\mathcal{O}_K \to \mathcal{O}_K/\pi\mathcal{O}_K$ of a subset $\overline{D} \subseteq \mathbb{F}_q^n$, and let $f(x) \in \mathcal{O}_K[x]$ be a polynomial such that $Sing_f(\mathcal{O}_K) \cap D = \emptyset$, then*

(i) *$L(f_{P_1,\dots,P_k},0) \leqq L(f, P_1 + \pi P_2 + \dots + \pi^{k-1}P_k) - k$, for every P_k, $k \geq 1$, satisfying*: (H1) *P_k is in the $(k-1)^{th}$-generation of descendants of P_1*; (H2) *P_k has at least one descendant in the k^{th}-generation of descendants of P_1.*

(ii) *For any $P = P_1 \in S(f,D)$, if $k \geq C(f,D)+1$ then $S(f_{P_1,P_2,\dots,P_k}) = \emptyset$.*

Proof. First, we observe that

$$f(P_1 + \pi P_2 + \dots + \pi^{k-1}P_k + \pi^k x) = \pi^{E(P_1,\dots,P_k)} f_{P_1,\dots,P_k}(x), \tag{2.5}$$

where $E(P_1,\dots,P_k) = e_{P_1} + e_{P_1,P_2} + e_{P_1,\dots,P_k}$. The result follows from (2.5), if

$$e_{P_1,\dots,P_l} \geq 2, \quad \text{for } l = 1,2,\dots,k.$$

This last fact follows from the following reasoning.

By applying the Taylor formula to $f_{P_1,\dots,P_{l-1}}(P_l + \pi x)$, we obtain

$$f_{P_1,\dots,P_{l-1}}(P_l + \pi x) = f_{P_1,\dots,P_{l-1}}(P_l) + \pi \sum_j \frac{\partial f_{P_1,\dots,P_{l-1}}}{\partial x_j}(P_l)x_j + \pi^2(\text{degree} \geq 2). \tag{2.6}$$

From hypothesis (H1) follows that $v(f_{P_1,\dots,P_{l-1}}(P_l)) \geq 1$ and

$$v\Big(\frac{\partial f_{P_1,\dots,P_{l-1}}}{\partial x_j}(P_l)\Big) \geq 1,$$

and from hypothesis (H1) and (H2) that

$$v(f_{P_1,\dots,P_{l-1}}(P_l)) \geq 2;$$

therefore (2.6) implies that $e_{P_1,\dots,P_l} \geq 2$, $l = 1,2,\dots,k$.

(ii) The second part of the lemma follows immediately from (i). ∎

We observe that if $P_l \in S(f_{P_1,\dots,P_{l-1}})$ does not have descendants in the l^{th}-generation (i.e. $S(f_{P_1,\dots,P_{l-1},P_l}) = \emptyset$), then the polynomial

$$f_{P_1,\dots,P_{l-1},P_l}(P_{l+1} + \pi x) = \\ f_{P_1,\dots,P_l}(P_{l+1}) + \pi \sum_j \frac{\partial f_{P_1,\dots,P_l}}{\partial x_j}(P_{l+1}) x_j + \pi^2(\text{degree} \geq 2)$$

satisfies $\overline{f_{P_1,\dots,P_l}(P_{l+1})} \neq 0$, or $\overline{\frac{\partial f_{P_1,\dots,P_l}}{\partial x_{j_0}}(P_{l+1})} \neq 0$, for some j_0. Thus for any P_{l+1} satisfying $\overline{f_{P_1,\dots,P_l}(P_{l+1})} = 0$, it holds that $\overline{f_{P_1,\dots,P_{l+1}}(x)}$ is a polynomial of degree at most one.

Lemma 2.4. *Let $D \subseteq \mathcal{O}_K^n$ be the preimage under the canonical homomorphism $\mathcal{O}_K \to \mathcal{O}_K/\pi\mathcal{O}_K$ of a subset $\overline{D} \subseteq \mathbb{F}_q^n$. Let $f(x) \in \mathcal{O}_K[x]$ be a polynomial such that $Sing_f(K) \cap D = \emptyset$, then*

$$\int_D \chi(acf(x))|f(x)|_K^s |dx| = \begin{cases} \dfrac{T(q^{-s})}{1 - q^{-1}q^{-s}}, & \chi = \chi_{triv}, \\ L(q^{-s}), & \chi \neq \chi_{triv}, \end{cases}$$

where T and L are polynomials in q^{-s} with rational coefficients. Furthermore, in the case $\chi \neq \chi_{triv}$, the degree of the polynomial $L(q^{-s})$ is bounded by a constant depending only on f and D.

Proof. We define inductively I_k as follows:

$$I_1 := S(f, D),$$

$$I_k := \{(P_1, P_2, \dots, P_k) \mid (P_1, P_2, \dots, P_{k-1}) \in I_{k-1}, P_k \in S(f_{P_1,P_2,\dots,P_{k-1}})\}, \\ k \geq 2.$$

We set $E(P_1, \dots, P_k) := e_{P_1} + e_{P_1,P_2} + \cdots + e_{P_1,P_2,\dots,P_k}$.

If $m = C(f, D) + 1$, then $I_{m+1} = \emptyset$, because Lemma 2.3 (ii) implies that $S(f_{P_1,P_2,\dots,P_m}) = \emptyset$, for every $(P_1, P_2, \dots, P_m) \in I_m$. By applying the stationary phase formula $m+1$-times, we obtain

(2.7)

$$Z(D,s,f,\chi)=\nu(\bar{f},D,\chi)+\sigma(\bar{f},D,\chi)\frac{(1-q^{-1})q^{-s}}{(1-q^{-1-s})}$$
$$+\sum_{k=1}^{m}q^{-kn}\left(\sum_{(P_1,\ldots,P_k)\in I_k}\nu(\bar{f}_{P_1,\ldots,P_k},\chi)q^{-E(P_1,\ldots,P_k)s}\right)$$
$$+\frac{(1-q^{-1})q^{-s}}{(1-q^{-1-s})}\sum_{k=1}^{m}q^{-kn}\left(\sum_{(P_1,\ldots,P_k)\in I_k}\sigma(\bar{f}_{P_1,\ldots,P_k},\chi)q^{-E(P_1,\ldots,P_k)s}\right).$$

In the case $\chi\neq\chi_{triv}$, all $\sigma(\bar{f}_{P_1,\ldots,P_k},\chi)=0$, thus $Z(D,s,f,\chi)$ is a polynomial in q^{-s} and its degree is bounded by the maximum of the $E(P_1,\ldots,P_m)$, where P_m runs through the descendants of the $C(f,D)+1$-generation of $S(f,D)$. ∎

Corollary 2.5. *Let $D\subseteq\mathcal{O}_K^n$ be the preimage under the canonical homomorphism $\mathcal{O}_K\to\mathcal{O}_K/\pi\mathcal{O}_K$ of a subset $\overline{D}\subseteq\mathbb{F}_q^n$. Let $F(x)=f(x)+\pi^\beta g(x)\in\mathcal{O}_K[x]$ be a polynomial such that $\beta\geqq C(f,D)+1$, and*

$$Sing_F(K)\cap D=Sing_f(K)\cap D=\emptyset.$$

Then

$$Z(D,s,F,\chi)=Z(D,s,f,\chi). \tag{2.8}$$

Proof. The result follows immediately from expansion (2.7) and the fact that $C(f,D)=C(F,D)$. ∎

§3. Newton polyhedra

In this section we review some well-known results about Newton polyhedra that we shall use in this paper (see e.g. [K-M-S], [D3]).

We set $\mathbb{R}_+=\{x\in\mathbb{R}\mid x\geqq 0\}$. Let $f(x)=\sum_l a_lx^l\in K[x]$, $x=(x_1,x_2,\ldots,x_n)$ be a polynomial in n variables satisfying $f(0)=0$. The set $supp(f)=\{l\in\mathbb{N}^n\mid a_l\neq 0\}$ is called the *support* of f. *The Newton polyhedron* $\Gamma(f)$ of f is defined as the convex hull in $\mathbb{R}_+^n$ of the set

$$\bigcup_{l\in supp(f)}\left(l+\mathbb{R}_+^n\right).$$

By a *proper face* γ of $\Gamma(f)$, we mean the non-empty convex set γ obtained by intersecting $\Gamma(f)$ with an affine hyperplane H, such that $\Gamma(f)$ is contained in one of two half-spaces determined by H. The hyperplane H is named *the supporting hyperplane* of γ. A face of codimension one is named a *facet*. We set $\langle\ ,\ \rangle$ for the usual inner product in $\mathbb{R}^n$, and identify the dual vector space with $\mathbb{R}^n$. For $a \in \mathbb{R}^n_+$, we define

$$m(a) := \inf_{x\in\Gamma(f)} \{\langle a, x\rangle\}.$$

The first meet locus of $a \in \mathbb{R}^n_+ \smallsetminus \{0\}$ is defined by

$$F(a) := \{x \in \Gamma(f) \mid \langle a, x\rangle = m(a)\}.$$

The first meet locus $F(a)$ of a is a proper face of $\Gamma(f)$.

We define an equivalence relation on $\mathbb{R}^n_+ \smallsetminus \{0\}$ by

$$a \backsimeq a' \ \text{ if and only if } \ F(a) = F(a').$$

If γ is a face of $\Gamma(f)$, we define the cone associated to γ as

$$\Delta_\gamma := \{a \in (\mathbb{R}_+)^n \smallsetminus \{0\} \mid F(a) = \gamma\}.$$

The following two propositions describe the geometry of the equivalences classes of $\backsimeq$ (see e.g. [D3]).

Proposition 3.1. *Let γ be a proper face of $\Gamma(f)$. Let $w_1, w_2, \ldots, w_e$ be the facets of $\Gamma(f)$ which contain γ. Let $a_1, a_2, \ldots, a_e$ be vectors which are perpendicular to respectively $w_1, w_2, \ldots, w_e$. Then*

$$\Delta_\gamma = \left\{ \sum_{i=1}^{e} \alpha_i a_i \;\middle|\; \alpha_i \in \mathbb{R},\ \alpha_i > 0 \right\}.$$

If $a_1, a_2, \ldots, a_e \in \mathbb{R}^n$, we call $\left\{\sum_{i=1}^{e} \alpha_i a_i \mid \alpha_i \in \mathbb{R},\ \alpha_i > 0\right\}$ *the cone strictly positive spanned* by the vectors $a_1, a_2, \ldots, a_e$. Let Δ be a cone strictly positive spanned by the vectors $a_1, a_2, \ldots, a_e$. If $a_1, a_2, \ldots, a_e$ are linearly independent over $\mathbb{R}$, the cone Δ is called *a simplicial cone*. In this last case, if $a_1, a_2, \ldots, a_e \in \mathbb{Z}^n$, the cone Δ is called a rational simplicial cone. If $\{a_1, a_2, \ldots, a_e\}$ can be completed to be a basis of $\mathbb{Z}$-module $\mathbb{Z}^n$, the cone Δ is named *a simple cone*.

A vector $a \in \mathbb{R}^n$ is called *primitive* if the components of a are positive integers whose greatest common divisor is one.

For every facet of $\Gamma(f)$ there is a unique primitive vector in $\mathbb{R}^n$ which is perpendicular to this facet. Let $\mathcal{D}$ be the set of all these vectors.

PROPOSITION 3.2. *Let Δ be the cone strictly positively spanned by vectors $a_1, a_2, \ldots, a_e \in \mathbb{R}_+^n \smallsetminus \{0\}$. Then there is a partition of Δ into cones Δ_i, such that each Δ_i is strictly positively spanned by some vectors from $\{a_1, a_2, \ldots, a_e\}$ which are linearly independent over $\mathbb{R}$.*

The two previous propositions imply the existence of a partition of Δ_γ into rational simplicial cones.

PROPOSITION 3.3. ([K-M-S], p. 32–33) *Let Δ be a rational simplicial cone. Then there exists a partition of Δ into simple cones.*

Summarizing, given a polynomial $f(x) \in K[x]$, $f(0) = 0$, with Newton polyhedron $\Gamma(f)$, there exists a finite partition of $\mathbb{R}_+^n$ of the form:

$$\mathbb{R}_+^n = \{(0, \ldots, 0)\} \cup \bigcup_i \Delta_i,$$

where each Δ_i is a simplicial cone contained in an equivalence class of $\backsimeq$. Furthermore, by Proposition 3.3, it is possible to refine this partition in such a way that each Δ_i is a simple cone contained in an equivalence class of $\backsimeq$.

§4. Local zeta functions of globally non-degenerate polynomials

In this section we prove Theorem A. First, we give some preliminary results.

If $A \subseteq \mathbb{Z}_+^n$, we set

$$E_A := \{(x_1, \ldots, x_n) \in \mathcal{O}_K^n \mid (v(x_1), \ldots, v(x_n)) \in A\},$$

and

$$Z_A(s, f, \chi) := \int_{E_A} \chi(acf(x)) |f(x)|_K^s |dx|.$$

Also, if $B \subseteq \mathcal{O}_K^n$, we set

$$Z(B, s, f, \chi) := \int_B \chi(acf(x)) |f(x)|_K^s |dx|.$$

Thus $Z_A(s, f, \chi) = Z(E_A, s, f, \chi)$.

PROPOSITION 4.1. *Let $f(x) \in \mathcal{O}_K[x]$ be a globally non-degenerate polynomial with respect to its Newton polyhedron $\Gamma(f)$, $\gamma \subseteq \Gamma(f)$ a proper face, and Δ_γ its associated cone. If Δ_γ is a simple cone spanned by $a_1, a_2, \ldots, a_e \in \mathcal{D}$, and $f(x) = f_\gamma(x) + \pi^{g_0} H(x)$, where $g_0 \geq C(f_\gamma, \mathcal{O}_K^\times) + 1$ (the constant whose existence was established in Proposition 2.2), and all monomials of $H(x)$ are not in γ, then*

$$Z_{\Delta_\gamma}(s, f, \chi) = Z(\mathcal{O}_K^{\times\, n}, s, f_\gamma, \chi) \frac{q^{-\sum_{j=1}^{e}(|a_j| + m(a_j)s)}}{\prod_{j=1}^{e}\left(1 - q^{-|a_j| - m(a_j)s}\right)}. \tag{4.1}$$

Proof. The hypothesis Δ_γ is a simple cone spanned by $a_j = (a_{1,j}, a_{2,j}, \ldots, a_{n,j})$, $j = 1, 2, \ldots, e$, implies that

$$\Delta_\gamma \cap \mathbb{N}^n = \bigoplus_{j=1}^{e} a_j(\mathbb{N} \smallsetminus \{0\}). \tag{4.2}$$

From (4.2), we obtain the following expansion for $Z_\Delta(s, f, \chi)$:

$$Z_{\Delta_\gamma}(s, f, \chi) = \sum_{y_1=1}^{\infty} \cdots \sum_{y_e=1}^{\infty} \int_{\omega_{(y_1,\ldots,y_e)}} \chi(ac f(x)) |f(x)|_K^s |dx|, \tag{4.3}$$

where

$$\omega_{(y_1,\ldots,y_e)} := \{(x_1, \ldots, x_n) \in \mathcal{O}_K^n \mid x_i = \pi^{\sum_j a_{i,j} y_j} \mu_i,\ \mu_i \in \mathcal{O}_K^\times,\ i = 1, 2, \ldots, n\}.$$

In order to compute the integral in (4.3), we introduce the dilatation

$$\Phi_{(y_1,\ldots,y_e)}(x) = (\Phi_1(x), \ldots, \Phi_n(x)) : K^n \longrightarrow K^n,$$

where

$$\Phi_i(x) = \pi^{\sum_j a_{i,j} y_j} x_i, \quad i = 1, 2, \ldots, n. \tag{4.4}$$

By using the dilatation (4.4) as a change of variables in (4.3), it holds that

$$\int_{\omega_{(y_1,\ldots,y_e)}} \chi(ac f(x)) |f(x)|_K^s |dx| = q^{-\sum_{j=1}^{e} y_j(|a_j| + m(a_j)s)} \left(\int_{\mathcal{O}_K^{\times\, n}} \chi(ac(f_{(y_1,\ldots,y_e)}(x))) |f_{(y_1,\ldots,y_e)}(x)|_K^s |dx| \right), \tag{4.5}$$

where $f_{(y_1,\dots,y_e)}(x) = f_\gamma(x) + \pi^{g(y_1,\dots,y_e)+g_0} H_{(y_1,\dots,y_e)}(x)$, and $g(y_1,\dots,y_e) \geq 1$. The result follows from (4.5) by using Corollary 2.5 and expansion (4.3). □

PROPOSITION 4.2. *Let $f(x) \in \mathcal{O}_K[x]$ be a globally non-degenerate polynomial with respect to its Newton polyhedron $\Gamma(f)$, $\gamma \subseteq \Gamma(f)$ a proper face, and Δ_γ its associated cone. If Δ_γ is a simple cone spanned by $a_1, a_2, \dots, a_e \in \mathcal{D}$, then*

$$Z_{\Delta_\gamma}(s,f,\chi) = \sum_{y \text{ finite}} A_y(q^{-s}) Z(\mathcal{O}_K^{\times n}, s, f_y, \chi) + \sum_{I \subseteq \{1,2,\dots,e\}} \frac{A_I(q^{-s}) Z(\mathcal{O}_K^{\times n}, s, f_I, \chi)}{\prod_{j\in I}\left(1 - q^{-|a_j m(a_j) s)}\right)},$$

where y runs through a finite number of points in $\mathbb{N}^n$, $A_y(q^{-s})$, $A_I(q^{-s}) \in \mathbb{Q}[q^{-s}]$, $f_y(x)$ and $f_I(x)$ are polynomials in $\mathcal{O}_K[x]$ satisfying $Sing_{f_y}(K) \cap (K \smallsetminus \{0\})^n = \emptyset$, for every $y \in \mathbb{N}$, $Sing_{f_I}(K) \cap (K \smallsetminus \{0\})^n = \emptyset$, for every I, respectively. Furthermore, if γ_{a_i} denotes the facet with perpendicular a_i, and $\gamma_I = \bigcap_{i\in I} \gamma_{a_i}$, then $f_I(x) = f_{\gamma_I}(x)$.

Proof. By induction on l, the number of generators of the simple cone Δ_γ.

Case $l = 1$.

Let $m_0 = C(f_\gamma, \mathcal{O}_K^\times) + 1$, and

$$S := \Delta_\gamma \cap \mathbb{N}^n = \{a_1 y \mid y \in \mathbb{N},\ y \geq 1\}.$$

The set S can be partitioned into the subsets S_0, S_1, defined as follows:

$$S_0 := \{a_1 y \mid y = 1, 2, \dots, m_0 - 1\}, \quad S_1 := \{a_1 y \mid y \in \mathbb{N},\ y \geqq m_0\}.$$

Also we define

$$E_0 := \{(x_1,\dots,x_n) \in \mathcal{O}_K^n \mid (v(x_1),\dots,v(x_n)) \in S_0\},$$
$$E_1 := \{(x_1,\dots,x_n) \in \mathcal{O}_K^n \mid (v(x_1),\dots,v(x_n)) \in S_1\}.$$

Thus $Z_{\Delta_\gamma}(s,f,\chi) = Z(E_0,s,f,\chi) + Z(E_1,s,f,\chi)$, and by making a change of variables of type (4.4), we obtain

$$(4.6) \quad Z_{\Delta_\gamma}(s,f,\chi) = \sum_{y=1}^{m_0-1} q^{-y(|a_1|+m(a_1)s)} Z(\mathcal{O}_K^\times, s, f_y, \chi) + q^{-m_0(|a_1|+m(a_1)s)} Z_{\Delta_\gamma}(s, f_{a_1}(x) + \pi^{m_0} H(x), \chi),$$

where $f_y(x)$ are obtained from $f(x)$ by a change of variables of type (4.4) followed by a division by a power of π, $f_{a_1}(x)$ is the restriction of $f(x)$ to the facet γ_{a_1} with perpendicular a_1, and all monomials of $H(x)$ are not in γ_{a_1}. The result follows from (4.6), by means of the following equality (cf. Proposition 4.1)

$$q^{-m_0(|a_1|+m(a_1)s)}Z_{\Delta_\gamma}(s, f_{a_1}(x) + \pi^{m_0}H(x), \chi)$$
$$= \frac{q^{-(m_0+1)(|a_1|+m(a_1)s)}}{1 - q^{-(|a_1|+m(a_1)s)}} Z(\mathcal{O}_K^\times, s, f_{a_1}, \chi).$$

Induction hypothesis. Suppose that the lemma is valid for every polynomial $f(x)$ globally non-degenerate with respect its Newton polyhedron, and for every simple cone spanned by at most $e-1$ vectors of $\mathcal{D}$.

Case $l > 1$.

Let $f(x)$ be globally non-degenerate polynomial and Δ_γ a simple cone spanned by $a_1, a_2, \ldots, a_e$, satisfying the conditions of Proposition 4.2.

We set $m_0 = C(f_\gamma, \mathcal{O}_K^\times) + 1$, and

$$S := \Delta_\gamma \cap \mathbb{N}^n = \bigoplus_{j=1}^{e} a_j(\mathbb{N} \smallsetminus \{0\}), \tag{4.7}$$

$a_j = (a_{1,j}, \ldots, a_{n,j})$, $j = 1, 2, \ldots, e$. For each subset $I \subseteq \{1, 2, \ldots, e\}$, we put $r_I \in \mathbb{N}^{e-\mathrm{Card}(I)}$, $r_I = (r_{i_1}, r_{i_2}, \ldots, r_{i_{e-\mathrm{Card}(I)}})$, with $0 < r_{i_l} \le m_0 - 1$, $l = 1, 2, \ldots, e - \mathrm{Card}(I)$. The set S admits the following partition:

$$S = \bigcup_{I, r_I} S_{I, r_I}, \tag{4.8}$$

with

$$S_{I,r_I} = \left\{ \sum_{j\in I} a_j y_j + \sum_{j\notin I} a_j r_j \;\middle|\; y_j \ge m_0, \text{ if } j \in I, \text{ and } y_j = r_{i_j}, \text{ if } j \notin I \right\},$$

where for each $I \subseteq \{1, 2, \ldots, e\}$, the corresponding r_I's run through all possible different integer vectors satisfying the above mentioned conditions. We set

$$E_{I,r_I} := \{(x_1, \ldots, x_n) \in \mathcal{O}_K^n \mid (v(x_1), \ldots, v(x_n)) \in S_{I,r_I}\}.$$

It follows from partition (4.8) that

$$Z_{\Delta_\gamma}(s, f, \chi) = \sum_{I, r_I} Z(E_{I,r_I}, s, f, \chi). \tag{4.9}$$

By a change of variables of type

$$\Phi_i(x) = \pi^{\left(\sum_{j\in I} a_{i,j}y_j + \sum_{j\notin I} a_{i,j}r_j\right)} x_i, \quad i = 1, \ldots, n;$$

the integral $Z(E_{I,r_I}, s, f, \chi)$ equals

$$q^{-m_0 \sum_{j\in I}(|a_j|+m(a_j)s) - \sum_{j\notin I} r_j(|a_j|+m(a_j)s)} Z_{\Delta_I}(s, f_I, \chi), \tag{4.10}$$

where Δ_I is a simple cone generated by a_i, $i \in I$, and $f_I(x)$ is obtained from $f(\Phi_i(x))$ by division by a power of π. From these observations and (4.9), we obtain

$$\begin{aligned} Z_{\Delta_\gamma}(s, f, \chi) = &\sum_{I\subset\{1,2,\ldots,e\}} A_I(q^{-s}) Z_{\Delta_I}(s, f_I, \chi) \\ &+ q^{-m_0 \sum_{j=1}^{e}(|a_j|+m(a_j)s)} Z_{\Delta_\gamma}(s, f_\gamma + \pi^{g_0} H(x), \chi), \end{aligned} \tag{4.11}$$

where I runs through all proper subsets of $\{1, 2, \ldots, e\}$, $A_I(q^{-s}) = \sum_k q^{-a_k(I)-b_k(I)s}$, $a_k(I), b_k(I) \in \mathbb{N}$, $g_0 \geqq m_0$, and all monomials of $H(x)$ are not in γ. From (4.11) and Proposition 4.1, we obtain

$$\begin{aligned} Z_{\Delta_\gamma}(s, f, \chi) = &\sum_{I\subset\{1,2,\ldots,e\}} A_I(q^{-s}) Z_{\Delta_I}(s, f_I, \chi) \\ &+ q^{-(1+m_0)\sum_{i=1}^{e}(|a_i|+m(a_i)s)} Z(\mathcal{O}_K^{\times n}, s, f_\gamma, \chi) \frac{1}{\prod_{j=1}^{e}(1 - q^{-|a_j|-m(a_j)s})}. \end{aligned} \tag{4.12}$$

The result follows from the induction hypothesis and (4.12). □

We observe that each $A_I(q^{-s})$ in Proposition 4.1 is a finite sum of monomials of type $q^{-a_I - b_I s}$, with a_I, $b_I > 0$. We also note that a facet with supporting hyperplane $x_{i_0} = 0$ contributes to the denominator of $Z_{\Delta_\gamma}(s, f, \chi)$ with a constant factor $1/(1 - q^{-1})$.

The proof of Proposition 4.2 can be easily adapted to state the following more general result.

Corollary 4.3. *Let $f(x) \in \mathcal{O}_K[x]$ be a globally non-degenerate polynomial with respect to its Newton polyhedron $\Gamma(f)$, $\gamma \subseteq \Gamma(f)$ a proper face, and Δ_γ its associated cone. Let $\{a_1, a_2, \ldots, a_f\} \subset \mathcal{D}$ be a set of generators of Δ_γ, $\{a_1, a_2, \ldots, a_e\} \subset \{a_1, a_2, \ldots, a_f\}$ of e $\mathbb{R}$-linearly independent* *vectors, and* $b \in \Delta_\gamma \cap (\mathbb{N} \smallsetminus \{0\})^n$*. We set* $\Delta := b + \bigoplus_{j=1}^{e} a_j\mathbb{N}$*. Then*

$$Z_\Delta(s,f,\chi) = \sum_y A_y(q^{-s}) Z(\mathcal{O}_K^{\times n}, s, f_y, \chi) + \sum_{I \subseteq \{1,2,\ldots,e\}} \frac{A_I(q^{-s}) Z(\mathcal{O}_K^{\times n}, s, f_I, \chi)}{\prod_{j\in I}(1 - q^{-|a_j| - m(a_j)s})},$$

where y *runs through a finite number of points in* $\mathbb{N}^n$*,* $A_y(q^{-s})$*,* $A_I(q^{-s}) \in \mathbb{Q}[q^{-s}]$*, with* $A_I(q^{-s}) = \sum_k q^{-a_k(I) - b_k(I)s}$*,* $a_k(I), b_k(I) \in \mathbb{N}$*,* $f_y(x)$ *and* $f_I(x)$ *are polynomials in* $\mathcal{O}_K[x]$ *satisfying* $Sing_{f_y}(K) \cap (K \smallsetminus \{0\})^n = \emptyset$*, for every* y*,* $Sing_{f_I}(K) \cap (K \smallsetminus \{0\})^n = \emptyset$*, for every* I*, respectively. Furthermore, if* γ_{a_i} *denotes the facet with perpendicular* a_i*, and* $\gamma_I = \bigcap_{i\in I} \gamma_{a_i}$*, then* $f_I(x) = f_{\gamma_I}(x)$*.*

LEMMA 4.4. *Let* $f(x) \in \mathcal{O}_K[x]$ *be a globally non-degenerate polynomial with respect to its Newton polyhedron* $\Gamma(f)$*,* $\gamma \subseteq \Gamma(f)$ *a proper face, and* Δ_γ *its associated cone. Let* $\{a_1, a_2, \ldots, a_e\} \subset \mathcal{D}$ *be a set of generators of* Δ_γ*. Then*

$$Z_{\Delta_\gamma}(s,f,\chi) = \sum_y A_y(q^{-s}) Z(\mathcal{O}_K^{\times n}, s, f_y, \chi) + \sum_{I \subseteq \{1,2,\ldots,e\}} \frac{A_I(q^{-s}) Z(\mathcal{O}_K^{\times n}, s, f_I, \chi)}{\prod_{j\in I}(1 - q^{-|a_j| - m(a_j)s})}, \tag{4.13}$$

where y *runs through a finite number of points in* $\mathbb{N}^n$*,* $A_y(q^{-s})$*,* $A_I(q^{-s}) \in \mathbb{Q}[q^{-s}]$*, with* $A_I(q^{-s}) = \sum_k q^{-a_k(I) - b_k(I)s}$*,* $a_k(I), b_k(I) \in \mathbb{N}$*,* $f_y(x)$ *and* $f_I(x)$ *are polynomials in* $\mathcal{O}_K[x]$ *satisfying* $Sing_{f_y}(K) \cap (K \smallsetminus \{0\})^n = \emptyset$*, for every* y*, and* $Sing_{f_I}(K) \cap (K \smallsetminus \{0\})^n = \emptyset$*, for every* I*, respectively. Furthermore, if* γ_{a_i} *denotes the facet with perpendicular* a_i*, and* $\gamma_I = \bigcap_{i\in I} \gamma_{a_i}$*, then* $\Gamma(f_I) = \gamma_I$*.*

Proof. By Proposition 3.2 there exists a finite partition of Δ_γ into cones Δ_j, such that each Δ_j is strictly positively spanned by some vectors from $\{a_1, a_2, \ldots, a_e\}$ which are linearly independent over $\mathbb{R}$. Now, each cone Δ_j can be partitioned into a finite number of cones satisfying the conditions of Corollary 4.3. In order to verify this last assertion, we observe that the set $\Delta_j \cap \mathbb{N}^n$ admits the following partition:

$$\Delta_j \cap \mathbb{N}^n = \left(\bigoplus_{i=1}^{e} a_i (\mathbb{N} \smallsetminus \{0\}) \right) \cup \bigcup_b \left(b + \bigoplus_{i=1}^{e} a_j \mathbb{N} \right), \tag{4.14}$$

where b runs through a finite number of vectors in

$$\mathbb{N}^n \cap \left\{ \sum_{i=1}^{e} a_i \lambda_i \;\middle|\; \lambda_i \in \mathbb{R},\, 0 \leq \lambda_i < 1,\, i = 1, \ldots, e \right\}.$$

Now the result follows from Corollary 4.3. □

In the proof of the above result, we did not use a partition of the cone Δ into simple cones, because this approach produces a bigger list of candidates for the poles of $Z_{\Delta_\gamma}(s, f, \chi)$.

Proof of Theorem A. (i) Given a polynomial $f(x) \in \mathcal{O}_K[x]$, $f(0) = 0$, there exists a partition of $\mathbb{R}_+^n$ of the form:

$$\mathbb{R}_+^n = \{(0, \ldots, 0)\} \cup \bigcup_{\gamma} \Delta_\gamma, \tag{4.15}$$

where γ runs through all proper faces of $\Gamma(f)$, and Δ_γ is a cone strictly positive spanned by some vectors $a_1, \ldots, a_e \in \mathcal{D}$. In addition, Δ_γ is contained in an equivalence class of $\backsimeq$. From the above partition we obtain the following expression for $Z(s, f, \chi)$:

$$Z(s, f, \chi) = \int_{\mathcal{O}_K^{\times\, n}} \chi(ac f(x)) |f(x)|_K^s |dx| + \sum_{\gamma} Z_{\Delta_\gamma}(s, f, \chi). \tag{4.16}$$

In (4.16) there are two different types of integrals: $Z(\mathcal{O}_K^{\times\, n}, s, f, \chi)$, and $Z_{\Delta_\gamma}(s, f, \chi)$. The integrals of the first type are rational functions of q^{-s} with poles satisfying $\mathrm{Re}(s) = -1$ (cf. Lemma 2.4). The second type of integrals are rational functions of q^{-s} with poles satisfying condition (i) in the statement of Theorem A (cf. Lemma 4.4).

(ii) If $\chi \neq \chi_{triv}$, from (4.16) and Lemma 2.4 follow that $Z(s, f, \chi)$ is equal to a polynomial, with degree bounded by a constant independent of χ, plus a finite sum of functions of the form

$$\frac{A_I(q^{-s} Z(\mathcal{O}_K^{\times\, n}, s, f_I, \chi))}{\prod_{j \in I} \left(1 - q^{-|a_j| - m(a_j)s}\right)}, \tag{4.17}$$

where $f_I(x)$ denotes the restriction of $f(x)$ to the face $\gamma_I = \bigcap_{i \in I} \gamma_{a_i}$, and γ_{a_i} denotes the facet with perpendicular a_i. The second part of the theorem follows from (4.17) by the following fact: if the order of χ does not divide some $m(a_j) \neq 0$, $j \in I$, then

$$Z(\mathcal{O}_K^{\times\, n}, s, f_I, \chi) = 0. \tag{4.18}$$

If the order of χ does not divide $m(a_j)$, with $a_j = (a_{1,j}, a_{2,j}, \dots, a_{n,j})$, then there exists an $u \in \mathcal{O}_K^\times$ such that

$$\chi^{m(a_j)}(u) \neq 1. \tag{4.19}$$

We set

$$\begin{array}{rcl} \phi_u : \mathcal{O}_K^{\times\, n} & \longrightarrow & \mathcal{O}_K^{\times\, n} \\ (x_1, x_2, \dots, x_n) & \longrightarrow & (x_1 u^{a_{1,j}}, x_2 u^{a_{2,j}}, \dots, x_n u^{a_{n,j}}). \end{array} \tag{4.20}$$

The map ϕ_u establishes a bijection of $\mathcal{O}_K^{\times\, n}$ to itself that preserves the Haar measure. By using (4.20) as change of variables in the integral $Z(\mathcal{O}_K^{\times\, n}, s, f_I, \chi)$, it verifies that

$$\left(1 - \chi^{m(a_j)}(u)\right) Z(\mathcal{O}_K^{\times\, n}, s, f_I, \chi) = 0.$$

Therefore, (4.19) implies $Z(\mathcal{O}_K^{\times\, n}, s, f_I, \chi) = 0$. ∎

§5. The largest pole of $Z(s, f, \chi_{triv})$

In this section we prove Theorem B. Its proof will be accomplished by means of three preliminary results.

For a polynomial $f(x) \in \mathcal{O}_K[x]$ globally non-degenerate with respect to its Newton polyhedron $\Gamma(f)$, we set

$$\beta(f) := \max_{\tau_j} \left\{ -\frac{|a_j|}{m(a_j)} \right\},$$

where τ_j runs through all facets of $\Gamma(f)$ satisfying $m(a_j) \neq 0$. The point

$$T_0 = \left(-\beta(f)^{-1}, \dots, -\beta(f)^{-1}\right) \in \mathbb{Q}^n$$

is the intersection point of the boundary of the Newton polyhedron $\Gamma(f)$ with the diagonal $\Delta = \{(t, \dots, t) \mid t \in \mathbb{R}\}$ in $\mathbb{R}^n$. Let τ_0 be the face of smallest dimension of $\Gamma(f)$ containing T_0, and ρ its codimension, i.e. $\rho = \dim \Delta_{\tau_0}$.

Proposition 5.1. *Let $f(x) \in \mathcal{O}_K[x]$ be a globally non-degenerate polynomial with respect to its Newton polyhedron $\Gamma(f)$. If $\beta(f) > -1$, then $\beta(f)$ is a pole of $Z(s, f, \chi_{triv})$ and its multiplicity is equal to ρ.*

Proof. First, we note that the multiplicity of the possible pole $\beta(f)$ is less then or equal to $\dim \Delta_{\tau_0} = \rho$ (cf. formulas (4.16), (4.13), (2.7)). In order to prove that $\beta(f)$ is a pole of $Z(s, f, \chi_{triv})$, it is sufficient to show that

$$\lim_{s \to \beta(f)} \left(1 - q^{\beta(f)-s}\right)^{\rho} Z(s, f, \chi_{triv}) > 0. \tag{5.1}$$

This last assertion is a consequence of the following result (cf. (4.16), (4.13)):

Claim A. (i)

$$\lim_{s \to \beta(f)} \left(1 - q^{\beta(f)-s}\right)^{\rho} \left(\frac{A_I(q^{-s}) Z(\mathcal{O}_K^{\times\, n}, s, f_I, \chi_{triv})}{\prod_{j \in I} \left(1 - q^{-|a_j| - m(a_j)s}\right)} \right) \geq 0, \tag{5.2}$$

for every cone $\Delta_\gamma = \left\{ \sum_{i=1}^{e} a_i y_i \mid y_i \geq 0, \text{ for all } i \right\}$, and every $I \subseteq \{1, 2, \ldots, e\}$.

(ii) There is a cone Δ_0 and a subset I_0 of generators of this cone such that inequality (5.2) is strict.

The first part of the previous claim follows from the following two facts. The first fact is

$$\lim_{s \to \beta(f)} \left(A_I(q^{-s}) Z(\mathcal{O}_K^{\times\, n}, s, f_I, \chi_{triv}) \right) > 0. \tag{5.3}$$

Since $A_I(q^{-s}) = \sum_k q^{a_k(I) - b_k(I)s}$, with $a_k(I), b_k(I) \in \mathbb{N}$, inequality (5.3) follows from noticing that

$$\lim_{s \to \beta(f)} \left(\frac{(1 - q^{-1}) q^{-s}}{1 - q^{-1-s}} \right) > 0, \text{ when } \beta(f) > -1.$$

The second fact is

$$\lim_{s \to \beta(f)} \left(1 - q^{\beta(f)-s}\right)^{\rho} \left(\frac{1}{\prod_{j \in I} \left(1 - q^{-|a_j| - m(a_j)s}\right)} \right) \geq 0. \tag{5.4}$$

The second part of the claim follows from the following reasoning. Let $a_1, a_2, \ldots, a_e$ be the unique primitive vectors perpendicular to the facets which contain τ_0. There exists a cone Δ_0 in the partition into simplicial cones of Δ_{τ_0} given by Proposition3.2 and $I_0 \subseteq \{1, 2, \ldots, e\}$ such that $\{a_i \mid i \in I_0\}$ is a set of ρ linearly independent generators of Δ_0, because the dimension of Δ_{τ_0} is ρ. Then inequality (5.2) is strict for the cone Δ_0 and I_0. Thus, $\beta(f)$ is a pole of $Z(s, f, \chi_{triv})$ of multiplicity ρ. ∎

PROPOSITION 5.2. *Let $f(x) \in \mathcal{O}_K[x]$ be a globally non-degenerate polynomial with respect to its Newton polyhedron $\Gamma(f)$, and $\gamma \subseteq \Gamma(f)$ a proper face. If $\sigma(\bar{f}_\gamma, \mathcal{O}_K^{\times n}) = \sigma(\bar{f}_\gamma, \mathcal{O}_K^{\times n}, \chi_{triv}) > 0$ then*

$$\lim_{s \to -1} \left(1 - q^{-1-s}\right) Z(\mathcal{O}_K^{\times n}, s, f_\gamma, \chi_{triv}) \neq 0. \tag{5.5}$$

Proof. By using expansion (2.7), with $D = \mathcal{O}_K^{\times n}$, and $m = C(f_\gamma, \mathcal{O}_K^{\times n}) + 1$, we have that

$$\begin{aligned} \lim_{s \to -1} \left(1 - q^{-1-s}\right) Z(\mathcal{O}_K^{\times n}, s, f_\gamma, \chi_{triv}) &= (q-1)\sigma(\bar{f}_\gamma, \mathcal{O}_K^{\times n}, \chi_{triv}) \\ &+ (q-1) \sum_{k=1}^{m} q^{-kn} \left(\sum_{(P_1,\ldots,P_k) \in I_k} \sigma(\bar{f}_{\gamma P_1,\ldots,P_k}, \chi_{triv}) q^{E(P_1,\ldots,P_k)} \right). \end{aligned} \tag{5.6}$$

Since the right side of (5.6) is a sum of positive numbers, the result follows from the hypothesis $\sigma(\bar{f}_\gamma, \mathcal{O}_K^{\times n}, \chi_{triv}) > 0$. ∎

PROPOSITION 5.3. *Let $f(x) \in \mathcal{O}_K[x]$ be a globally non-degenerate polynomial with respect to its Newton polyhedron $\Gamma(f)$. Let $a_1, a_2, \ldots, a_e$ be the unique primitive vectors perpendicular to the facets which contain τ_0. If $\beta(f) = -1$, then $\beta(f)$ is a pole of $Z(s, f, \chi_{triv})$ with multiplicity less than or equal to $\rho + 1$. Furthermore, if every face $\gamma \supsetneqq \tau_0$ satisfies $\sigma(\bar{f}_\gamma, \mathcal{O}_K^{\times n}) > 0$, then the multiplicity of the pole $\beta(f)$ is $\rho + 1$.*

Proof. In the case $\beta(f) = -1$ the multiplicity of the possible pole $\beta(f)$ is less than or equal to $\rho + 1$ because $Z(\mathcal{O}_K^{\times n}, s, f_I, \chi_{triv})$ may have a pole at $s = -1$ (cf. formulas (4.16), (4.13), (2.7)). As in the case $\beta(f) > -1$, the result follows from inequality (5.1) by Claim A. In the case $\beta(f) = -1$, we may suppose that

$$Z(\mathcal{O}_K^{\times n}, s, f_I, \chi_{triv}) = \frac{c_I(q^{-s})}{(1 - q^{-1-s})}, \tag{5.7}$$

where $c_I(q^{-s})$ is a polynomial with positive coefficients (cf. expansion (2.7)). The proof of Claim A, for $\beta(f) = 1$, involves the same ideas as in the case $\beta(f) > -1$.

The second part of the proposition is proved as follows. There exists a simplicial cone $\Delta_0 \subseteq \Delta_{\tau_0}$ with $\dim \Delta_0 = \rho$ (cf. final part of the proof of Proposition 5.1). Let I_0 be a set of ρ linearly independent generators of Δ_0.

By duality this cone corresponds to a face $\gamma \supseteqq \tau_0$, and $Z(\mathcal{O}_K^{\times n}, s, f_{I_0}, \chi_{triv})$ has a pole of multiplicity 1 at $s = -1$ (cf. Proposition 5.2), thus

$$\lim_{s \to -1} \left(1 - q^{-1-s}\right)^{\rho+1} \left(\frac{A_{I_0}(q^{-s}) Z(\mathcal{O}_K^{\times n}, s, f_{I_0}, \chi_{triv})}{\prod_{j \in I_0} \left(1 - q^{-|a_j| - m(a_j)s}\right)} \right) > 0. \tag{5.8}$$

□

Proof of Theorem B. The theorem follows from Proposition 5.1 and Proposition 5.3. □

§6. Exponential sums

Let Ψ be an additive character trivial on $\mathcal{O}_K$ but not on $\mathcal{P}_K^{-1}$. A such character is named *standard*. We put $z = u\pi^{-m}$, $m \in \mathbb{N} \setminus \{0\}$, $u \in \mathcal{O}_K^{\times}$. To these data one associates the following exponential sum:

$$E(z, K, f) = q^{-nm} \sum_{x \bmod \mathcal{P}_K^m} \Psi(uf(x)/\pi^m).$$

The following corollary follows Theorem A, Theorem B above, and Proposition 1.4.5 of [D2], by writing $Z(s, f, \chi)$ in partial fractions.

COROLLARY 6.1. (i) *Let $f(x) \in \mathcal{O}_K[x]$ be a globally non-degenerate polynomial with respect to its Newton polyhedron $\Gamma(f)$, then for $|z|$ big enough $E(z, K, f)$ is a finite $\mathbb{C}$-linear combination of functions of the form*

$$\chi(ac(z)) |z|_K^{\lambda} (\log_q(|z|_K))^{\beta},$$

with coefficients independent of z, and with $\lambda \in \mathbb{C}$ a pole of $(1 - q^{-1-s})$ $Z(s, f, \chi_{triv})$ or of $Z(s, f, \chi)$, $\chi \neq \chi_{triv}$, and $\beta \in \mathbb{N}$, $\beta \leqq$ (multiplicity of pole λ) $- 1$. Moreover all poles λ appear effectively in this linear combination.

(ii) *Let L be a global field, and let $f(x) \in L[x]$ be a globally non-degenerate polynomial with respect to its Newton polyhedron $\Gamma(f)$, and suppose that $\beta(f) > -1$. For almost all non-archimedean completions L_v of L, there exists a constant $C(L_v) \in \mathbb{R}$ satisfying*

$$|E(z, L_v, f)| \leqq C(L_v) |z|_{L_v}^{\beta(f)} \log_q(|z|_{L_v})^{\rho - 1}, \quad \textit{for all } z \in L_v.$$

Igusa has conjectured that $C(L_v) = 1$ for almost all v [I2]. This conjecture was proved by Denef and Sperber when K has characteristic zero, f is a non-degenerate polynomial, and the face of the Newton polyhedron which cuts the diagonal does not have vertex in $\{0, 1\}^n$ [D-Sp]. Corollary 6.1 permits us to extent the result of Denef and Sperber to positive characteristic using the methods in [D-Sp].

§7. Examples

EXAMPLE 7.1. In this example, we compute $Z(s, f, \chi_{triv}) = Z(s, f)$, for $f(x, y) = x^2 + xy + y^2$, when the characteristic of K is different from 2, 3, and analyze the behavior of the pole $s = -1$. In this case $Sing_f(K) = \{(0, 0)\}$, and the Newton polygon has only a compact segment with supporting hyperplane $x + y = 2$. The polynomial f is globally non-degenerate with respect to its Newton polygon.

One easily verifies that $\mathbb{R}_+^2 \smallsetminus \{(0, 0)\}$ can be partitioned into equivalence classes modulo $\backsimeq$, as follows.

If

$$\begin{aligned}
\Delta_1 &:= \{(0, a) \mid a > 0\},\\
\Delta_2 &:= \{(b, a + b) \mid a, b > 0\},\\
\Delta_3 &:= \{(a, a) \mid a > 0\},\\
\Delta_4 &:= \{(a + b, a) \mid a, b > 0\},\\
\Delta_5 &:= \{(a, 0) \mid a > 0\},
\end{aligned}$$

then

$$\mathbb{R}_+^2 = \{(0, 0)\} \cup \bigcup_{i=1}^{5} \Delta_i,$$

and

$$Z(s, f) = Z({\mathcal{O}_K^\times}^2, s, f) + \sum_{i=1}^{5} Z_{\Delta_i}(s, f).$$

Calculation of $Z({\mathcal{O}_K^\times}^2, s, f)$**, and** $Z_{\Delta_1}(s, f)$.

By using the stationary phase formula, we obtain

$$Z({\mathcal{O}_K^\times}^2, s, f) = \nu(\bar{f}, {\mathcal{O}_K^\times}^2) + \sigma(\bar{f}, {\mathcal{O}_K^\times}^2) \frac{(1 - q^{-1})q^{-1}}{(1 - q^{-1-s})}. \tag{7.1}$$

On the other hand, it is simple to verify that $Z_{\Delta_1}(s, f) = q^{-1}(1 - q^{-1})$.

Calculation of $Z_{\Delta_2}(s, f)$ **and** $Z_{\Delta_3}(s, f)$.

$$\begin{aligned}
Z_{\Delta_2}(s, f) &= \sum_{a,b=1}^{\infty} q^{-a-2b} \int_{{\mathcal{O}_K^\times}^2} |\pi^{2b}x^2 + \pi^{a+2b}xy + \pi^{2a+2b}y^2|_K^s |dxdy| \\
&= \frac{q^{-3-2s}(1 - q^{-1})}{(1 - q^{-1-s})(1 + q^{-1-s})}.
\end{aligned} \tag{7.2}$$

(7.3)

$$Z_{\Delta_3}(s,f)=\sum_{a\geq 1}^{\infty} q^{-2a}\int_{\mathcal{O}_K^{\times 2}} |\pi^{2a}x^2+\pi^{2a}xy+\pi^{2a}y^2|_K^s |dxdy|$$

$$=\frac{q^{-2-2s}}{(1-q^{-1-s})(1+q^{-1-s})}\left(\nu(\bar{f},\mathcal{O}_K^{\times 2})+\sigma(\bar{f},\mathcal{O}_K^{\times 2})\frac{(1-q^{-1})q^{-s}}{(1-q^{-1-s})}\right).$$

Calculation of $Z_{\Delta_4}(s,f)$ and $Z_{\Delta_5}(s,f)$.

$$Z_{\Delta_4}(s,f)=\sum_{a,b\geq 1}^{\infty} q^{-2a-b}\int_{\mathcal{O}_K^{\times 2}} |\pi^{2a+2b}x^2+\pi^{2a+b}xy+\pi^{2a}y^2|_K^s |dxdy| \tag{7.4}$$

$$=\frac{q^{-3-2s}(1-q^{-1})}{(1-q^{-1-s})(1+q^{-1-s})}.$$

$$Z_{\Delta_5}(s,f)=q^{-1}(1-q^{-1}). \tag{7.5}$$

From the above calculations, we obtain

$$\lim_{s\to -1}\big(1-q^{-1-s}\big)^2 Z(s,f)=\frac{\sigma(\bar{f},\mathcal{O}_K^{\times 2})(q-1)}{2}. \tag{7.6}$$

Now suppose that $K=\mathbb{Q}_p$, with $p\neq 2,3$. Since

$$\sigma(f,\mathcal{O}_K^{\times 2})=p^2\operatorname{Card}\big(\{(u,v)\in\mathbb{F}_p^{\times 2}\mid \bar{f}(u,v)=0\}\big)$$
$$=\begin{cases}0, & \text{if } p\equiv 5,11 \bmod 12,\\ 2p^{-2}(p-1), & \text{if } p\equiv 1,7 \bmod 12,\end{cases}$$

it follows from (7.6) that

$$\lim_{s\to -1}\big(1-p^{-1-s}\big)^2 Z(s,f)=\begin{cases}0, & \text{if } p\equiv 5,11 \bmod 12,\\ p^{-2}(p-1)^2, & \text{if } p\equiv 1,7 \bmod 12.\end{cases} \tag{7.7}$$

Thus $Z(s,f)$ has a pole at $s=-1$ of multiplicity $\rho+1=2$, when

$$\operatorname{Card}\big(\{(u,v)\in\mathbb{F}_p^{\times 2}\mid \bar{f}_{\tau_0}(u,v)=0\}\big)$$
$$=\operatorname{Card}\big(\{(u,v)\in\mathbb{F}_p^{\times 2}\mid \bar{f}(u,v)=0\}\big)>0.$$

Otherwise the multiplicity is $\rho=1$.

EXAMPLE 7.2. In this example, by using the method of Lemma 4.4, we compute the local zeta function attached to the polynomial $f(x,y) = x^2y^2 + x^5 + y^5 \in K[x,y]$, when the characteristic of K is different from 2, 5. This polynomial is globally non-degenerate with respect to its Newton polyhedron.

One easily verifies that $\mathbb{R}_+^2 \setminus \{(0,0)\}$ can be partitioned into equivalence classes modulo $\simeq$, as follows.

If

$$\begin{aligned}
\Delta_1 &:= \{(0,a) \mid a > 0\},\\
\Delta_2 &:= \{(2b, a+3b) \mid a,b > 0\},\\
\Delta_3 &:= \{(2a, 3a) \mid a > 0\},\\
\Delta_4 &:= \{(2a+3b, 3a+2b) \mid a,b > 0\},\\
\Delta_5 &:= \{(3a, 2a) \mid a > 0\},\\
\Delta_6 &:= \{(3a+b, 2a) \mid a,b > 0\},\\
\Delta_7 &:= \{(a,0) \mid a > 0\},
\end{aligned}$$

then

$$\mathbb{R}_+^2 = \{(0,0)\} \cup \bigcup_{i=1}^{7} \Delta_i,$$

where each Δ_i is exactly an equivalence class modulo $\simeq$.

Calculation of $Z(\mathcal{O}_K^{\times 2}, s, f)$, and $Z_{\Delta_1}(s,f)$.

By using the stationary phase formula, we obtain

$$Z(\mathcal{O}_K^{\times 2}, s, f) = \nu(\bar{f}, \mathcal{O}_K^{\times 2}) + \sigma(\bar{f}, \mathcal{O}_K^{\times 2}) \frac{(1-q^{-1-s})}{1-q^{-1-s}}. \tag{7.8}$$

On the other hand, it is simple to verify that $Z_{\Delta_1}(s,f) = q^{-1}(1-q^{-1})$.

Calculation of $Z_{\Delta_2}(s,f)$ and $Z_{\Delta_3}(s,f)$.

The cone Δ_2 is not a simple. In this case, one verifies that there is only one element in $\Delta_2 \cap \mathbb{N}^2$ satisfying $0 \le a < 1$, $0 \le b < 1$. This element is $(1,2) = (0,1)\frac{1}{2} + (2,3)\frac{1}{2}$. Thus

$$\begin{aligned}
\Delta_2 \cap \mathbb{N}^2 = \{&(0,1)(\mathbb{N} \setminus \{0\}) + (2,3)(\mathbb{N} \setminus \{0\})\}\\
&\cup \{(1,2) + (0,1)\mathbb{N} + (2,3)\mathbb{N}\}.
\end{aligned} \tag{7.9}$$

From the partition (7.9), we obtain that

(7.10)
$$Z_{\Delta_2}(s,f)=\sum_{a,b=1}^{\infty}q^{-a-5b}\int_{\mathcal{O}_K^{\times\,2}}|\pi^{2a+10b}x^2y^2+\pi^{10b}x^5+\pi^{5a+15b}y^5|_K^s|dxdy|$$
$$+\sum_{a,b=0}^{\infty}q^{-a-5b-3}\int_{\mathcal{O}_K^{\times\,2}}|\pi^{2a+10b+6}x^2y^2+\pi^{10b+5}x^5+\pi^{5a+15b+10}y^5|_K^s|dxdy|$$
$$=\frac{q^{-5-10s}}{1-q^{-5-10s}}q^{-1}(1-q^{-1})+\frac{q^{-3-5s}}{1-q^{-5-10s}}(1-q^{-1})$$
$$=\frac{(1-q^{-1})(q^{-3-5s}+q^{-6-10s})}{1-q^{-5-10s}}.$$

By applying Proposition 4.1, and then the stationary phase formula to $Z_{\Delta_3}(s,f)$, one obtains

$$Z_{\Delta_3}(s,f)=\sum_{a=1}^{\infty}q^{-5a-10as}\int_{\mathcal{O}_K^{\times\,2}}|y^2+x^3|_K^s|dxdy|$$
$$=\frac{q^{-5-10s}}{1-q^{-5-10s}}\left(\nu(\bar{f},\mathcal{O}_K^{\times\,2})+\sigma(\bar{f},\mathcal{O}_K^{\times\,2})\frac{(1-q^{-1})q^{-s}}{(1-q^{-1-s})}\right). \tag{7.11}$$

Calculation of $Z_{\Delta_4}(s,f)$ and $Z_{\Delta_5}(s,f)$.

The cone Δ_4 is not a simple, thus we proceed as in the computation of $Z_{\Delta_2}(s,f)$, i.e. we find $0\leq a<1$, $0\leq b<1$, such that

$$(2,3)a+(3,2)b\in\mathbb{N}^2\cap\Delta_4.$$

If $a=b$, one finds immediately that $(2,3)\frac{i}{5}+(3,2)\frac{i}{5}\in\mathbb{N}^2\cap\Delta_4$, $i=1,2,3,4$. The case $a\neq b$ cannot occur. Suppose that $(m,n)\in\mathbb{N}^2\cap\Delta_4$, with $b>a$, $a\neq 0$, $b\neq 0$, ($a=0$ or $b=0$ cannot occur), i.e.

$$m=2a+3b,\ \ n=3a+2b,\ \ m,n\in\mathbb{N}\setminus\{0\},\ \ 0<a<b<1. \tag{7.12}$$

From (7.12), we get $b-a=m-n$, but this is impossible because $0<b-a<1$, and $m-n\geq 1$. If $a>b$ then $a-b=n-m$ and the same argument applies.

Therefore, we have the following partition for $\mathbb{N}^2\cap\Delta_4$:

$$\mathbb{N}^2\cap\Delta_4=\{(2,3)(\mathbb{N}\setminus\{0\})+(3,2)(\mathbb{N}\setminus\{0\})\}\cup\bigcup_{i=1}^{4}\{(i,i)+(2,3)\mathbb{N}+(3,2)\mathbb{N}\}. \tag{7.13}$$

From the partition (7.13), we obtain that
(7.14)

$$Z_{\Delta_4}(s,f) = \left(\frac{(1-q^{-1})(q^{-5-10s})}{1-q^{-5-10s}}\right)^2 + \left(\sum_{i=1}^{4} q^{-2i-4is}\right)\left(\frac{1-q^{-1}}{1-q^{-5-10s}}\right)^2.$$

For $Z_{\Delta_5}(s,f)$, we get

$$Z_{\Delta_5}(s,f) = \frac{q^{-5-10s}}{1-q^{-5-10s}}\left(\nu(\bar{f},\mathcal{O}_K^{\times 2}) + \sigma(\bar{f},\mathcal{O}_K^{\times 2})\frac{(1-q^{-1})q^{-s}}{1-q^{-1-s}}\right). \tag{7.15}$$

Calculation of $Z_{\Delta_6}(s,f)$.

In the computation of the integral $Z_{\Delta_6}(s,f)$, we use the following partition:

$$\begin{aligned} \Delta_6 \cap \mathbb{N}^2 = \{&(3,2)(\mathbb{N}\smallsetminus\{0\}) + (1,0)(\mathbb{N}\smallsetminus\{0\})\} \\ &\cup \{(2,1) + (3,2)\mathbb{N} + (1,0)\mathbb{N}\}. \end{aligned} \tag{7.16}$$

From the above partition, we get

$$Z_{\Delta_6}(s,f) = (1-q^{-1})\frac{q^{-3-5s} + q^{-6-10s}}{1-q^{-5-10s}}. \tag{7.17}$$

Calculation of $Z_{\Delta_7}(s,f)$.

$$Z_{\Delta_7}(s,f) = q^{-1}(1-q^{-1}). \tag{7.18}$$

Now, with $\beta(f) = -1/2$, and $\rho = 2$, it holds that

$$\begin{aligned} \lim_{s\to\beta(f)} \left(1-q^{\beta(f)-s}\right)^{\rho} Z(s,f) &= \lim_{s\to\beta(f)} \left(1-q^{\beta(f)-s}\right)^{\rho} Z_{\Delta_4}(s,f) \\ &= \frac{(1-q^{-1})^2}{50}. \end{aligned}$$

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