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MODIFICATION OF BALAYAGE SPACES BY TRANSITIONS WITH APPLICATION TO COUPLING OF PDE'S

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Abstract. Modifications of balayage spaces are studied which, in probabilistic terms, correspond to killing and transitions (creation of mass combined with jumps). This is achieved by a modification of harmonic kernels for sufficiently small open sets. Applications to coupling of elliptic and parabolic partial differential equations of second order are discussed.

§1. Introduction

Balayage spaces provide a potential theory which is as rich as that of harmonic spaces, the only difference being that harmonic measures for open sets may live on the entire complement instead of being concentrated on the boundary (see [BH86]). While harmonic spaces are designed for a unified discussion of solutions to large classes of linear elliptic and parabolic partial differential equations of second order, the notion of a balayage space covers, in addition, Riesz potentials, Markov chains on discrete spaces, and integro-differential equations.

In this paper we shall study modifications of balayage spaces which, in probabilistic terms, correspond to killing and transitions (creation of mass combined with jumps). This will be achieved by a modification of harmonic kernels for sufficiently small open sets. Considering transitions on direct sums we obtain coupling of balayage spaces.

For Markov processes, semigroups and resolvents such procedures have been developed in a series of papers [Bou79a], [Bou79b], [Bou80], [Bou81], [Bou82] and recently (apparently without knowledge of the work of N. Bouleau) in [CZ96]. So it should come as no surprise that our application to PDE's leads to similar results. We would like to stress, however, that our method yields an immediate solution to Dirichlet problems for coupled

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PDE's since we may directly apply the general theory of balayage spaces, whereas in [Bou81] and [CZ96] additional considerations are necessary.

To give a first idea of our approach let us look at a very simple example where the transition merely consists in jumping back and forth between two copies of an open set: Consider two global Kato measures $\mu_1, \mu_2 \geq 0$ on a Green domain D in $\mathbb{R}^d$, $d \geq 1$, (i.e., we have a Green function G_D on D and $G_D^{\mu_j} = \int G_D(\,\cdot\,, y)\, \mu_j(dy)$ is a bounded continuous real function on D, $j = 1, 2$) and assume that $\|G_D^{\mu_1}\|_\infty \|G_D^{\mu_2}\|_\infty < 1$. Let U be a regular relatively compact open subset of D and fix continuous real functions φ_1, φ_2 on the boundary ∂U. Suppose we want to solve the coupled Dirichlet problem

$$\Delta h_1 = -h_2 \mu_1 \quad \text{on } U, \quad h_1 = \varphi_1 \quad \text{on } \partial U, \tag{1.1}$$

$$\Delta h_2 = -h_1 \mu_2 \quad \text{on } U, \quad h_2 = \varphi_2 \quad \text{on } \partial U. \tag{1.2}$$

Note that e.g. the biharmonic problem

$$\Delta(\Delta h) = 0 \quad \text{on } U, \quad h = \varphi_1 \quad \text{on } \partial U, \quad -\Delta h = \varphi_2 \quad \text{on } \partial U \tag{1.3}$$

is a special case (take $\mu_1 = \lambda^d$, $\mu_2 = 0$).

Let X be the topological sum of two copies X_1, X_2 of D, each equipped with the harmonic structure given by the Laplacian and let π denote the canonical mapping between these two copies (in Section 8 we shall do this more formally). Let U_j be the set U in X_j, $j = 1, 2$. Taking μ on X, h on $\overline{U}_1 \cup \overline{U}_2$, φ on $\partial U_1 \cup \partial U_2$ such that

$$\mu|_{X_j} = \mu_j\,, \quad h|_{\overline{U}_j} = h_j\,, \quad \varphi|_{\partial U_j} = \varphi_j \quad (j = 1, 2) \tag{1.4}$$

the equations (1.1) and (1.2) may be rewritten as a single equation

$$\Delta h = -(h \circ \pi)\mu \quad \text{on } U_1 \cup U_2\,, \quad h = \varphi \quad \text{on } \partial(U_1 \cup U_2). \tag{1.5}$$

For $j = 1, 2$, let G_{U_j} denote the Green function on U_j and define a kernel $K_{U_j}^\mu$ by

$$K_{U_j}^\mu \psi := G_{U_j}^{\psi\mu} = \int G_{U_j}(\,\cdot\,, z) \psi(z)\, d\mu(z).$$

Then $\Delta h = -(h \circ \pi)\mu$ if and only if

$$\Delta\big(h - K_{U_j}^\mu (h \circ \pi)\big) = 0 \qquad \text{on } U_j\,,\ j = 1, 2. \tag{1.6}$$

The idea is now the following: Given $j \in \{1,2\}$ and a regular subset V of X_j, let H_V denote the harmonic kernel of V (i.e., H_V is a kernel on X such that, for every continuous function φ on X, the function $H_V\varphi$ is continuous on X, harmonic on V, and equal to φ on $X \setminus V$) and define a new kernel $\widetilde{H}_V$ on X by

$$\widetilde{H}_V\varphi = H_V\varphi + K_V^\mu(\varphi \circ \pi). \tag{1.7}$$

The family of all $\widetilde{H}_V$, V regular, $V \subset X_1$ or $V \subset X_2$, yields a balayage space $(X, \widetilde{\mathcal{W}})$ (this requires some proof, see Example 7.3) and then there are corresponding harmonic kernels $\widetilde{H}_U$ for *every* open subset U of X. In particular, $U_1 \cup U_2$ is regular with respect to $(X, \widetilde{\mathcal{W}})$ and then

$$h := \widetilde{H}_{U_1 \cup U_2}\varphi$$

is the solution of (1.5). Indeed, clearly $h = \varphi$ on $\partial(U_1 \cup U_2)$. And, for every $j \in \{1,2\}$, we have $\widetilde{H}_{U_1 \cup U_2} = \widetilde{H}_{U_j}\widetilde{H}_{U_1 \cup U_2}$, hence

$$h = \widetilde{H}_{U_j}h = H_{U_j}h + K_{U_j}^\mu(h \circ \pi).$$

Since $H_{U_j}h$ is harmonic on U_j, this implies that $\Delta\big(h - K_{U_j}^\mu(h \circ \pi)\big) = 0$ on U_j, i.e., (1.6) holds.

This paper is organized as follows: First we shall briefly recall some basic definitions for balayage spaces (Section 2) and discuss stability with respect to increasing limits of harmonic kernels (necessary for Section 9). Section 3 presents some fundamental properties of parabolic balayage spaces (applied in Sections 7 and 8). In Section 4 we shall generalize definition (1.7) to study a first modification of balayage spaces by transitions. A short discussion of perturbed balayage spaces in Section 5 will allow us to combine transitions with (positive or negative) perturbations (Section 6). In Section 7 we consider the special case of coupling in direct sums of balayage spaces, and in Section 8 we apply these results to coupling of partial differential equations. The most general modification of balayage spaces will be studied in Section 9 (which is independent of Sections 7 and 8). An appendix on lifting of potentials and potential kernels finishes the paper.

§2. Balayage spaces

There are various ways of describing a balayage space: By its cone $\mathcal{W}$ of positive hyperharmonic functions, by a family of harmonic kernels, by a corresponding semigroup, by an associated Hunt process (see [BH86, Theorem IV.8.1] or the survey article [Han87]). For our purpose the description using harmonic kernels is very appropriate.

We begin by introducing some notation: Let X be a locally compact space with countable base. For every open set U in X, let $\mathcal{B}(U)$ denote the set of all numerical Borel measurable functions on U. Further, $\mathcal{C}(U)$ will denote the space of all real continuous functions on U and $\mathcal{K}(U)$ ($\mathcal{C}_0(U)$ resp.) the set of all functions in $\mathcal{C}(U)$ having compact support (vanishing at infinity) with respect to U. Occasionally, functions on U will be identified with functions on X which are zero on U^c. Finally, given any set $\mathcal{A}$ of functions let $\mathcal{A}_b$ ($\mathcal{A}^+$ resp.) denote the set of all functions in $\mathcal{A}$ which are bounded (positive resp.)

Let $\mathcal{U}$ be a base of relatively compact open subsets of X and, for every $U \in \mathcal{U}$, let H_U be a kernel on X such that $H_U(x, \cdot\,) = \varepsilon_x$ for every $x \in U^c$ and $H_U 1_U = 0$. It will be convenient to assume that $\mathcal{U}$ is stable with respect to finite intersections (by [BH86, Remark VII.3.2.4] this is no restriction of generality). Define

$$\mathcal{W} := \{v \mid v : X \to [0,\infty] \text{ l.s.c., } H_U v \le v \text{ for every } U \in \mathcal{U}\} \tag{2.1}$$

and, for every numerical function $f \ge 0$ on X, let

$$R_f := \inf\{v \in \mathcal{W} : v \ge f\}.$$

A function $s \in \mathcal{C}^+(X)$ is called *strongly* ($\mathcal{W}$-)*superharmonic* if, for every $U \in \mathcal{U}$, $H_U s < s$ on U.

Then $(H_U)_{U\in\mathcal{U}}$ is a family of (regular) harmonic kernels and $(X, \mathcal{W})$ is a balayage space provided the following holds (where $U, V \in \mathcal{U}$):

(H_1) Given $x \in X$, $\lim_{U\downarrow\{x\}} H_U\varphi(x) = \varphi(x)$ for all $\varphi \in \mathcal{K}(X)$ or $R_{1_{\{x\}}}$ is l.s.c. at x.

(H_2') $H_V H_U = H_U$ if $V \subset U$.

(H_3) For every $f \in \mathcal{B}_b(X)$ with compact support, the function $H_U f$ is continuous on U.

(H_4') For every $\varphi \in \mathcal{K}(X)$, the function $H_U\varphi$ is continuous on $\overline{U}$.

(H_5') There exists a strongly superharmonic function $s \in \mathcal{C}^+(X)$.

Remarks 2.1. 1. Let f be a strictly positive continuous function on X and define kernels H_U' on X by $H_U'(x, \cdot\,) := (f/f(x))H_U(x, \cdot\,)$. Obviously $(H_U)_{U\in\mathcal{U}}$ is a family of harmonic kernels if and only if $(H_U')_{U\in\mathcal{U}}$ is a family of harmonic kernels, and the corresponding set $\mathcal{W}'$ is related to $\mathcal{W}$ by $\mathcal{W}' = (1/f)\mathcal{W}$. If $f = s$, s being a strongly superharmonic function in $\mathcal{C}^+(X)$, we have $1 \in \mathcal{W}'$ (even strongly $\mathcal{W}'$-superharmonic). This implies that for the proof of many results on general balayage spaces we may assume without loss of generality that $1 \in \mathcal{W}$.

2. It will be clear to the specialist how to proceed if we would not assume having a base of regular sets, i.e., if instead of (H_4') we would only suppose that the following property (H_4) holds: For every $x \in U$ there exists a l.s.c. function $w \geq 0$ on U such that $w(x) < \infty$, $H_V w \leq w$ if $\overline{V} \subset U$, and $\lim_{\mathcal{F}} w = \infty$ for every non-regular ultrafilter $\mathcal{F}$ on U (see [BH86, p. 94]).

Moreover, properties (H_1)–(H_5') imply the following property (H_5): $\mathcal{W}$ is linearly separating (i.e., for $x, y \in X$, $x \neq y$, and $\lambda \in \mathbb{R}_+$ there exists $v \in \mathcal{W}$ such that $v(x) \neq \lambda v(y)$) and there exists a strictly positive function $s_0 \in \mathcal{W} \cap \mathcal{C}(X)$. Indeed, let $s \in \mathcal{C}^+(X)$ be strongly superharmonic. Then of course $s > 0$ and $s \in \mathcal{W}$. Furthermore, $H_U s \in \mathcal{W}$ for every $U \in \mathcal{U}$: Because of (H_4') the function $H_U s$ is l.s.c. Given $V \in \mathcal{U}$, we have to show that $H_V H_U s \leq H_U s$. Since $H_U s \leq s$ and $H_V s \leq s$, we obtain first that

$$H_V H_U s \leq H_V s \leq s = H_U s \quad \text{on } U^c.$$

In addition, $H_V H_U s = H_U s$ on V^c. Since $(U \cap V)^c = U^c \cup V^c$, we conclude that

$$H_V H_U s = H_{U \cap V} H_V H_U s \leq H_{U \cap V} H_U s = H_U s.$$

It is now easily seen that $\mathcal{W}$ is linearly separating: Fix $x, y \in X$, $x \neq y$. Choose $U \in \mathcal{U}$ such that $x \in U$, $y \notin U$. For every $\lambda \in \mathbb{R}_+$, $s(x) \neq \lambda s(y)$ or $H_U s(x) \neq \lambda s(y) = \lambda H_U s(y)$.

We finally note that (H_5') holds for every balayage space by [BH86, pp. 17, 118].

3. It will be useful to know that $\mathcal{W}$ as defined by (2.1) does not change if we replace $\mathcal{U}$ by a smaller base $\mathcal{U}'$ (see [BH86, Remark III.6.13]).

As for harmonic spaces continuous potentials play an important role. The convex cone $\mathcal{P}(X)$ of all continuous real potentials can be defined and characterized in several ways:

$$\begin{aligned}
\mathcal{P}(X) &= \Big\{p \in \mathcal{W} \cap \mathcal{C}(X) : \inf_{K \text{ compact} \subset X} R_{1_{K^c} p} = 0\Big\} \\
&= \big\{p \in \mathcal{W} \cap \mathcal{C}(X) : \tfrac{p}{q} \in \mathcal{C}_0(X) \text{ for some } q \in \mathcal{W} \cap \mathcal{C}(X)\big\} \\
&= \{p \in \mathcal{W} \cap \mathcal{C}(X) : 0 \leq g \leq p,\ g \in \mathcal{H}^+(X) \Rightarrow g = 0\}
\end{aligned}$$

where $\mathcal{H}^+(X)$ denotes the set of all positive harmonic functions on X, i.e.,

$$\mathcal{H}^+(X) = \{g \in \mathcal{C}^+(X) : H_U g = g \text{ for every } U \in \mathcal{U}\}.$$

Moreover, we have a Riesz decomposition

$$\mathcal{W} \cap \mathcal{C}(X) = \mathcal{H}^+(X) \oplus \mathcal{P}(X).$$

A function f on X is called *$\mathcal{P}$-bounded* if $|f| \le p$ for some $p \in \mathcal{P}(X)$.

For every open subset V of X, the set ${}^*\mathcal{H}^+(V)$ of all positive functions which are hyperharmonic on V is defined by

$${}^*\mathcal{H}^+(V) := \left\{ s \in \mathcal{B}^+(X) : \begin{array}{l} s \text{ l.s.c. on } V, \\ H_U s \le s \text{ for every } U \in \mathcal{U} \text{ with } \overline{U} \subset V \end{array} \right\}.$$

(see [BH86, p. 94]). Of course, ${}^*\mathcal{H}^+(X) = \mathcal{W}$ and, by [BH86, Corollary III.4.5],

$$(2.2) \qquad {}^*\mathcal{H}^+\Big(\bigcup_{i \in I} V_i\Big) = \bigcap_{i \in I} {}^*\mathcal{H}^+(V_i)$$

for every family $(V_i)_{i \in I}$ of open subsets of X. Note that $H_U(\mathcal{B}^+(X)) \subset {}^*\mathcal{H}^+(U)$ for every $U \in \mathcal{U}$ (consequence of (H_2') and (H_3)).

It is easily seen that we may restrict the balayage space $(X, \mathcal{W})$ on any open subset Y of X defining kernels

$$H_U^Y(x, \cdot\,) := H_U(x, \cdot\,)|_Y \quad (x \in U \in \mathcal{U}\,, \overline{U} \subset Y).$$

The corresponding cone $\mathcal{W}_Y$ is ${}^*\mathcal{H}^+(Y)|_Y$.

It is trivial that finite and countable direct sums of balayage spaces are balayage spaces as well: Let $(X_i, \mathcal{W}_i)$, $i \in I \subset \mathbb{N}$, be balayage spaces. If $X = \sum_{i \in I} X_i$ denotes the topological sum of all X_i, $i \in I$, and

$$\mathcal{W} = \sum_{i \in I} \mathcal{W}_i = \{v \mid v : X \to [0, \infty],\, v|_{X_i} \in \mathcal{W}_i \text{ for every } i \in I\}$$

(we identify $v_i \in \mathcal{W}_i$ with a function on X taking $v_i = 0$ on $X \setminus X_i$), then $(X, \mathcal{W})$ is a balayage space. To see this it suffices to take $\mathcal{U} = \bigcup_{i \in I} \mathcal{U}_i$ ($\mathcal{U}_i$ being a base of regular sets for the balayage space $(X_i, \mathcal{W}_i)$) and to extend the harmonic kernels H_U, $U \in \mathcal{U}_i$, defining $H_U(x, \cdot\,) = \varepsilon_x$ for all $x \in X \setminus X_i$. Of course, for every $i \in I$, the restriction of $(X, \mathcal{W})$ on X_i is $(X_i, \mathcal{W}_i)$.

In Section 9 we shall need the following stability result with respect to increasing limits which is of interest in itself:

PROPOSITION 2.2. *Let $\mathcal{U}$ be a base of relatively compact open sets in X and, for every $n \in \mathbb{N}$, let $(H_U^n)_{U\in\mathcal{U}}$ be a family of (regular) harmonic kernels on X. Suppose that, for every $U \in \mathcal{U}$, the sequence $(H_U^n)_{n\in\mathbb{N}}$ is increasing to a kernel H_U^∞. Then the following are equivalent:*

(1) *$(H_U^\infty)_{U\in\mathcal{U}}$ is a family of harmonic kernels on U.*
(2) *There exists $s \in \mathcal{C}^+(X)$ such that, for every $U \in \mathcal{U}$, the function $H_U^\infty s$ is continuous on X and $H_U^\infty s < s$ on U.*

Proof. (1) $\Rightarrow$ (2): By general properties of a family of harmonic kernels (see [BH86]).

(2) $\Rightarrow$ (1): For every $n \in \mathbb{N} \cup \{\infty\}$, define

$$\mathcal{W}^n := \{v \mid v : X \to [0,\infty],\ v \text{ l.s.c.},\ H_U^n v \le v \text{ for every } U \in \mathcal{U}\}.$$

Then

$$\mathcal{W}^\infty = \bigcap_{n=1}^{\infty} \mathcal{W}^n.$$

By assumption (2), the function s is strongly $\mathcal{W}^\infty$-superharmonic.

If $U, V \in \mathcal{U}$ and $V \subset U$, then $H_V^n H_U^n = H_U^n$ for every $n \in \mathbb{N}$, and hence

$$H_V^\infty H_U^\infty = H_U^\infty.$$

Fix a sequence (ψ_m) in $\mathcal{K}^+(X)$ which is increasing to 1, fix $U \in \mathcal{U}$ and $f \in \mathcal{B}_b^+(X)$ with compact support. Choose $\alpha \in \mathbb{R}_+$ such that $f \le \alpha s$. Then, for every $n \in \mathbb{N}$, the function $H_U^n f$ is continuous on U and the function $H_U^n(\alpha s - f) = \sup_m H_U^n(\psi_m(\alpha s - f))$ is l.s.c. on U. So the increasing limits $H_U^\infty f$ and $H_U^\infty(\alpha s - f)$ are l.s.c. on U. Knowing that their sum $H_U^\infty(\alpha s) = \alpha H_U^\infty s$ is continuous on U we obtain continuity of $H_U^\infty f$ and $H_U^\infty(\alpha s - f)$ on U. Now suppose that f is even continuous, i.e., that $f \in \mathcal{K}^+(X)$. Then we have the corresponding continuity properties on X. In particular, we see that $H_U^\infty f \in \mathcal{K}(X)$.

So we already know that $(H_U^\infty)_{U\in\mathcal{U}}$ has the properties (H_5'), (H_2), (H_3), and (H_4').

It remains to show that (H_1) is satisfied. So fix $x \in X$. Assume first that, for every $\varphi \in \mathcal{K}(X)$,

$$\lim_{V \downarrow \{x\}} H_V^1 \varphi(x) = \varphi(x).$$

Fix $\varphi_1 \in \mathcal{K}^+(X)$ and choose $\alpha \in \mathbb{R}_+$, $\varphi_2 \in \mathcal{K}^+(X)$ such that $\varphi_1 + \varphi_2 \leq \alpha s$, $(\varphi_1 + \varphi_2)(x) = \alpha s(x)$. Then

$$\liminf_{V \downarrow \{x\}} H_V^\infty \varphi_j(x) \geq \lim_{V \downarrow \{x\}} H_V^1 \varphi_j(x) = \varphi_j(x), \quad j = 1, 2$$

and, for every $x \in V \in \mathcal{U}$,

$$H_V^\infty \varphi_1(x) + H_V^\infty \varphi_2(x) \leq H_V^\infty(\alpha s)(x) \leq \alpha s(x) = \varphi_1(x) + \varphi_2(x).$$

Therefore

$$\lim_{V \downarrow \{x\}} H_V^\infty \varphi_j(x) = \varphi_j(x), \quad j = 1, 2.$$

Finally, define

$$r_1 = \inf\{v \in \mathcal{W}^1 : v(x) \geq 1\}, \quad r_\infty = \inf\{v \in \mathcal{W}^\infty : v(x) \geq 1\},$$

and suppose that r_1 is l.s.c. at x. Since $\mathcal{W}^\infty$ is contained in $\mathcal{W}^1$, we have $r_1 \leq r_\infty$. Moreover, obviously $r_\infty \leq s/s(x)$. Therefore

$$1 = \liminf_{y \to x} r_1(y) \leq \liminf_{y \to x} r_\infty(y) \leq \liminf_{y \to x} s(y)/s(x) = 1 = r_\infty(x),$$

i.e., r_∞ is l.s.c. at x. $\square$

Given a balayage space $(X, \mathcal{W})$, a kernel K_X on X is called a *potential kernel* provided

$$K_X f \in \mathcal{P}(X) \cap \mathcal{H}(X \setminus \operatorname{supp}(f)) \quad \text{for } f \in \mathcal{B}_b^+(X) \text{ with compact support.} \tag{2.3}$$

For $\varphi \in \mathcal{B}^+(X)$ let M_φ denote the multiplication operator $f \mapsto \varphi f$. It follows immediately from the definition that $K_X M_\varphi$ is a potential kernel on X if K_X is a potential kernel on X and $\varphi \in \mathcal{B}^+(X)$ is locally bounded.

Moreover, for every potential kernel K_X, a general minimum principle implies that $v \geq K_X f$ whenever $v \in \mathcal{W}$ and $f \in \mathcal{B}^+(X)$ such that $v \geq K_X f$ on $\operatorname{supp}(f)$.

For every $U \in \mathcal{U}$, the equation

$$K_U \varphi := K_X \varphi - H_U K_X \varphi \quad (\varphi \in \mathcal{K}(X))$$

defines a kernel K_U on X such that

$$K_X = K_U + H_U K_X \tag{2.4}$$

and

$$K_U f = K_U(1_U f) \in \mathcal{C}_0(U) \cap {}^*\mathcal{H}^+(U) \quad \text{for every } f \in \mathcal{B}_b^+(X). \tag{2.5}$$

In particular, K_U may be regarded as a kernel on U. Furthermore,

$$K_U = K_V + H_V K_U \quad \text{for all } U, V \in \mathcal{U} \text{ with } V \subset U. \tag{2.6}$$

(All this follows immediately from (H_2'), (H_3), and (H_4).)

Remarks 2.3. 1. If we have a Green function G_X for X, then $K_X f = G_X^{f\mu}$ for some measure $\mu \geq 0$ on X and $K_U f = G_U^{f\mu}$ where $G_U(\,\cdot\,, y) = G_X(\,\cdot\,, y) - H_U G_X(\,\cdot\,, y)$ for $y \in X$, $U \in \mathcal{U}$.

2. For every $p \in \mathcal{P}(X)$, there exists a unique potential kernel K_X^p such that $K_X^p 1 = p$ (see [BH86, p. 75]). It is called the *potential kernel associated with* p.

3. Conversely, for every potential kernel K_X, there exists $p \in \mathcal{P}(X)$ and a strictly positive function $\varphi \in \mathcal{C}^+(X)$ such that

$$K_X = K_X^p M_\varphi.$$

Indeed, fix a sequence (ψ_n) in $\mathcal{K}^+(X)$ such that $X = \bigcup_{n=1}^\infty \{\psi_n > 0\}$. Since $p_n := K_X \psi_n \in \mathcal{P}(X)$, we may choose reals $\alpha_n > 0$, $n \in \mathbb{N}$, such that

$$\psi := \sum_{n=1}^\infty \alpha_n \psi_n \in \mathcal{C}^+(X), \quad p := \sum_{n=1}^\infty \alpha_n p_n \in \mathcal{P}(X).$$

Obviously, $K_X \psi = p$ and hence $K_X M_\psi = K_X^p$ by Remarks 2.3, 2. So $\varphi := 1/\psi$ has the desired properties.

4. If K_X is a potential kernel on X, then every K_U, $U \in \mathcal{U}$, is a potential kernel on U. For the converse, i.e., for the construction of K_X from a compatible family of potential kernels $(K_U)_{U \in \mathcal{U}}$, see Section 10.

§3. Parabolic balayage spaces

Extending the notion used in [HH88] for harmonic spaces let us say that the balayage space $(X, \mathcal{W})$ is *parabolic*, if for every non-empty compact subset C of X there exists $x \in C$ such that $\liminf_{y \to x} R_{1_C}(y) = 0$. To get equivalent properties we shall need the following result on compactness of operators K_X^q which is of independent interest:

Lemma 3.1. *Suppose that there exists a strictly positive bounded function in $\mathcal{W}$ and let $p \in \mathcal{P}(X)$ such that p is harmonic outside a compact set C. Then K_X^p is a compact operator on $\mathcal{B}_b(X)$.*

Proof (cf. also [Han81, p. 504]). Let $K := K_X^p$ and let us fix $w \in \mathcal{W}$ such that $0 < w \leq 1$. There exists $\alpha > 0$ such that $p \leq \alpha w$ on C and hence $p \leq \alpha w$ on X. So p is bounded. We intend to show first that the subset $\{Kf : f \in \mathcal{B}(X),\, 0 \leq f \leq 1\}$ of $\mathcal{P}_b(X)$ is equicontinuous. Fix $x \in X$, $\varepsilon > 0$, and let L be a compact neighborhood of x. By Dini's theorem, there exists an open neighborhood U of x in L such that $K1_{U\setminus\{x\}} < \varepsilon$ on L. For every $f \in \mathcal{B}(X)$ such that $0 \leq f \leq 1$,

$$Kf = f(x)K1_{\{x\}} + K(1_{U\setminus\{x\}}f) + K(1_{U^c}f)$$

where $K1_{\{x\}}$ is continuous (it vanishes if $\{x\}$ is semi-polar), $0 \leq K(1_{U\setminus\{x\}}f) < \varepsilon$ on L, and the functions $K(1_{U^c}f)$ are equicontinuous on U, since they are harmonic on U and bounded by p. So there exists a neighborhood V of x in U such that, for every $f \in \mathcal{B}(X)$ with $0 \leq f \leq 1$,

$$|Kf - Kf(x)| < 3\varepsilon \quad \text{on } V.$$

Fix a sequence (f_n) in $\mathcal{B}(X)$ such that $0 \leq f_n \leq 1$ for every $n \in \mathbb{N}$. By our preceding considerations, there exist a subsequence (g_n) of (f_n) such that the sequence (Kg_n) is locally uniformly convergent on X. Fix $\delta > 0$. There exists a natural n_0 such that, for all $n, m \geq n_0$,

$$|Kg_n - Kg_m| < \delta w \quad \text{on } C.$$

Fix $n, m \geq n_0$. Having $Kg_n \leq \delta w + Kg_m$ on C and knowing that Kg_n is harmonic outside C, we conclude that $Kg_n \leq \delta w + Kg_m$ on X. Similarly, $Kg_m \leq \delta w + Kg_n$ on X. Thus

$$|Kg_n - Kg_m| \leq \delta w \leq \delta \quad \text{on } X.$$

∎

Remark 3.2. If follows easily that for every potential kernel K_X and for every $U \in \mathcal{U}$ (even for every relatively compact open U in X) the kernel K_U is a compact operator on $\mathcal{B}_b(U)$.

Theorem 3.3. *Suppose that there exists a strictly positive bounded function in $\mathcal{W}$ and let $p \in \mathcal{P}(X)$ be strongly superharmonic. Then the following statements are equivalent*:

(1) *$(X, \mathcal{W})$ is parabolic.*
(2) *For every $q \in \mathcal{P}(X)$ and for every non-empty compact subset C of X, there exists $x \in C$ such that $K_X^q 1_C(x) = 0$.*
(2′) *For every non-empty compact subset C of X, there exists $x \in C$ such that $K_X^p 1_C(x) = 0$.*
(3) *For every $q \in \mathcal{P}_b(X)$ such that K_X^q is a compact operator on $\mathcal{B}_b(X)$, the operator $I - K_X^q$ is invertible.*
(3′) *For every compact subset C of X and for every $\alpha > 0$, the operator $I - \alpha K_X^p M_{1_C}$ on $\mathcal{B}_b(X)$ is invertible.*

Proof. (1) ⇒ (2): Fix $q \in \mathcal{P}(X)$ and a non-empty compact subset C of X. There exists $\alpha > 0$ such that $\alpha q \le 1$ on C and hence $\alpha K_X^q 1_C \le R_{1_C}$. By (1), there exists $x \in C$ such that $\liminf_{y \to x} R_{1_C}(y) = 0$ and therefore

$$\alpha K_X^q 1_C(x) = \lim_{y \to x} \alpha K_X^q 1_C(y) \le \liminf_{y \to x} R_{1_C}(y) = 0$$

whence $K_X^q 1_C(x) = 0$.

(2) ⇒ (2′): Trivial.

(2′) ⇒ (1): Suppose that there is a non-empty compact subset C of X such that $\liminf_{y \to x} R_{1_C}(y) > 0$ for every $x \in C$. Then there exists a compact neighborhood C' of C such that $R_{1_C} > 0$ on C'. Define $q' := K_X^p 1_{C'}$. Since p is strongly superharmonic, we know that $q' > 0$ on the interior of C' whence $\beta q' \ge 1$ on C for some $\beta > 0$. This implies that $\beta q' \ge R_{1_C}$. In particular, $q' > 0$ on C'.

(2) ⇒ (3): Fix $q \in \mathcal{P}_b(X)$ such that $K := K_X^q$ is a compact operator on $\mathcal{B}_b(X)$. Assume that $I - K$ is not invertible. Then there exists a function $f \in \mathcal{B}_b(X) \setminus \{0\}$ such that $f = Kf$, and we may assume without loss of generality that $|f| \le 1$ and $\{f > 0\} \ne \emptyset$. Since the kernel K is a compact operator on $\mathcal{B}_b(X)$, there exist a real $\varepsilon > 0$ and a compact subset C of $\{f \ge \varepsilon\}$ such that

$$K1_{\{0<f<\varepsilon\}} < 1/2 \quad \text{and} \quad K1_{\{f \ge \varepsilon\} \setminus C} < \varepsilon/2.$$

By (2), there exists $x \in C$ such that $K1_C(x) = 0$ and therefore

$$\varepsilon \le f(x) = Kf(x) \le K(f1_{\{f>0\}})(x) \le \varepsilon K1_{\{0<f<\varepsilon\}}(x) + K1_{\{f \ge \varepsilon\} \setminus C}(x) < \varepsilon.$$

This contradiction shows that $I - K$ is invertible.

(3) ⇒ (3′): Trivial, since, for every compact subset C of X, $K_X^p M_{1_C}$ is the operator K_X^q for $q := K_X^p 1_C \in \mathcal{P}_b(X)$ (see Remarks 2.3, 2) and K_X^q is compact by Lemma 3.1.

(3′) ⇒ (2′): Suppose that there exists a non-empty compact subset C of X such that $K_X^p 1_C > 0$ on C. Then there exists a real $\gamma > 0$ such that $\gamma K_X^p 1_C \ge 1$ on C. Defining $q := \gamma K_X^p 1_C$ we already noted before that $K_X^q = \gamma K_X^p M_{1_C}$. In particular, $K_X^q 1 = q \ge 1$ on C and $K_X^q 1_{C^c} = 0$. Therefore $(K_X^q)^n 1 \ge 1$ on C whence $\sum_{n=0}^\infty (K_X^q)^n 1 = \infty$ on C. Thus the following lemma implies that (3′) does not hold. □

LEMMA 3.4. *Let K be a bounded kernel on X and $\gamma > 0$ such that $I - \alpha K$ is invertible for every $0 < \alpha \le \gamma$. Then $(I - \gamma K)^{-1} = \sum_{n=0}^\infty (\gamma K)^n$.*

Proof. Let

$$\beta := \sup\{\alpha \in [0, \gamma] : (I - \alpha K)^{-1} f \ge 0 \text{ for every } f \in \mathcal{B}_b^+(X)\}.$$

By continuity, $(I - \beta K)^{-1} f \ge 0$ for every $f \in \mathcal{B}_b^+(X)$. So

$$(I - \beta K)^{-1} = \sum_{n=0}^\infty (\beta K)^n$$

by [HH88, Lemma 1.3]. If $\beta < \gamma$, then by continuity again, there exists $\beta < \beta' \le \gamma$ such that

$$(I - \beta' K)^{-1} = \sum_{n=0}^\infty (\beta' K)^n$$

and therefore $(I - \beta' K)^{-1} f \ge 0$ for every $f \in \mathcal{B}_b^+(X)$. This contradicts the definition of β. Thus $\beta = \gamma$ and the proof is finished. □

§4. First modification by transitions

In the following $(X, \mathcal{W})$ will always denote a balayage space associated with a family $(H_U)_{U \in \mathcal{U}}$ of regular harmonic kernels and K_X a potential kernel for $(X, \mathcal{W})$. Moreover, we fix a kernel T on X and assume that, for some sequence (W_n) of open sets increasing to X,

$$(4.1) \qquad T1_{W_n} < \infty, \quad K_X(1_{W_n} T 1_{W_n}) \in \mathcal{C}(X) \quad (n \in \mathbb{N}).$$

Such a kernel T will be called an *admissible transition kernel*.

Remarks 4.1. 1. If the sets W_n are relatively compact and the functions $T1_{W_n}$ are bounded on W_n, then (4.1) is already a consequence of (2.3).

So every kernel T on X such that $T\varphi$ is locally bounded for every $\varphi \in \mathcal{K}(X)$ is an admissible transition kernel.

2. It is easily seen that (4.1) implies that, for all $U \in \mathcal{U}$,

$$K_U(Tf) \in \mathcal{C}_0(U) \quad f \in \mathcal{B}_b(X) \text{ with compact support.} \tag{4.2}$$

Indeed, choosing $n \in \mathbb{N}$ such that $\overline{U} \subset W_n$ and $\operatorname{supp}(f) \subset W_n$, the lower semi-continuity of the functions $K_X(1_{W_n}Tf^{\pm})$, $K_X(1_{W_n}T(\|f\|_\infty 1_{W_n} - f^{\pm}))$ and the continuity of the sum $\|f\|_\infty K_X(1_{W_n}T(1_{W_n}))$ implies that the functions $K_X(1_{W_n}Tf^{\pm})$ are continuous. Thus by (2.6)

$$\begin{aligned} K_U(Tf) &= K_X(Tf) - H_U K_X(Tf) \\ &= K_X(1_{W_n}Tf) - H_U K_X(1_{W_n}Tf) \in \mathcal{C}_0(U) \end{aligned}$$

(the harmonicity of $K_X(1_{W_n^c}Tf)$ on W_n implies that $H_U K_X(1_{W_n^c}Tf) = K_X(1_{W_n^c}Tf)$).

3. Using lifting of potentials (see Remarks 2.3, 4) it can be shown that, conversely, (4.2) implies (4.1).

Let $\mathcal{U}^T$ be the set of all $U \in \mathcal{U}$ such that T is a transition from U to the complement of U, i.e.,

$$\mathcal{U}^T = \{U \in \mathcal{U} : 1_U T 1_U = 0\}.$$

In this section we shall assume that

$$\mathcal{U}^T \text{ is a base of } X \tag{4.3}$$

(in Section 9 we shall deal with the general case by approximation). We define

$$K_U^T := K_U T, \quad H_U^T := H_U + K_U^T \quad (U \in \mathcal{U}^T)$$

(cf. definition (1.7)) and

$$\mathcal{W}^T := \{v \mid v : X \to [0,\infty] \text{ l.s.c., } H_U^T v \le v \text{ for every } U \in \mathcal{U}^T\}.$$

By Remarks 2.1, 3,

$$\mathcal{W}^T \subset \mathcal{W}.$$

Let us check that most of the axioms of a family of harmonic kernels are satisfied by $(H_U^T)_{U \in \mathcal{U}^T}$ without any further assumption: Fix $U, V \in \mathcal{U}^T$, $V \subset U$. Then

$$K_V^T 1_U = K_V T 1_U = K_V(1_V T 1_U) = 0, \tag{4.4}$$

hence (taking $V = U$)

$$H_U^T 1_U = H_U 1_U = 0.$$

Let $f \in \mathcal{B}_b(X)$ with compact support. Then

$$H_U^T f = H_U f = f \quad \text{on } U^c \tag{4.5}$$

showing that $H_U^T(x, \cdot) = \varepsilon_x$ for every $x \in U^c$. Since $K_U^T f \in \mathcal{C}_0(U)$, we obtain by (H_3) that $H_U^T f$ is continuous on U. And if $f \in \mathcal{K}(X)$, then $H_U^T f \in \mathcal{K}(X)$ by (H_4'). Thus the family $(H_U^T)_{U \in \mathcal{U}^T}$ satisfies (H_3) and (H_4').

Moreover, by (4.4) and (4.5), $K_V^T H_U^T f = K_V^T(1_{U^c} H_U^T f) = K_V^T(1_{U^c} f) = K_V^T f$, i.e.,

$$K_V^T H_U^T = K_V^T. \tag{4.6}$$

Since $H_V H_U = H_U$ by (H_2), we obtain by (4.6) and (2.6) that

$$\begin{aligned} H_V^T H_U^T &= H_V(H_U + K_U^T) + K_V^T H_U^T = H_V H_U + H_V K_U^T + K_V^T \\ &= H_U + K_U^T = H_U^T. \end{aligned}$$

So $(H_U^T)_{U \in \mathcal{U}^T}$ satisfies (H_2) as well.

Given $x \in U$ and $\varphi \in \mathcal{K}^+(X)$, we obtain by (2.6) that $\lim_{V \downarrow \{x\}} K_V^T \varphi(x) = 0$, since $\lim_{V \downarrow \{x\}} H_V K_U(T\varphi)(x) = K_U(T\varphi)(x)$. Hence

$$\lim_{V \downarrow \{x\}} H_V^T \varphi(x) = \varphi(x) \quad \text{if} \quad \lim_{V \downarrow \{x\}} H_V \varphi(x) = \varphi(x).$$

Moreover, defining

$$r := R_{1_{\{x\}}}, \quad r^T := R^T_{1_{\{x\}}} = \inf\{v \in \mathcal{W}^T : v(x) \geq 1\}$$

we have $r^T \geq r$, since $\mathcal{W}^T \subset \mathcal{W}$. Hence $\liminf_{y \to x} r^T(y) \geq \liminf_{y \to x} r(y) = 1$, if r is l.s.c. at x. And then r^T is l.s.c. at x provided there exists $v \in \mathcal{W}^T$ with $v(x) < \infty$ (since then $v/v(x) \geq r^T$, $1 \geq r^T(x)$).

Thus we have the following result:

Theorem 4.2. *If $\mathcal{U}^T$ is a base of X, the following properties are equivalent:*

(1) *$(X, \mathcal{W}^T)$ is a balayage space (i.e., $(H_U^T)_{U \in \mathcal{U}^T}$ is a family of harmonic kernels on X).*
(2) *There exists a strongly $\mathcal{W}^T$-superharmonic function $s \in \mathcal{C}^+(X)$.*

Remark 4.3. Let T' be a kernel on X such that $T' \le T$, $\mathcal{U}^T$ is a base of X, and $(X, \mathcal{W}^T)$ is a balayage space. Then T' is admissible and every $\mathcal{W}^T$-strongly superharmonic function is obviously $\mathcal{W}^{T'}$-strongly superharmonic. So Theorem 4.2 implies that $(X, \mathcal{W}^{T'})$ is a balayage space as well.

COROLLARY 4.4. *Suppose that $\mathcal{U}^T$ is a base of X and that there exist $s \in \mathcal{W}$ and $u \in \mathcal{B}^+(X)$ such that*

$$v := s + K_X u \in \mathcal{C}(X), \quad Tv \le u$$

and, for every $U \in \mathcal{U}^T$,

$$\{H_U s < s\} \cup \{K_U(u - Tv) > 0\} = U.$$

Then $(X, \mathcal{W}^T)$ is a balayage space and v is strongly $\mathcal{W}^T$-superharmonic.

Proof. It suffices to note that, for every $U \in \mathcal{U}^T$,

$$v - H_U^T v = v - H_U v - K_U(Tv) = s - H_U s + K_U(u - Tv) > 0 \quad \text{on } U.$$

∎

Remarks 4.5. 1. For a version not assuming that $\mathcal{U}^T$ is a base see Theorem 9.2.

2. If $K_X = K_X^p$ for some strongly superharmonic $p \in \mathcal{P}(X)$, then $TK_X u < u$ implies that taking $s = 0$ we have $K_U(u - Tv) > 0$ on $U \in \mathcal{U}$.

3. For some applications (see e.g. Corollary 7.9) it will be useful to keep in mind that, given any strictly positive locally bounded function $\varphi \in \mathcal{B}(X)$, we may replace the potential kernel K_X by the potential kernel $f \mapsto K_X(\varphi f)$ and the transition kernel T by the transition kernel $f \mapsto (Tf)/\varphi$ without changing $(X, \mathcal{W}^T)$.

COROLLARY 4.6. *Suppose that $\mathcal{U}^T$ is a base of X, K_X is associated with $p \in \mathcal{P}(X)$, and that for some $s \in \mathcal{W} \cap \mathcal{C}(X)$ the function $v := p + s$ is strongly superharmonic and $Tv < 1$. Then $(X, \mathcal{W}^T)$ is a balayage space and v is strongly $\mathcal{W}^T$-superharmonic.*

Proof. Fix $U \in \mathcal{U}$ and $x \in U$. By assumption, $H_U v(x) < v(x)$. Suppose that $H_U s(x) = s(x)$. Then $H_U p(x) < p(x)$, i.e., $K_U 1(x) > 0$. Since $1 - Tv > 0$, this implies that $K_U(1 - Tv)(x) > 0$. So the statement follows from Corollary 4.4. ∎

If $(X, \mathcal{W}^T)$ is a balayage space, then, for every $U \in \mathcal{U}^T$, H_U^T is the kernel solving the Dirichlet problem for U with respect to $(X, \mathcal{W}^T)$. We may, however, solve the Dirichlet problem with respect to $(X, \mathcal{W}^T)$ for *any* $U \in \mathcal{U}$ (if we wanted to we could even solve it for any open set U in X, see [BH86, VII.2]). This leads to the larger family $(H_U^T)_{U \in \mathcal{U}}$ where H_U^T for arbitrary $U \in \mathcal{U}$ can be characterized in the following way:

PROPOSITION 4.7. *Suppose that $(X, \mathcal{W}^T)$ is a balayage space. Then, for every $U \in \mathcal{U}$, the harmonic kernel H_U^T for U with respect to $(X, \mathcal{W}^T)$ has the following property*:

For every $\varphi \in \mathcal{K}^+(X)$, the function $H_U^T\varphi$ is the unique function h in $\mathcal{K}^+(X)$ such that

$$h - K_U^T h = H_U \varphi.$$

Proof. 1. Fix $\varphi \in \mathcal{K}^+(X)$ and define $h := H_U^T\varphi$. Then $h \in \mathcal{K}^+(X)$ and hence $K_U^T h \in \mathcal{C}_0(U)$. So

$$g := h - K_U^T h \in \mathcal{K}(X), \quad g = \varphi \ \text{ on } U^c.$$

For every $V \in \mathcal{U}^T$ with $\overline{V} \subset U$,

$$h = H_V^T h = H_V h + K_V^T h,$$

hence

$$g = h - K_V^T h - H_V K_U^T h = H_V (h - K_U^T h)$$

is harmonic on V. Thus g is harmonic on U, $g = H_U \varphi$.

2. Now let h be any function in $\mathcal{K}^+(X)$ such that

$$h - K_U^T h = H_U \varphi.$$

Then $h = \varphi$ on U^c and, for every $V \in \mathcal{U}^T$ with $\overline{V} \subset U$,

$$H_V^T h = H_V h + K_V^T h = H_V H_U \varphi + H_V K_U^T h + K_V^T h = H_U \varphi + K_U^T h = h.$$

Thus $h = H_U^T \varphi$. ∎

Remark 4.8. Assuming that $(X, \mathcal{W}^T)$ is a balayage space we may show in the same way that, for every $\varphi \in \mathcal{K}(X)$, $H_U^T \varphi$ is the unique function $h \in \mathcal{K}(X)$ such that $K_U^T |h| \in \mathcal{C}_0(U)$ and $h - K_U^T h = H_U \varphi$.

PROPOSITION 4.9. *Let v be a positive numerical function on X. Then $v \in \mathcal{W}^T$ if and only if there exists a function $w \in \mathcal{W}$ such that $v = K_X^T v + w$. In particular, the fine topologies for $(X, \mathcal{W})$ and $(X, \mathcal{W}^T)$ coincide.*

Proof. Suppose first that $w \in \mathcal{W}$ and $v = K_X^T v + w$. Then v is l.s.c. Fix $U \in \mathcal{U}^T$ and $x \in U$. We have to show that $H_U^T v(x) \le v(x)$. To that end we may assume that $v(x) < \infty$ and hence $H_U K_X^T v(x) \le K_X^T v(x) \le v(x) < \infty$. Then

$$\begin{aligned} H_U^T v(x) &= H_U v(x) + K_U^T v(x) = H_U v(x) - H_U K_X^T v(x) + K_X^T v(x) \\ &= H_U w(x) + K_X^T v(x) \le w(x) + K_X^T v(x) = v(x). \end{aligned}$$

Thus $v \in \mathcal{W}^T$.

Suppose now conversely that $v \in \mathcal{W}^T$. Then $v \in \mathcal{W}$, so v is finely continuous. Let us choose an increasing sequence (W_n) of relatively compact open sets satisfying (4.1). Defining

$$\varphi_n := 1_{W_n} T(1_{W_n} \inf(v, n)) \quad (n \in \mathbb{N})$$

we then have $K_X \varphi_n \in \mathcal{P}(X)$ for every $n \in \mathbb{N}$ and

$$K_X \varphi_n \uparrow K_X^T v, \quad K_U \varphi_n \uparrow K_U^T v$$

for every $U \in \mathcal{U}^T$. Define

$$w_n := v - K_X \varphi_n \quad (n \in \mathbb{N}).$$

For every $U \in \mathcal{U}^T$,

$$H_U w_n + K_X \varphi_n = H_U v + K_U \varphi_n \le H_U v + K_U^T v = H_U^T v \le v,$$

i.e., $H_U w_n \le w_n$. Since w_n is l.s.c. and $w_n \ge -K_X \varphi_n$, we therefore obtain that $w_n \in \mathcal{W}$. The sequence (w_n) is decreasing and the function w defined by

$$w(x) = \text{f-}\liminf_{y \to x} \inf_n w_n(y), \quad x \in X,$$

is contained in $\mathcal{W}$. Since the functions v and $K_X^T v$ are finely continuous and obviously

$$v = K_X^T v + \inf_n w_n,$$

we finally obtain that $v = K_X^T v + w$. ∎

§5. Perturbation of balayage spaces

In order to get further possibilities for transitions let us briefly discuss perturbation of $(X, \mathcal{W})$. To that end we fix a real function $k \in \mathcal{B}(X)$ such that, for every $U \in \mathcal{U}$,

$$K_U|k| \in \mathcal{C}_0(U).$$

Such a function will be called a *Kato function* (with respect to K_X). Let us note that, given $U \in \mathcal{U}$, the kernels

$$K_U M_{k^\pm} : f \longmapsto K_U(k^\pm f)$$

are the potential kernels associated with $K_U k^\pm$ (see Remarks 2.3, 2).

LEMMA 5.1. *For every $U \in \mathcal{U}$, the mapping $I + K_U M_{k^+}$ is a bijection on $\mathcal{B}_b(X)$. For every bounded $s \in {}^*\mathcal{H}^+(U)$,*

$$0 \le (I + K_U M_{k^+})^{-1} s \le s \quad on\ X, \quad (I + K_U M_{k^+})^{-1} s > 0 \quad on\ \{s > 0\}.$$

Proof. Obviously, $(I + K_U M_{k^+})f = f$ on U^c, $(I + K_U M_{k^+})f = (I + K_U M_{k^+})(1_U f)$ on U, and the claim follows as for harmonic spaces (see [BHH87, p. 104], or [HM90, p. 558]). ∎

In particular, for every $U \in \mathcal{U}$, the operator

$$L_U := (I + K_U M_{k^+})^{-1} K_U M_{k^-}$$

on $\mathcal{B}_b(X)$ defines a kernel on X. Obviously, L_U lives on U, i.e., $L_U 1_U = 0$ on U^c and $L_U 1_{U^c} = 0$. As for harmonic spaces we obtain (see [HM90]):

LEMMA 5.2. *For every $U \in \mathcal{U}$, the following statements are equivalent:*

(1) *The operator $I - L_U$ is invertible on $\mathcal{B}_b(X)$ and $(I - L_U)^{-1} f \ge 0$ for every $f \in \mathcal{B}_b^+(X)$.*
(2) *$\sum_{n=0}^\infty L_U^n 1$ is bounded on U.*

*If (2) holds, then U is called k-*bounded *and*

$$(I + K_U M_k)^{-1} = \sum_{n=1}^\infty L_U^n (I + K_U M_{k^+})^{-1}.$$

THEOREM 5.3. *$\big((I+K_UM_{k^+})^{-1}H_U\big)_{U\in\mathcal{U}}$ is a family of harmonic kernels on X.*

More generally:

THEOREM 5.4. *Suppose that there exist $s\in\mathcal{W}$ and $u\in\mathcal{B}^+(X)$ such that*

$$v:=s+K_Xu\in\mathcal{C}(X),\quad 0\le u+kv,$$

and, for every $U\in\mathcal{U}$, $\{H_Us<s\}\cup\{K_U(u+kv)>0\}=U$. Then every $U\in\mathcal{U}$ is k-bounded and defining

$$\widetilde{H}_U:=(I+K_UM_k)^{-1}H_U\quad(U\in\mathcal{U}) \tag{5.1}$$

and

$$\widetilde{\mathcal{W}}:=\{w\mid w:X\to[0,\infty]\ \text{l.s.c.},\ \widetilde{H}_Uw\le w\ \text{for every}\ U\in\mathcal{U}\} \tag{5.2}$$

the family $(\widetilde{H}_U)_{U\in\mathcal{U}}$ is a family of harmonic kernels on X, the pair $(X,\widetilde{\mathcal{W}})$ is a balayage space, and v is strongly $\widetilde{\mathcal{W}}$-superharmonic.

Proof. For the moment fix $U\in\mathcal{U}$ and define

$$f:=1_Uv-L_Uv=1_Uv-L_U(1_Uv).$$

By induction $1_Uv=\sum_{n=0}^{m-1}L_U^nf+L_U^m(1_Uv)$ for every $m\in\mathbb{N}$ and therefore

$$\sum_{n=0}^{\infty}L_U^nf\le 1_Uv. \tag{5.3}$$

To prove that $\inf f(U)>0$ we note that

$$\begin{aligned}(I+K_UM_{k^+})f&=1_Uv+K_UM_{k^+}v-K_UM_{k^-}v\\&=1_Uv+K_U(kv)=1_Us+H_UK_Xu+K_U(u+kv)\end{aligned}$$

is a bounded function in ${}^*\mathcal{H}^+(U)$ and strictly positive on U. So we conclude by Lemma 5.1 that $f>0$ on U. Moreover, $L_Uv\in\mathcal{C}_0(U)$ and $\inf v(U)>0$. Therefore $\inf f(U)>0$, and (5.3) shows that U is k-bounded. We define a kernel $\widetilde{H}_U$ by

$$\widetilde{H}_U:=(I+K_UM_k)^{-1}H_U=\sum_{n=0}^{\infty}L_U^n(I+K_UM_{k^+})^{-1}H_U. \tag{5.4}$$

and observe that

$$(I+K_UM_k)(v-\widetilde{H}_Uv) = v+K_U(kv)-H_Uv = (s-H_Us)+K_U(u+kv) =: t$$

is a bounded function in ${}^*\mathcal{H}^+(U)$ which is strictly positive on U. Applying Lemma 5.1 once more we obtain that

$$v-\widetilde{H}_Uv = (I+K_UM_k)^{-1}t \geq (I+K_UM_{k^+})^{-1}t > 0.$$

In particular, $(\widetilde{H}_U)_{U\in\mathcal{U}}$ satisfies (H_5').

Obviously, $\widetilde{H}_U 1_U = 0$ and $\widetilde{H}_U(x,\,\cdot\,) = \varepsilon_x$ for all $U \in \mathcal{U}$ and $x \in U^c$. If $f \in \mathcal{B}_b(X)$ with compact support, then $\widetilde{H}_U f \in \mathcal{B}_b(X)$, hence $K_U(k\widetilde{H}_Uf) \in \mathcal{C}_0(U)$. So the equality

$$\widetilde{H}_Uf + K_U(k\widetilde{H}_Uf) = H_Uf$$

immediately implies that $(\widetilde{H}_U)_{U\in\mathcal{U}}$ satisfies (H_3) and (H_4'). Applied to functions in $\mathcal{K}(X)$ we have for all $U, V \in \mathcal{U}$ with $V \subset U$

$$\begin{aligned}(I+K_VM_k)\widetilde{H}_U &= \widetilde{H}_U + (K_U - H_VK_U)M_k\widetilde{H}_U\\ &= H_U - H_VK_UM_k\widetilde{H}_U = H_V(H_U - K_UM_k\widetilde{H}_U) = H_V\widetilde{H}_U,\end{aligned}$$

i.e.,

$$\widetilde{H}_U = (I+K_VM_k)^{-1}H_V\widetilde{H}_U = \widetilde{H}_V\widetilde{H}_U.$$

So $(\widetilde{H}_U)_{U\in\mathcal{U}}$ satisfies (H_2').

To show that (H_1) holds let us fix $x \in X$ and assume first that $\lim_{U\downarrow\{x\}} H_U\varphi(x) = \varphi(x)$ for every $\varphi \in \mathcal{K}(X)$. Let W be a neighborhood of x. Then, for every $U \in \mathcal{U}$ with $\overline{U} \subset W$,

$$K_U(|k|\widetilde{H}_Uv) \leq K_U(|k|v) \leq \sup(v(W))K_U|k|$$

and $\lim_{U\downarrow\{x\}} \|K_U|k|\|_\infty = 0$. So we conclude that, for every $\varphi \in \mathcal{K}(X)$,

$$\lim_{U\downarrow\{x\}} \widetilde{H}_U\varphi(x) = \lim_{U\downarrow\{x\}} H_U\varphi(x) = \varphi(x).$$

By [BH86, Proposition III.2.7], it remains to consider the case where x is ($\mathcal{W}$-)finely isolated. Let

$$\tilde{r} = \inf\{w \in \widetilde{\mathcal{W}} : w(x) \geq 1\}.$$

By Choquet's lemma, there exist $w_n \in \widetilde{\mathcal{W}}$, such that $w_n(x) \geq 1$ for every $n \in \mathbb{N}$ and
$$\hat{\tilde{r}} = \widehat{\inf w_n}.$$

Of course we may assume without loss of generality that $w_{n+1} \leq w_n \leq v/v(x)$ for every $n \in \mathbb{N}$. Define

$$s_n := w_n + K_U(k^+ w_n) \quad (n \in \mathbb{N}).$$

Then s_n is l.s.c. and, for every $V \in \mathcal{U}$ with $\overline{V} \subset U$,

$$\begin{aligned} H_V s_n &= \widetilde{H}_V w_n + K_V(k\widetilde{H}_V w_n) + H_V K_U(k^+ w_n) \\ &\leq w_n + K_V(k^+ w_n) + H_V K_U(k^+ w_n) = s_n, \end{aligned}$$

i.e., $s_n \in {}^*\mathcal{H}^+(U)$. Defining $s := \inf s_n$, we hence know that $\hat{s}^{\mathrm{f}} = \hat{s}$ (see [BH86, p. 58]). Let $w = \inf w_n$. Then $s = w + K_U(k^+ w)$ and the continuity of $K_U(k^+ w)$ implies that

$$\hat{w}^{\mathrm{f}} + K_U(k^+ w) = \hat{s}^{\mathrm{f}} = \hat{s} = \hat{w} + K_U(k^+ w),$$

i.e., $\hat{w}^{\mathrm{f}} = \hat{w}$. Since x is finely isolated, we conclude that

$$\hat{\tilde{r}}(x) = \hat{w}(x) = \hat{w}^{\mathrm{f}}(x) = \operatorname*{f\text{-}liminf}_{y \to x} w(y) = w(x) = 1 = \tilde{r}(x).$$

Thus $\tilde{r}$ is l.s.c. at x. This finishes the proof of Theorem 5.4. $\square$

Theorem 5.3 is a special case: If $k \geq 0$, then we may take $u = 0$ and any strongly superharmonic $s \in \mathcal{C}^+(X)$. But of course we may as well take the preceding proof and omit its first part noting that, by Lemma 5.1, the operators $(I + K_U M_k)^{-1} H_U$, $U \in \mathcal{U}$, yield kernels $\widetilde{H}_U$ and that $\mathcal{W} \subset \widetilde{\mathcal{W}}$ if $k \geq 0$.

Moreover we shall need the following:

PROPOSITION 5.5. *If every $U \in \mathcal{U}$ is k-bounded and $(\widetilde{H}_U)_{U \in \mathcal{U}}$ is a family of harmonic kernels on X, then there exists a (unique) potential kernel $\widetilde{K}_X$ on X with respect to $\widetilde{\mathcal{W}}$ such that*

$$\widetilde{K}_X - \widetilde{H}_U \widetilde{K}_X = (I + K_U M_k)^{-1} K_U \quad \textit{for every } U \in \mathcal{U}.$$

Proof. Define

$$\widetilde{K}_U = (I + K_U M_k)^{-1} K_U \quad (U \in \mathcal{U}).$$

If $U, V \in \mathcal{U}$ with $V \subset U$, we have $I + K_V M_k = I + K_U M_k - H_V K_U M_k$, hence

$$\begin{aligned}
&(I + K_V M_k)(\widetilde{K}_V + \widetilde{H}_V \widetilde{K}_U - \widetilde{K}_U) \\
&\quad = K_V + H_V \widetilde{K}_U - (K_U - H_V K_U M_k \widetilde{K}_U) \\
&\quad = K_V - K_U + H_V(I + K_U M_k)\widetilde{K}_U = K_V - K_U + H_V K_U = 0,
\end{aligned}$$

i.e.,

$$\widetilde{K}_V = \widetilde{K}_U - \widetilde{H}_V \widetilde{K}_U. \tag{5.5}$$

By Remarks 2.3, 4, it therefore suffices to show that every $\widetilde{K}_U$ is a potential kernel on U with respect to $\widetilde{\mathcal{W}}$.

So fix $U \in \mathcal{U}$ and $f \in \mathcal{B}_b^+(U)$. If $V \in \mathcal{U}$ with $\overline{V} \subset U$, then (5.5) implies that $\widetilde{H}_V \widetilde{K}_U f \le \widetilde{K}_U f$ with equality if $f = 0$ on V. If $0 \le h \le \widetilde{K}_U f$ such that h is harmonic on U with respect to $(\widetilde{H}_V)_{V \in \mathcal{U}}$, then $g := h + K_U(kh)$ is harmonic on U and $0 \le g \le K_U f$, hence $g = 0$, $h = 0$. ∎

§6. Perturbation and transitions in balayage spaces

We shall now combine assumptions of Section 4 and Section 5: Let us assume that k is a Kato function on X (with respect to K_X) and that T is an admissible transition kernel on the balayage space $(X, \mathcal{W})$. In this section we shall still assume that $\mathcal{U}^T$ is a base of X (we shall get rid of this assumption in Section 9).

For every k-bounded $U \in \mathcal{U}^T$ we define a kernel $\widetilde{H}_U^T$ by

$$\widetilde{H}_U^T = (I + K_U M_k)^{-1}(H_U + K_U T). \tag{6.1}$$

We shall simply say that $(\widetilde{H}_U^T)_{U \in \mathcal{U}^T}$ is a family of harmonic kernels if every $U \in \mathcal{U}^T$ is k-bounded and $(\widetilde{H}_U^T)_{U \in \mathcal{U}^T}$ is a family of harmonic kernels, and then we define

$$\widetilde{\mathcal{W}}^T := \{v \mid v : X \to [0, \infty] \text{ l.s.c.},\ \widetilde{H}_U^T v \le v \text{ for every } U \in \mathcal{U}^T\}. \tag{6.2}$$

The following result generalizes Corollary 4.4:

THEOREM 6.1. *Suppose that there exist $s \in \mathcal{W}$ and $u \in \mathcal{B}^+(X)$ such that*

$$v := s + K_X u \in \mathcal{C}(X), \quad Tv \le u + kv,$$

and, for every $U \in \mathcal{U}$,

$$\{H_U s < s\} \cup \{K_U(u + kv - Tv) > 0\} = U.$$

Then every $U \in \mathcal{U}$ is k-bounded, $(\widetilde{H}_U^T)_{U \in \mathcal{U}^T}$ is a family of harmonic kernels on X, $(X, \widetilde{\mathcal{W}}^T)$ is a balayage space, and v is strongly $\widetilde{\mathcal{W}}^T$-superharmonic.

Proof. By Theorem 5.4, every $U \in \mathcal{U}$ is k-bounded and $\widetilde{H}_U := (I + K_U M_k)^{-1} H_U$, $U \in \mathcal{U}$, defines a family of harmonic kernels on X. By Proposition 5.5, there exists a potential kernel $\widetilde{K}_X$ with respect to $(\widetilde{H}_U)_{U \in \mathcal{U}}$ such that, for every $U \in \mathcal{U}$,

$$\widetilde{K}_U := \widetilde{K}_X - \widetilde{H}_U \widetilde{K}_X = (I + K_U M_k)^{-1} K_U.$$

Fix $U \in \mathcal{U}$ and let

$$f := v - \widetilde{H}_U^T v = v - (I + K_U M_k)^{-1}(H_U v + K_U(Tv)).$$

Then

$$\begin{aligned} t := (I + K_U M_k) f &= v + K_U(kv) - H_U v - K_U(Tv) \\ &= s - H_U s + K_U(u + kv - Tv) \end{aligned}$$

is a positive superharmonic function on U, hence $f \ge 0$. By assumption $t > 0$ and therefore $f > 0$. The proof is finished by an application of Theorem 4.2. $\square$

COROLLARY 6.2. *Assume that, for every $U \in \mathcal{U}$, the function $K_U 1$ is strictly positive on U. Then the following holds:*

(1) *If $1 \in \mathcal{W}$ and $k > T1$, then the assumptions of Theorem 6.1 are satisfied and 1 is strongly $\widetilde{\mathcal{W}}^T$-superharmonic.*
(2) *If $u \in \mathcal{B}^+(X)$ such that $q := K_X u \in \mathcal{C}(X)$ and $Tq < u + kq$, then the assumptions of Theorem 6.1 are satisfied and q is strongly $\widetilde{\mathcal{W}}^T$-superharmonic.*

Proposition 6.3. *Suppose that* $(\widetilde{H}_U^T)_{U\in\mathcal{U}^T}$ *is a family of harmonic kernels. Then, for every* $U \in \mathcal{U}$, *the harmonic kernel* $\widetilde{H}_U^T$ *for* U *with respect to* $(X, \widetilde{\mathcal{W}}^T)$ *has the following property: For every* $\varphi \in \mathcal{K}^+(X)$, *the function* $\widetilde{H}_U^T\varphi$ *is the unique function* $h \in \mathcal{K}^+(X)$ *such that*

$$h + K_U(kh - Th) = H_U\varphi.$$

Proof (*see the proof of Proposition* 4.7). 1. Fix $\varphi \in \mathcal{K}^+(X)$ and define $h := \widetilde{H}_U^T\varphi$. Then $h \in \mathcal{K}^+(X)$, hence $K_U(kh - Th) \in \mathcal{C}_0(U)$. So

$$g := h + K_U(kh - Th) \in \mathcal{K}(X), \quad g = \varphi \quad \text{on } U^c.$$

For every $V \in \mathcal{U}^T$ with $\overline{V} \subset U$,

$$h = \widetilde{H}_V^T h = (I + K_V M_k)^{-1}(H_V\varphi + K_V(T\varphi))$$

and therefore

$$\begin{aligned} g &= h + K_V(kh) + H_V K_U(kh) - K_U(Th) \\ &= H_V\varphi + K_V(T\varphi) + H_V K_U(kh) - K_U(Th) = H_V(\varphi + K_U(kh - Th)) \end{aligned}$$

is harmonic on V (note that $\varphi = h$ on U^c implies that $T\varphi = Th$ on V, since $1_V T 1_V = 0$). Thus g is harmonic on U, $g = H_U\varphi$.

2. Now let h be any function in $\mathcal{K}^+(X)$ such that

$$h + K_U(kh - Th) = H_U\varphi.$$

Then $h = \varphi$ on U^c and, for every $V \in \mathcal{U}^T$ with $\overline{V} \subset U$,

$$\begin{aligned} (I + K_V M_k)\widetilde{H}_V^T h &= H_V h + K_V^T h = H_V H_U\varphi - H_V K_U(kh - Th) + K_V^T h \\ &= H_U\varphi + K_U(Th) - H_V K_U(kh) = h + K_V(kh), \end{aligned}$$

i.e., $\widetilde{H}_V^T h = h$. Thus $h = \widetilde{H}_U^T\varphi$. ∎

§7. Coupling in direct sums of balayage spaces

In this section we shall first consider general transitions between spaces forming a direct sum and then study the important case of direct sums with the same underlying topological space Y and transition between corresponding points in the copies of Y.

Let $I = \{1, 2, \dots, n\}$, $n \in \mathbb{N}$, or $I = \mathbb{N}$ and let $(X, \mathcal{W})$ be the direct sum of balayage spaces $(X_i, \mathcal{W}_i)$, $i \in I$ (see Section 2). Let K_X be the potential kernel associated with a potential $p \in \mathcal{P}(X)$ and fix an admissible kernel T on X satisfying

$$(7.1) \qquad T(x, X_i) = 0 \quad \text{for every } i \in I \text{ and } x \in X_i.$$

Clearly $\mathcal{U}^T = \mathcal{U} = \bigcup_{i \in I} \mathcal{U}_i$ is a base of X. Sometimes a very coarse consideration of the transitions may already lead to the conclusion that $(X, \mathcal{W}^T)$ is a balayage space: Let $s_0 \in \mathcal{W} \cap \mathcal{C}(X)$ be strongly superharmonic and let us define kernels P and P' on I by

$$\begin{aligned} P(i, \{j\}) &:= \|1_{X_i} T(1_{X_j} p)\|_\infty = \sup_{x \in X_i} \int_{X_j} p(z)\, T(x, dz), \\ \widetilde{P}(i, \{j\}) &:= \|1_{X_i} T(1_{X_j}(s_0 + p))\|_\infty \end{aligned}$$

for $i, j \in I$ where of course $P(i, \{i\}) = 0$ by (7.1). Then Corollary 4.4 leads to the following result:

THEOREM 7.1. *If there exists a positive real function t on I such that $\widetilde{P}t \le t$, then $(X, \mathcal{W}^T)$ is a balayage space.*

Remark 7.2. It is sufficient to know that $Pt < t$ if I is finite and if, moreover, there exists a strictly positive $w \in \mathcal{W}_b$ such that Tw is bounded. Indeed, then there exists $\varepsilon > 0$ such that $Pt + \varepsilon n \|Tw\|_\infty \|t\|_\infty < t$ (n being the number of elements in I), we may choose a strongly $\mathcal{W}$-superharmonic function $s_0 \in \mathcal{W} \cap \mathcal{C}(X)$ with $s_0 \le \varepsilon w$, and obtain that $\widetilde{P}t \le Pt + \varepsilon n \|Tw\|_\infty \|t\|_\infty < t$.

Proof of Theorem 7.1. We define functions s and u on X by

$$s(x) := t(i) s_0(x), \quad u(x) := t(i) \quad (i \in I, x \in X_i)$$

and take $v := s + K_X u$. Then $v \in \mathcal{C}(X)$, s is strongly superharmonic, and, for every $i \in I$ and $x \in X_i$,

$$\begin{aligned} Tv(x) &= \sum_{j \in I} t(j) T(1_{X_j}(s_0 + p))(x) \le \sum_{j \in I} t(j) \widetilde{P}(i, \{j\}) \\ &= \widetilde{P}t(i) \le t(i) = u(x). \end{aligned}$$

The proof is completed by Corollary 4.4. $\square$

EXAMPLE 7.3. Let us consider the example given in the introduction. There we have $I = \{1, 2\}$ and $T(x, \cdot\,) = \varepsilon_{\pi(x)}$, hence $P(i, \{j\}) = (1 - \delta_{ij})\|G_D^{\mu_j}\|_\infty$ so that by assumption $P(1, \{2\})P(2, \{1\}) < 1$. If $P(1, \{2\}) > 0$, then $Pt < t$ if we take $t(1) = 1$ and $P(2, \{1\}) < t(2) < P(1, \{2\})^{-1}$. Similarly, if $P(2, \{1\}) > 0$. The case $P(1, \{2\}) = P(2, \{1\}) = 0$ (which is of no interest, since we have no transition at all) can be dealt with taking $t = 1$. Thus $(X, \mathcal{W}^T)$ is a balayage space by Theorem 7.1 and Remark 7.2.

COROLLARY 7.4. *Suppose that* $I = \{1, \dots, n\}$ *and that* $T(x, X_j) = 0$ *for all* $x \in X_i$ *and* $1 \le j \le i \le n$. *Moreover, assume that* $p > 0$ *and* Tp *is bounded. Then* $(X, \mathcal{W}^T)$ *is a balayage space.*

Proof. In view of Theorem 7.1 and Remark 7.2 it suffices to note that we may easily find a positive real function t on I satisfying $Pt < t$: Having $P(i, \{j\}) = 0$ for $1 \le j \le i$ and $P(i, \{j\}) < \infty$ for $1 \le i < j \le n$ we may take $t(n) = 1$ and choose $t(i) > \sum_{j=i+1}^n P(i, \{j\})t(j)$ recursively for $i = n-1, n-2, \dots, 1$. ∎

Remark 7.5. Using the results of [Bou84] it can easily be seen that (strong) biharmonic spaces as introduced by [Smy75], [Smy76] (or, more generally, polyharmonic spaces) are a special case. They are balayage spaces if interpreted in the right way.

Let us now suppose that all X_i, $i \in I$, are copies of a space Y and that we have transitions only between corresponding points in these copies: Let $\mathcal{W}_i$, $i \in I$, be convex cones of l.s.c. positive numerical functions on Y such that every $(Y, \mathcal{W}_i)$ is a balayage space. For every $i \in I$, let p_i be a strongly superharmonic continuous real potential for $(Y, \mathcal{W}_i)$, $K_{\mathcal{W}_i}^{p_i}$ the corresponding potential kernel and g_{ij}, $j \in I$, Kato functions with respect to $K_{\mathcal{W}_i}^{p_i}$, positive for $j \neq i$. We define

$$T((y, i), \cdot\,) := \sum_{j \in I \setminus \{i\}} g_{ij}(y)\varepsilon_{(y,j)}, \quad k(y, i) := -g_{ii}(y) \quad (y \in Y, i \in I).$$

The potentials p_i define a strongly superharmonic continuous real potential p for the direct sum $(X, \mathcal{W})$, the restriction of K_X^p on the copy of Y corresponding to $(Y, \mathcal{W}_i)$ is the kernel $K_{\mathcal{W}_i}^{p_i}$, T is admissible, and k is a Kato function with respect to K_X. Therefore Theorem 6.1 immediately leads to the following result:

THEOREM 7.6. *If there exist functions* $u_i \in \mathcal{B}^+(Y)$ *such that* $K_{\mathcal{W}_i}^{p_i} u_i \in \mathcal{C}(Y)$ *and*

$$\sum_{j\in I} g_{ij} K_{\mathcal{W}_j}^{p_j} u_j < u_i$$

for every $i \in I$, *then* $(\widetilde{H}_U^T)_{U\in\mathcal{U}^T}$ *is a family of harmonic kernels.*

COROLLARY 7.7. *Assume that* $\mathcal{W}_i = \mathcal{W}_1$ *and* $p_i = p_1$ *for every* $i \in I$. *Then* $(\widetilde{H}_U^T)_{U\in\mathcal{U}^T}$ *is a family of harmonic kernels if there exists a strictly positive function* $u \in \mathcal{B}^+(Y)$ *and strictly positive reals* b_i *such that* $K_{\mathcal{W}_1}^{p_1} u \in \mathcal{C}(Y)$ *and, for all* $i \in I$,

$$\sum_{j\in I} g_{ij} b_j < b_i u / K_{\mathcal{W}_1}^{p_1} u. \tag{7.2}$$

Remark 7.8. Suppose that $I = \{1, \ldots, n\}$, $a_{ij} := \sup g_{ij}(Y) < \infty$ for all i, j and denote $A := (a_{ij})$. Assume that $u \in \mathcal{B}^+(Y)$ is strictly positive and $\alpha > 0$ such that

$$\alpha K_{\mathcal{W}_1}^{p_1} u \le u.$$

Then (7.2) is satisfied if there exists $b \in \mathbb{R}^n$, $b > 0$, such that

$$Ab < \alpha b. \tag{7.3}$$

(Note that $a_{ij} \ge 0$ for $i \ne j$. If, in addition, $a_{ii} \ge 0$ for all i, then (7.3) holds if and only if the spectral radius of A is strictly less than α.)

COROLLARY 7.9. *Assume that* $\mathcal{W}_i = \mathcal{W}_1$ *and* $p_i = p_1$ *for all* $i \in I$. *Then* $(\widetilde{H}_U^T)_{U\in\mathcal{U}^T}$ *is a family of harmonic kernels if* $(Y, \mathcal{W}_1)$ *is parabolic and the function* $\psi := \max_{i\in I} \sum_{j\in I} |g_{ij}|$ *is a Kato function with respect to* $K_{\mathcal{W}_1}^{p_1}$ *having compact support.*

Proof. It is no restriction of generality if we assume that there exists a strictly positive bounded function in $\mathcal{W}_1$ (even that $1 \in \mathcal{W}_1$, see Remarks 2.1, 1). Moreover, we may assume without loss of generality that $\psi \le 1$ and that $K := K_{\mathcal{W}_1}^{p_1}$ is a compact operator on $\mathcal{B}_b(Y)$. Indeed, using Lemma 3.1 we may find a strictly positive $\psi_0 \in \mathcal{B}_b(Y)$ such that $f \mapsto K_{\mathcal{W}_1}^{p_1}(\psi_0 f)$ is a compact operator on $\mathcal{B}_b(Y)$. It now suffices to replace p_1 by $K(\psi_0 + \psi)$ and the functions g_{ij} by $g_{ij}/(\psi_0 + \psi)$.

Then $u := \sum_{n=0}^{\infty} K^n 1 \in \mathcal{B}_b^+(Y)$, $Ku \in \mathcal{C}_b(Y)$, and, for all $i \in I$, $\sum_{j\in I} g_{ij} K u \le Ku = u - 1 < u$. By Theorem 7.6 we conclude that $(\widetilde{H}_U^T)_{U\in\mathcal{U}^T}$ is a family of harmonic kernels. ∎

Proposition 6.3 can be expressed as follows:

PROPOSITION 7.10. *Let $I = \{1, \dots, n\}$. Suppose that $(\widetilde{H}_U^T)_{U \in \mathcal{U}^T}$ is a family of harmonic kernels and that U is a relatively compact open subset of Y which is $\mathcal{W}_i$-regular for every $1 \le i \le n$.*

Then, for any choice of functions $\varphi_1, \dots, \varphi_n \in \mathcal{K}(Y)$, there exist unique functions $h_1, \dots, h_n \in \mathcal{K}(Y)$ such that, for every $1 \le i \le n$,

$$h_i - \sum_{j \in I} K_{\mathcal{W}_j}^{p_j}(g_{ij} h_j) \text{ is } \mathcal{W}_i\text{-harmonic on } U, \quad h_i = \varphi_i \quad \text{on } U^c.$$

Moreover, the functions $h_1, \dots, h_n$ are positive, if the functions $\varphi_1, \dots, \varphi_n$ are positive.

§8. Application to coupling of PDE's

Let D be a domain in $\mathbb{R}^d$, $d \ge 1$, let $n \in \mathbb{N}$, and let L_i, $1 \le i \le n$, be second order (elliptic or parabolic) linear partial differential operators on D leading to harmonic spaces $(D, \mathcal{H}_{L_i})$. (For the definition of harmonic spaces and various sufficient conditions for the differential operators the reader might consult [Her62], [CC72], [BH86], [Kro88], [Her68], [Bon70]). Moreover, we assume that, for every $1 \le i \le n$, we have a base of L_i-regular sets for D, a Green function G_{L_i} for $(D, \mathcal{H}_{L_i})$, and a Radon measure $\mu_i \ge 0$ on D such that $G_{L_i}^{\mu_i} \in \mathcal{C}_b(D)$ and $(G_{L_i})_V^{\mu_i} > 0$ on V for every (L_i-regular) open subset V of D.

We want to study the coupled system

$$L_i h_i + \sum_{j=1}^n g_{ij} h_j \mu_i = 0 \quad (1 \le i \le n)$$

where $g_{ij} \in \mathcal{B}(D)$ such that $g_{ij} \ge 0$ for $i \ne j$ and $G_{L_i}^{1_A |g_{ij}| \mu_i} \in \mathcal{C}(D)$ for every compact subset A of D and all $i, j \in \{1, \dots, n\}$.

This will be possible by introducing associated transitions on the direct sum of the spaces $(D, \mathcal{H}_{L_i})$ (cf. the example given in the introduction). Our formal procedure is as follows: For every $1 \le i \le n$, let

$$X_i := D \times \{i\}$$

and let π_i denote the canonical projection from X_i on D. Then the direct sum $(X, \mathcal{H})$ of the spaces $(X_i, \mathcal{H}_{L_i} \circ \pi_i)$, $1 \le i \le n$, is a harmonic space (with the subspace $X = D \times \{1, 2, \dots, n\}$ of $\mathbb{R}^d \times \mathbb{N}$). (If $\mathcal{W}_i$ denotes the convex cone of all positive hyperharmonic functions for $(X_i, \mathcal{H}_{L_i} \circ \pi_i)$ and $\mathcal{W}$ the convex cone of all positive hyperharmonic functions for $(X, \mathcal{H})$, then of course $(X, \mathcal{W})$ is the direct sum of $(X_1, \mathcal{W}_1), \dots, (X_n, \mathcal{W}_n)$.)

We define a continuous bounded potential p, a kernel T and a function k on X by

$$p(x,i) = G_{L_i}^{\mu_i}(x), \quad T((x,i), \cdot\,) = \sum_{j \neq i} g_{ij}(x)\, \varepsilon_{(x,j)}, \quad k(x,i) = -g_{ii}(x).$$

Then T is admissible, k is a Kato function with respect to K_X^p, and the results of the preceding section can be applied.

Suppose for a moment that $(\widetilde{H}_U^T)_{U \in \mathcal{U}^T}$ is a family of harmonic kernels. Fix a relatively compact subset U of D and functions $\varphi_1, \dots, \varphi_n \in \mathcal{K}(D)$. For simplicity suppose that U is L_i-regular for every $1 \le i \le n$ (it will be clear for the specialist how to proceed if this does not hold). Then

$$\widetilde{U} := \bigcup_{i=1}^{n} U \times \{i\}$$

is a regular subset of X. Defining

$$\varphi(x,i) := \varphi_i(x) \quad (x \in D,\, 1 \le i \le n)$$

we obtain a function $\varphi \in \mathcal{K}(X)$. By Proposition 4.7, there is a unique function $h \in \mathcal{K}(X)$ such that

$$h + K_{\widetilde{U}}(kh - Th) = H_{\widetilde{U}}\varphi.$$

Of course, $h|_{\widetilde{U}}$ depends only on $\varphi|_{\partial\widetilde{U}}$, since $T(\widetilde{U}) \subset \widetilde{U}$ and $H_{\widetilde{U}}\varphi$ depends only on $\varphi|_{\partial\widetilde{U}}$. Define

$$h_i := h \circ \pi_i^{-1} \quad (1 \le i \le n)$$

and fix $1 \le i \le n$. Clearly, $h_i \in \mathcal{K}(D)$ and $h_i = \varphi_i$ on $D \setminus U$, since $h = \varphi$ on $X \setminus \widetilde{U}$. Furthermore, $L_i((H_{\widetilde{U}}\varphi) \circ \pi_i^{-1}) = 0$ on U, since $H_{\widetilde{U}}\varphi \in \mathcal{H}(\widetilde{U})$ and hence $(H_{\widetilde{U}}\varphi) \circ \pi_i^{-1} \in \mathcal{H}_{L_i}(U)$. And

$$(K_{\widetilde{U}}(kh - Th)) \circ \pi_i^{-1} = (G_{L_i})^{(kh - Th) \circ \pi_i^{-1} \mu_i}$$

where, for every $x \in D$, by definition of T and k

$$(kh - Th) \circ \pi_i^{-1}(x) = (kh - Th)(x,i) = \sum_{j=1}^{n} g_{ij}(x) h(x,j) = \sum_{j=1}^{n} g_{ij}(x) h_j(x).$$

Thus

$$0 = L_i((H_{\widetilde{U}}\varphi) \circ \pi_i^{-1}) = L_i[(h + K_{\widetilde{U}}(kh - Th)) \circ \pi_i^{-1}] = L_i h_i + \sum_{j=1}^{n} g_{ij} h_j \mu_j$$

and we obtain the following consequence of Proposition 7.10 (cf. [Bou81, Proposition 11.5]):

PROPOSITION 8.1. *Assume that $(\widetilde{H}_U^T)_{U \in \mathcal{U}^T}$ is a family of harmonic kernels. Let U be a relatively compact open subset of D which is L_i-regular for every $1 \le i \le n$ and $\varphi_1, \dots, \varphi_n \in \mathcal{C}(\partial U)$. Then there exist unique functions $h_1, \dots, h_n \in \mathcal{C}(\overline{U})$ such that*

$$L_i h_i + \sum_{j=1}^{n} h_j g_{ij} \mu_i = 0 \quad \text{on } U, \qquad h_i|_{\partial U} = \varphi_i \quad (1 \le i \le n).$$

Further, if $\varphi_1, \dots, \varphi_n$ are positive, then $h_1, \dots, h_n$ are positive.

If the functions φ_i are bounded and measurable, but not necessarily continuous and/or if the set U is not L_i-regular, we still have a unique generalized solution of the Dirichlet problem (see [BH86, Chapter VII]).

By Corollary 7.4, $(\widetilde{H}_U^T)_{U \in \mathcal{U}^T}$ is a family of harmonic kernels provided $g_{ij} = 0$ for all $1 \le i \le j \le n$. A very special case is the situation where all operators L_i are equal and $g_{ij}\mu_i = \delta_{i+1,j}\lambda$:

COROLLARY 8.2. *Let D be a bounded domain in $\mathbb{R}^d$, $d \ge 1$, and let L be a second order linear partial differential operator on D leading to a harmonic space $(D, \mathcal{H}_L)$ with Green function G_L such that G_L^λ is continuous and bounded. Let U be a relatively compact $(L-)$regular subset of D, $n \in \mathbb{N}$, and $\varphi_1, \dots, \varphi_n \in \mathcal{C}(\partial U)$. Then there exists a unique function $h \in \mathcal{C}(U)$ such that $Lh, L^2 h, \dots, L^{n-1} h \in \mathcal{C}(U)$,*

$$L^n h = 0 \quad \text{on } U, \quad \lim_{x \to z} (-L)^{i-1} h(x) = \varphi_i(z)$$

$$\text{for every } 1 \le i \le n \text{ and for all } z \in \partial U.$$

Further, $h, -Lh, L^2, \dots, (-L)^{n-1} h$ are positive, if $\varphi_1, \dots, \varphi_n$ are positive.

We shall complete our application by giving further conditions implying that $(\widetilde{H}_U^T)_{U\in\mathcal{U}^T}$ is a family of harmonic kernels:

PROPOSITION 8.3. *Suppose that there exists a strictly positive real function s on D such that, for each $1 \le i \le n$, one of the following conditions is satisfied*:

(1) $\sum_{j=1}^n g_{ij} \le 0$ *and s is strongly L_i-superharmonic.*
(2) $\sum_{j=1}^n g_{ij} < 0$ *and s is L_i-superharmonic.*

Then $(\widetilde{H}_U^T)_{U\in\mathcal{U}^T}$ is a family of harmonic kernels.

Proof. Define $s \in \mathcal{W}$ by $s(x,i) = s(x)$ and fix $1 \le i \le n$. Then, for every $x \in D$,

$$(ks - Ts)(x,i) = -g_{ii}(x) - \sum_{j\neq i} g_{ij}(x) \ge 0.$$

So $(\widetilde{H}_U^T)_{U\in\mathcal{U}^T}$ is a family of harmonic kernels by Theorem 6.1 (taking $u = 0$). ∎

In [CZ96] it is assumed that, for every $1 \le i \le n$, the operator L_i is uniformly elliptic, $\mu_i = \lambda$, $\sum_{j=1}^n g_{ij} \le 0$, and 1 is L_i-superharmonic.

Moreover, Theorem 7.6 implies the following result involving μ_i-eigenfunctions for the operators L_i (cf. [Bou81, pp. 348–350] and [Bou82]):

PROPOSITION 8.4. *Suppose that there exist strictly positive $\mathcal{P}_{L_i}(D)$-bounded functions $u_i \in \mathcal{C}_b(D)$ and strictly positive real numbers α_i, β_{ij}, $i,j \in \{1,\dots,n\}$, such that*

$$L_i u_i + \alpha_i u_i \mu_i = 0,$$

and

$$u_j \le \beta_{ij} u_i, \quad \sum_{j=1}^n \beta_{ij} g_{ij}/\alpha_j < 1, \quad \beta_{ii} = 1.$$

Then $(\widetilde{H}_U^T)_{U\in\mathcal{U}^T}$ is a family of harmonic kernels.

Remark 8.5. If there exists an L_i-superharmonic function $s_i \ge 1$ on D, then every function $u \in \mathcal{C}_0(D)$ is $\mathcal{P}_{L_i}(D)$-bounded.

Proof of Proposition 8.4. For every $1 \le i \le n$,

$$\alpha_i G_{L_i}^{u_i\mu_i} = u_i,$$

since $u_i - \alpha_i G_{L_i}^{u_i\mu_i}$ is $\mathcal{P}_{L_i}(D)$-bounded and L_i-harmonic on D. Therefore

$$\sum_{j=1}^n g_{ij} G_{L_j}^{u_j\mu_j} = \sum_{j=1}^n g_{ij}\frac{u_j}{\alpha_j} \le \sum_{j=1}^n g_{ij}\frac{\beta_{ij}}{\alpha_j} u_i < u_i$$

for every $1 \le i \le n$. Thus $(\widetilde{H}_U^T)_{U\in\mathcal{U}^T}$ is a family of harmonic kernels by Theorem 7.6. □

PROPOSITION 8.6. *Suppose that $L_1 = \cdots = L_n =: L$. Then $(\widetilde{H}_U^T)_{U\in\mathcal{U}^T}$ is a family of harmonic kernels if one of the following conditions is satisfied*:

(1) $\mu_1 = \cdots = \mu_n =: \mu$ *and there exist $\alpha > 0$, a strictly positive $\mathcal{P}_L(D)$-bounded function $u \in \mathcal{C}_b(D)$, and strictly positive real numbers $b_1, \dots, b_n$ such that*

$$Lu + \alpha u\mu = 0 \quad \textit{and} \quad \sum_{j=1}^n g_{ij} b_j < \alpha b_i \quad \textit{for every } 1 \le i \le n.$$

(2) $(D, \mathcal{H}_L)$ *is parabolic, the functions g_{ij} have compact support and the L-potentials $G_L^{|g_{ij}|\mu_i}$, $i, j \in \{1, \dots, n\}$, are continuous.*

Remark 8.7. Note that the harmonic space associated with the heat equation or a similar parabolic equation is parabolic. Moreover, the last property clearly holds if the functions g_{ij} are bounded. Finally, to obtain the conclusion of Proposition 8.1 we obviously may drop the assumption on the compact support (replacing g_{ij} by $1_U g_{ij}$).

Proof of Proposition 8.6. By Proposition 8.4, (1) implies that $(\widetilde{H}_U^T)_{U\in\mathcal{U}^T}$ is a family of harmonic kernels (take $u_i = b_i u$ and $\beta_{ij} = b_j/b_i$).

So suppose that (2) holds. Since of course $g_{ij}\mu_i = \tilde{g}_{ij}(\mu_1 + \cdots + \mu_n)$ for some Borel functions $\tilde{g}_{ij}$ such that $\tilde{g}_{ij} \ge 0$ for $j \ne i$ and $|\tilde{g}_{ij}| \le |g_{ij}|$ for all i, j, we may assume without loss of generality that $\mu_1 = \cdots = \mu_n$. Thus Corollary 7.9 implies that $(\widetilde{H}_U^T)_{U\in\mathcal{U}^T}$ is a family of harmonic kernels. □

§9. Perturbation and general transitions

Let us go back to a general situation as considered in Section 6. So we have a balayage space $(X, \mathcal{W})$, a potential kernel K_X for $(X, \mathcal{W})$, a Kato function k and an admissible transition kernel T. However, we shall no longer assume that $\mathcal{U}^T$ is a base of X. So our result will be new even if there is no perturbation at all, i.e., if $k = 0$.

Let us suppose for the moment that at least $T(x, \{x\}) = 0$ for every $x \in X$ (we shall see that this is no restriction, since we may modify k). Moreover, assume that there exists $s \in \mathcal{C}^+(X)$ such that, for every $U \in \mathcal{U}$, $H_U s + K_U^T s < s$ on U.

Let ρ be a metric for X and define kernels T_n, T'_n on X by

$$T_n(x, \cdot\,) = 1_{B(x,1/n)^c}\, T(x, \cdot\,), \quad T'_n(x, \cdot\,) = 1_{B(x,1/n)}\, T(x, \cdot\,) \quad (n \in \mathbb{N}, x \in X)$$

(where of course $B(x, 1/n) = \{y \in X : \rho(x, y) < 1/n\}$). Then, for every $n \in \mathbb{N}$, the set $\mathcal{U}^{T_n} = \{U \in \mathcal{U} : 1_U T_n 1_U = 0\}$ is a base of X and we have kernels

$$K_U^{T_n} = K_U T_n, \quad H_U^{T_n} = H_U + K_U^{T_n} \quad (U \in \mathcal{U}^{T_n}).$$

Since obviously, for every $V \in \mathcal{U}^{T_n}$,

$$H_V^{T_n} s = H_V s + K_V^{T_n} s \le H_V s + K_V^T s < s \quad \text{on } V,$$

the function s is strongly $\mathcal{W}^{T_n}$-superharmonic and we conclude by Theorem 4.2 that $(H_U^{T_n})_{U \in \mathcal{U}^{T_n}}$ is a family of harmonic kernels and that $(X, \mathcal{W}^{T_n})$ is a balayage space. In particular, for every $n \in \mathbb{N}$ and for *every* $U \in \mathcal{U}$, we have a harmonic kernel $H_U^{T_n}$ solving the Dirichlet problem with respect to $(X, \mathcal{W}^{T_n})$ (see [BH86, Chapter VII]).

Clearly, $\mathcal{U}^{T_{n+1}} \subset \mathcal{U}^{T_n}$ and $H_U^{T_n} \le H_U^{T_{n+1}}$ for every $U \in \mathcal{U}^{T_{n+1}}$. We claim that in fact

$$H_U^{T_n} \le H_U^{T_{n+1}} \quad \text{for every } U \in \mathcal{U}. \tag{9.1}$$

Indeed, fix $U \in \mathcal{U}$, $\varphi \in \mathcal{K}^+(X)$, and define

$$t := H_U^{T_{n+1}} \varphi.$$

Then, for every $V \in \mathcal{U}^{T_{n+1}}$ with $\overline{V} \subset U$,

$$H_V^{T_n} t \le H_V^{T_{n+1}} t = t,$$

hence t is superharmonic on U with respect to $(X, \mathcal{W}^{T_n})$. Moreover, $t \in \mathcal{K}^+(X)$ and $t = \varphi$ on U^c. Therefore

$$H_U^{T_n}\varphi \leq t$$

proving (9.1). In particular, the sequence $(\mathcal{W}^{T_n})$ is decreasing and defining

$$H_U^T := \sup_n H_U^{T_n}$$

we have

$$\mathcal{W}^T := \{v \mid v : X \to [0,\infty] \text{ l.s.c.}, \ H_U^T v \leq v \text{ for every } U \in \mathcal{U}\} = \bigcap_{n\in\mathbb{N}} \mathcal{W}^{T_n}.$$

We now obtain the following extension of Theorem 4.2 (see also Remark 9.3):

THEOREM 9.1. *Let T be an admissible kernel such that $T(x,\{x\}) = 0$ for every $x \in X$. Suppose that there exists $s \in \mathcal{C}^+(X)$ such that, for every $U \in \mathcal{U}$, $K_U^T s$ is continuous on U and $H_U s + K_U^T s < s$ on U. Then the following holds*:

1. *$(X, \mathcal{W}^T)$ is a balayage space and s is strongly $\mathcal{W}^T$-superharmonic.*
2. *For every $U \in \mathcal{U}$ and for every $\varphi \in \mathcal{K}^+(X)$, the Dirichlet solution $H_U^T\varphi$ is the unique function $h \in \mathcal{K}^+(X)$ such that $h - K_U^T h = H_U\varphi$.*
3. *If v is any positive numerical function on X, then $v \in \mathcal{W}^T$ if and only if there exists a function $w \in \mathcal{W}$ such that*

$$v = K_X^T v + w.$$

Proof. 1. Fix $U \in \mathcal{U}$. By Proposition 2.2 it suffices to show that $H_U^T s$ is continuous on X and $H_U^T s < s$ on U. Let us note first that obviously $s \in \mathcal{W} \cap \mathcal{C}(X)$ and hence $H_U s \in \mathcal{C}(X)$ and $s - H_U s \in \mathcal{C}_0(X)$. Given $n \in \mathbb{N}$, we have $s \in \mathcal{W}^{T_n}$. So

$$h_n := H_U^{T_n} s \leq s.$$

and, by Proposition 4.7,

$$h_n = H_U s + K_U^{T_n} h_n.$$

Letting n tend to infinity we obtain that

$$h := H_U^T s = \lim_{n\to\infty} h_n = H_U s + K_U^T h \leq s$$

and hence

$$h \le H_U s + K_U^T s < s \quad \text{on } U.$$

Moreover, $K_U^T h \in \mathcal{C}(U)$, since $0 \le h \le s$ and $K_U^T s$ is continuous on U by assumption. Since $0 \le K_U^T h \le K_U^T s \le s - H_U s$, we know that $K_U^T h$ tends to zero at the boundary of U. Thus $K_U^T h \in \mathcal{C}_0(U)$ and $h = H_U s + K_U^T h \in \mathcal{C}(X)$.

2. Fix $\varphi \in \mathcal{K}^+(X)$. Since by Proposition 4.7

$$H_U^{T_n}\varphi - K_U^{T_n} H_U^{T_n}\varphi = H_U\varphi,$$

we immediately obtain that

$$H_U^T\varphi - K_U^T H_U^T\varphi = H_U\varphi. \tag{9.2}$$

Conversely, let h be any function in $\mathcal{K}^+(X)$ such that

$$h - K_U^T h = H_U\varphi. \tag{9.3}$$

Let C be the support of h. By (4.2), $K_U^T 1_C \in \mathcal{C}_0(U)$. Given $x \in U$, the functions

$$K_V^T 1_C = K_U^T 1_C - H_V K_U^T 1_C, \quad x \in V, \overline{V} \subset U$$

are uniformly decreasing to zero as V decreases to $\{x\}$. So we may choose $V_x \in \mathcal{U}$ such that $x \in V_x, \overline{V}_x \subset U$ and $K_V^T 1_C \le \gamma$ for some real $\gamma < 1$. Fix $V \in \mathcal{U}$ such that $x \in V \subset V_x$ and define a positive operator N on $\mathcal{B}_b(X)$ by $Nf := K_V^T(1_C f)$. Then the operator $I - N$ is invertible.

Applying H_V on both sides of (9.3) we obtain that

$$H_V h - H_V K_U^T h = H_V H_U \varphi = H_U\varphi = h - K_U^T h,$$

and therefore

$$H_V h = h - K_U^T h + H_V K_U^T h = h - K_V^T h = (I - N)h.$$

On the other hand,

$$H_V h = H_V^T h - K_V^T H_V^T h = (I - N) H_V^T h$$

(using (9.2) for h instead of φ and V instead of U). Since $I - N$ is invertible, we conclude that

$$h = H_V^T h.$$

By [BH86, Proposition III.4.4], this shows that h is harmonic on U with respect to $(X, \mathcal{W}^T)$. Thus $h = H_U^T\varphi$.

3. Suppose that $w \in \mathcal{W}$ such that $v = K_X^T v + w$. Then, for every $n \in \mathbb{N}$,

$$v = K_X^{T_n} v + K_X^{T_n'} v + w$$

where $K_X^{T_n'} v + w \in \mathcal{W}$. Thus Proposition 4.9 implies that

$$v \in \bigcap_{n=1}^{\infty} \mathcal{W}^{T_n} = \mathcal{W}^T.$$

Assume conversely that $v \in \mathcal{W}^T$. Then, for every $n \in \mathbb{N}$, there exists a function $w_n \in \mathcal{W}^{T_n}$ such that

$$K_X^{T_n} v + w_n = v.$$

Defining $w \in \mathcal{W}$ by

$$w(x) = \text{f-}\liminf_{y \to x} \inf_n w_n(y)$$

we finally get that $K_X^T v + w = v$. □

We now obtain the results of Theorem 6.1 and Proposition 6.3 not assuming any more that $\mathcal{U}^T$ is a base of X.

Theorem 9.2. *Let T be an admissible transition kernel and let k be a Kato function (with respect to K_X). Suppose that there exist $s \in \mathcal{W}$ and $u \in \mathcal{B}^+(X)$ such that*

$$v := s + K_X u \in \mathcal{C}(X), \quad Tv \leq u + kv,$$

and, for every $U \in \mathcal{U}$, $\{H_U s < s\} \cup \{K_U(u + kv - Tv) > 0\} = U$.

Then, for every $U \in \mathcal{U}$ and for every $\varphi \in \mathcal{K}^+(X)$, there exists a unique function $h = \widetilde{H}_U^T \varphi \in \mathcal{K}^+(X)$, such that

$$h + K_U(kh - Th) = H_U \varphi.$$

Moreover, $(\widetilde{H}_U^T)_{U \in \mathcal{U}}$ is a family of harmonic kernels on X for which v is strongly superharmonic.

Remark 9.3. Note that taking $k = 0$ we obtain the statements of Theorem 9.1 without the assumption that $T(x, \{x\}) = 0$ for $x \in X$.

Proof of Theorem 9.2. Replacing T by the kernel $x \mapsto T(x,\cdot) - T(x,\{x\})\varepsilon_x$, k by the function $x \mapsto k(x) - T(x,\{x\})$ we may assume that $T(x,\{x\}) = 0$ for every $x \in X$.

We now proceed as in the proof of Theorem 6.1: By Theorem 5.4, every $U \in \mathcal{U}$ is k-bounded and defining $\widetilde{H}_U$, $U \in \mathcal{U}$, by (5.1) and $\widetilde{\mathcal{W}}$ by (5.2) we obtain a family $(\widetilde{H}_U)_{U\in\mathcal{U}}$ of harmonic kernels and a balayage space $(X, \widetilde{\mathcal{W}})$ such that v is strongly $\widetilde{\mathcal{W}}$-superharmonic. Moreover, by Proposition 5.5, there exists a potential kernel $\widetilde{K}_X$ such that, for every $U \in \mathcal{U}$,

$$\widetilde{K}_U := \widetilde{K}_X - \widetilde{H}_U \widetilde{K}_X = (I + K_U M_k)^{-1} K_U. \tag{9.4}$$

We claim that, for every $U \in \mathcal{U}$,

$$\widetilde{H}_U v + \widetilde{K}_U^T v < v \quad \text{on } U.$$

Indeed, defining $f := v - \widetilde{H}_U v - \widetilde{K}_U^T v$ we obtain that

$$(I + K_U M_k) f = v + K_U(kv) - H_U v - K_U(Tv) = s - H_U s + K_U(u + kv - Tv)$$

is a strictly positive superharmonic function on U and hence $f > 0$ on U. Clearly, $K_X u \in \mathcal{C}(X)$ and hence $K_U u \in \mathcal{C}_0(U)$. Since $|kv| \le \sup v(U)|k|$ on U, we know that $K_U|kv| \in \mathcal{C}_0(U)$. Therefore the inequality $0 \le Tv \le u + kv$ implies that $K_U^T v \in \mathcal{C}_0(U)$ and hence $\widetilde{K}_U^T v \in \mathcal{C}_0(U)$.

Replacing $(H_U)_{U\in\mathcal{U}}$ by $(\widetilde{H}_U)_{U\in\mathcal{U}}$ and $(K_U)_{U\in\mathcal{U}}$ by $(\widetilde{K}_U)_{U\in\mathcal{U}}$ we get a balayage space $(X, \widetilde{\mathcal{W}^T})$ such that v is strongly $\widetilde{\mathcal{W}}^T$-superharmonic.

Moreover, for every $\varphi \in \mathcal{K}^+(X)$, the function

$$\widetilde{H}_U^T \varphi = \lim_{n\to\infty} \widetilde{H}_U^{T_n} \varphi$$

is the unique function $h \in \mathcal{K}^+(X)$ such that

$$h - \widetilde{K}_U^T h = \widetilde{H}_U \varphi.$$

By (5.1) and (9.4), the last equation is equivalent to

$$h + K_U(kh - Th) = H_U \varphi,$$

and the proof is finished. ∎

§10. Appendix: Lifting of potentials in balayage spaces

In this section we shall construct a potential kernel corresponding to a compatible family of potential kernels $(K_U)_{U\in\mathcal{U}}$ (see Remarks 2.3, 4). We shall need the following *lifting property*:

THEOREM 10.1. *Let U be an open subset of X and q a continuous real potential on U which is harmonic outside a compact subset C of U. Then there exists a unique $p \in \mathcal{P}(X)$ such that p is harmonic outside C and $p-q$ is harmonic on U.*

For harmonic spaces the proof is already fairly technical (see [Her62, Theorem 13.2]), for balayage spaces it is even more delicate. For every open subset V of X let $\mathcal{S}_{pb}(V)$ denote the set of all $\mathcal{P}$-bounded $s \in \mathcal{B}(X)$ such s is l.s.c. on V and $H_W s \le s$ for every $W \in \mathcal{U}$ with $\overline{W} \subset V$. An easy generalization of [BH86, Proposition II.4.4] yields the following sheaf property (cf. also (2.2)): For every family $(V_i)_{i\in I}$ of open subsets of X,

$$\mathcal{S}_{pb}\Big(\bigcup_{i\in I} V_i\Big) = \bigcap_{i\in I} \mathcal{S}_{pb}(V_i). \tag{10.1}$$

Proof of Theorem 10.1 (cf. [Alb95]). The uniqueness of p is easily established. Indeed, if p and p' have the desired properties, then $p-p'$ is harmonic on U and harmonic outside C. Therefore $p-p'$ is harmonic on X by (10.1) (applied to $p-p'$ and $p'-p$). Since $p-p'$ is of course $\mathcal{P}(X)$-bounded, we conclude that $p=p'$.

To prove the existence let us define

$$\mathcal{F} := \{p \in \mathcal{P}(X) : p-q \in \mathcal{S}_{pb}^{+}(U)\}.$$

We intend to show that there is a smallest element in $\mathcal{F}$ and that this function $\inf \mathcal{F}$ has the desired properties.

1. First we claim that the set $\mathcal{F}$ is non-empty: We choose an open set V and a compact set L such that $C \subset V \subset L \subset U$. By a general approximation property (see [BH86, I.1.2]) there exist $q_1, q_2 \in \mathcal{P}(X)$ such that

$$q_2 - q_1 \ge q \quad \text{on } V, \qquad q_1 = q_2 \quad \text{on } L^c.$$

Then

$$p_0 := \inf(q+q_1, q_2) \in \mathcal{S}_{pb}^{+}(U).$$

Moreover, $p_0 = q_2$ on L^c and $p_0 \le q_2$ on X whence $p_0 \in \mathcal{S}^+_{pb}(L^c)$. Thus $p_0 \in \mathcal{S}^+_{pb}(X)$ by (10.1). In fact, $p_0 \in \mathcal{P}(X)$, since p_0 is continuous.

Clearly, $p_0 - q = q_1 \ge 0$ on V. Therefore $p_0 - q \ge 0$, since q is harmonic outside C. Knowing that $p_0 - q \le q_1$ on X we conclude by (10.1) that

$$p_0 - q \in \mathcal{S}^+_{pb}(V) \cap \mathcal{S}^+_{pb}(U \setminus C) = \mathcal{S}^+_{pb}(U).$$

Thus $p_0 \in \mathcal{F}$.

2. Obviously $\mathcal{F}$ is stable with respect to finite infima, since both $\mathcal{P}(X)$ and $\mathcal{S}^+_{pb}(U)$ are.

3. Next we show that $\inf \mathcal{F}$ is harmonic outside C: Let us fix an open neighborhood W of C in U. Clearly it suffices to show that $\inf \mathcal{F}$ is harmonic outside the closure of W. For the present fix $p \in \mathcal{F}$. Then $K^p_X 1_W - q = (p - q) - K^p_X 1_{W^c} \in \mathcal{S}_{pb}(W)$ and $K^p_X 1_W - q \in \mathcal{S}_{pb}(U \setminus C)$, hence $K^p_X 1_W - q \in \mathcal{S}_{pb}(U)$ by (10.1). Since $q \in \mathcal{P}(U)$, we obtain that $K^p_X 1_W - q \ge 0$. Therefore $K^p_X 1_W \in \mathcal{F}$, i.e.,

$$\inf \mathcal{F} = \inf\{K^p_X 1_W : p \in \mathcal{F}\}.$$

Since $\mathcal{F}$ is stable with respect to finite infima, the set of all $K^p_X 1_W$, $p \in \mathcal{F}$, is decreasingly filtered and therefore contains a decreasing sequence (p_n) converging to $\inf \mathcal{F}$. Since all functions $K^p_X 1_W$, $p \in \mathcal{F}$, are harmonic outside $\overline{W}$, we conclude in particular that $\inf \mathcal{F}$ is harmonic outside $\overline{W}$ as well.

4. Moreover, $\inf \mathcal{F} - q$ is harmonic on U: Fix $p \in \mathcal{F}$, a compact neighborhood L of C in U and an open neighborhood W of C such that $\overline{W}$ is contained in the interior of L. Choose $\varphi \in \mathcal{C}(X)$ such that $0 \le \varphi \le 1$, $\varphi = 1$ on L^c, and $\varphi = 0$ on W. Define

$$p' := \inf(R_{\varphi p} + q, p).$$

Then $p' = p$ on L^c, so p' is continuous on L^c. Further, the continuity of the functions $R_{\varphi p}$, q, and p on U implies that p' is continuous on U. Therefore p' is continuous on X.

Clearly, $p' \in \mathcal{S}^+_{pb}(U)$. Moreover, $p' \in \mathcal{S}^+_{pb}(L^c)$, since $p' = p$ on L^c and $p' \le p$. Therefore $p' \in \mathcal{S}^+_{pb}$ by (10.1) and even $p' \in \mathcal{P}(X)$, since p' is continuous. Since $p - q \in \mathcal{S}^+_{pb}(U)$, we obtain that $p' - q = \inf(R_{\varphi p}, p - q) \in \mathcal{S}^+_{pb}(U)$. Thus $p' \in \mathcal{F}$.

Further, $R_{\varphi p} \le R_{1_{W^c} p} = H_W p$ whence $p' - q \le H_W p$. So, for every $n \in \mathbb{N}$ and for every $V \in \mathcal{U}$ with $\overline{V} \subset W$, we obtain that

$$p_n - q \ge H_V(p_n - q) \ge H_W(p_n - q) = H_W p_n - H_W q \ge p_n' - q - H_W q.$$

Since obviously $\inf \mathcal{F} = \inf p_n = \inf p_n'$, we conclude that

$$\inf \mathcal{F} - q \ge H_V(\inf \mathcal{F} - q) \ge \inf \mathcal{F} - q - H_W q.$$

Because of $\lim_{W \uparrow U} H_W q = 0$ this implies that

$$\inf \mathcal{F} - q = H_V(\inf \mathcal{F} - q)$$

for all $V \in \mathcal{U}$ with $\overline{V} \subset U$. Thus $\inf \mathcal{F} - q$ is harmonic on U.

Knowing that $\inf \mathcal{F} - q$ is harmonic on U and $\inf \mathcal{F}$ is harmonic on C^c we see immediately that $\inf \mathcal{F}$ is continuous on X. Thus $\inf \mathcal{F} \in \mathcal{P}(X)$, and the proof is finished. ∎

Proposition 10.2. *Let $(K_U)_{U \in \mathcal{U}}$ be a compatible family of potential kernels, i.e., for every $U \in \mathcal{U}$, we have a potential kernel K_U on U and $K_U = K_V + H_V K_U$ whenever $U, V \in \mathcal{U}$ with $V \subset U$. Then there exists a unique potential kernel K_X on X such that $K_U = K_X - H_U K_X$ for every $U \in \mathcal{U}$.*

Proof. Indeed, if $f \in \mathcal{B}_b^+(X)$ with compact support in some $U \in \mathcal{U}$, then $K_X f$ has to be the lifting of $K_U f$. So we have uniqueness of K_X.

To prove its existence we may choose a locally finite covering of X by a sequence (U_n) in $\mathcal{U}$ and continuous functions $\varphi_n \ge 0$ on X with compact support in U_n, $n \in \mathbb{N}$, such that $\sum_{n=1}^\infty \varphi_n = 1$. For every $n \in \mathbb{N}$, let p_n be the lifting of $K_{U_n} \varphi_n$ on X so that

$$K_X^{p_n} - H_{U_n} K_X^{p_n} = K_{U_n} M_{\varphi_n}. \tag{10.2}$$

Define

$$K_X := \sum_{n=1}^\infty K_X^{p_n}.$$

Clearly, K_X is a potential kernel on X. Fix $U \in \mathcal{U}$, $n \in \mathbb{N}$, and $f \in \mathcal{B}_b^+(X)$ with compact support in U. Then $\varphi_n f$ has compact support in $U_n \cap U$ and our compatibility assumption implies that $K_U(\varphi_n f)$ is the lifting of

$K_{U_n \cap U}(\varphi_n f)$ on U and $K_{U_n}(\varphi_n f)$ is the lifting of $K_{U_n \cap U}(\varphi_n f)$ on U_n. By (10.2), $K_X^{p_n} f$ is the lifting of $K_U(\varphi_n f)$ on X. Therefore

$$K_X^{p_n} f - H_U K_X^{p_n} f = K_U(\varphi_n f).$$

Taking the sum over all $n \in \mathbb{N}$ we finally conclude that $K_X - H_U K_X = K_U$. $\square$

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