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NUMERICAL CRITERIA FOR CERTAIN FIBER SPACES TO BE BIRATIONALLY TRIVIAL

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Abstract. Let $f\colon X \to B$ be a fiber space over a curve B whose general fiber F belongs to one of the following type: 1) F is of general type and satisfying some mild conditions, 2) F is with trivial canonical sheaf. In this note, a numerical characterization for $f\colon X \to B$ to be birationally trivial is given.

§1. Introduction

Let X be a complex projective manifold, and $f\colon X \to B$ be a morphism over a smooth projective curve B with connected fibers. A natural problem is to find a numerical characterization for $f\colon X \to B$ to be birationally trivial (see (2.1) for the definition).

When X is a surface, it is well-known that, if $g(F) \geq 2$, f is birationally trivial if and only if $q(X) - g(B) = g(F)$, where F is a general fiber of f, $g(F)$ (resp. $g(B)$) is the genus of F (resp. B), and $q(X)$ is the irregularity of X (cf. [2]).

In this note, we consider the higher dimensional case.

For any $1 \leq i \leq \dim X$, let $\mathcal{H}^i_X$ be the image of the map $H^0(\Omega^i_X) \otimes \mathcal{O}_X \to \Omega^i_X$, where Ω^i_X is the sheaf of holomorphic i-forms on X. Let $\operatorname{rk} \mathcal{H}^i_X$ be the rank of $\mathcal{H}^i_X$. It is easy to see that $\operatorname{rk} \mathcal{H}^i_X$ is a birational invariant. Let $h^{i,0}(X) = \dim H^0(\Omega^i_X)$ and $p_g(X)$ be the geometric genus of X. Our main result is the following.

Theorem 1.1. *Let X be a complex projective manifold of dimension $n+1$ ($n \geq 2$), and $f\colon X \to B$ be a morphism over a smooth projective curve B with connected fibers. Let F be a general fiber of f. Assume that $h^{n-1,0}(F) = 0$, and that either the canonical map ϕ_F of F is birational, or ϕ_F is generically finite of degree being a prime number and $p_g(\operatorname{Im} \phi_F) = 0$. Then f is birationally trivial if and only if $\operatorname{rk} \mathcal{H}^n_X = 1$ and $h^{n,0}(X) = p_g(F)$.*

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Theorem 1.1 will be proved in Section 2. In Section 3 we will give some criteria for fiber spaces whose general fibers have trivial canonical sheaf to be birationally trivial.

We use standard notations as in [3] or [10].

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§2. Proof of Theorem 1.1

2.1. A fiber space $f\colon X \to B$ of relative dimension n is a surjective morphism between smooth projective varieties X and B with connected geometric fibers of dimension n. We say that two fiber spaces $f_i\colon X \to B_i$ ($i = 1, 2$) are birationally equivalent if there are birational maps $\pi_1\colon X_1 \to X_2$ and $\pi_2\colon B_1 \to B_2$ such that $f_2\pi_1 = \pi_2 f_1$. A fiber space $f\colon X \to B$ is called birationally trivial, if it is birationally equivalent to the trivial fiber space $p\colon F\times B \to B$, where F is a general fiber of f and p is the projection.

2.2. Let $f\colon X \to B$ be a fiber space, and F a general fiber of f. We say that f has constant moduli, if any two smooth geometric fibers of f are birationally equivalent.

Assume that f has constant moduli and that the Kodaira dimension of F is non-negative. Then f admits a very concrete description, i.e., there exists a finite group G acting on F and on some smooth variety $\widetilde{B}$ such that f is birationally equivalent to (the smooth model of) the fiber space $p\colon (F \times \widetilde{B})/G \to \widetilde{B}/G$, where the action of G on the production $F \times \widetilde{B}$ is compatible with the actions on each factor and p is the projection to the second factor. (See [6, Theorem 2.11] or [7, Proposition 1] for a proof.)

2.3. Let $f\colon X \to B$ be a fiber space of relative dimension n, and F a general fiber of f. In what follows we always assume that B is a curve. Then $R^n f_*\mathcal{O}_X$ is a locally free sheaf of rank $p_g(F)$. By Theorem 3.1 [5], $\mathcal{O}_B^{\bigoplus h^0(R^n f_*\mathcal{O}_X)}$ is a direct factor of $R^n f_*\mathcal{O}_X$. By the Leray spectral sequence,

$$h^0(R^n f_*\mathcal{O}_X) + h^1(R^{n-1} f_*\mathcal{O}_X) = h^n(\mathcal{O}_X).$$

Combining these two facts, we get $h^n(\mathcal{O}_X) \le h^1(R^{n-1} f_*\mathcal{O}_X) + p_g(F)$.

NOTATION 2.4. Let X be a complex projective manifold. For any $0 \neq \alpha \in H^0(\Omega^i_X)$ $(1 \leq i \leq \dim X)$, we denote by $\mathrm{Z}(\alpha)$ the zero-locus of the holomorphic i-form α.

2.5. Let $f\colon X \to B$ and F be as in 2.3. Let ι be the embedding of F in X. We can factor the pullback of forms under the restriction map $\iota^*\colon \Omega^n_X \to \Omega^n_F$ by

$$\Omega^n_X \xrightarrow{r} \Omega^n_X|_F \longrightarrow \Omega^n_F.$$

Consider the long exact sequences associated with the exact sequences of sheaves

$$0 \longrightarrow \Omega^n_X(-F) \longrightarrow \Omega^n_X \xrightarrow{r} \Omega^n_X|_F \longrightarrow 0 \quad \text{and}$$
$$0 \longrightarrow \Omega^{n-1}_F \longrightarrow \Omega^n_X|_F \longrightarrow \Omega^n_F \longrightarrow 0.$$

Then we have that, if $h^{n-1,0}(F) = 0$, then for any $0 \neq \varphi \in H^0(\Omega^n_X)$, $\iota^*\varphi = 0$ if and only if $\varphi \in \mathrm{Ker}\ r$, i.e., $F \subset \mathrm{Z}(\varphi)$.

2.6. Let X be a complex projective manifold of dimension $n+1$ $(n \geq 2)$, with $h^{n,0}(X) \geq 2$. Assume that there are two linearly independent n-forms φ_1 and φ_2 such that $\varphi_1 \wedge \varphi_2 = 0$ in $H^0(\bigwedge^2 \Omega^n_X)$. Then there exists a non-constant rational function h on X such that $\varphi_2 = h\varphi_1$. Let $\pi\colon X' \to X$ be the blowing up of the locus of indeterminacy of the rational map

$$(1 : h)\colon X \dashrightarrow \mathbb{P}^1,$$

and $f_h\colon X' \to C$ the Stein factorization of $(1 : h) \circ \pi$. We have that h is constant along the fibers of f_h.

LEMMA 2.7. *Let X and f_h be as above. Then for any smooth fiber F of f_h, we have*

(i) $\iota_F^*(\pi^*\varphi_i) = 0$ *for $i = 1$ and 2, where we denote by ι_F the embedding of F in X',*

(ii) $h^{n-1,0}(F) > 0$.

Proof. (i) Indeed, for any $x \in F$, let $z_0, z_1, \ldots, z_n$ be a set of analytic local coordinates of X around x, such that z_0 is the pullback of a local coordinate of C around the image c of F by f_h. Then h is the pull-back of a non-constant holomorphic function of a neighborhood of c, and within an analytic neighborhood of x, we can write

$$\pi^*\varphi_1 = \sum_{i=0}^{n} A_i\, dz_0 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_n,$$
$$\pi^*\varphi_2 = \sum_{i=0}^{n} B_i\, dz_0 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_n,$$

($\widehat{dz_i}$ indicating the omission of the i-th factor dz_i) where A_i and B_i are holomorphic functions of this neighborhood. Clearly we have that

$$\iota_F^*(\pi^*\varphi_1) = A_0|_F\, dz_1 \wedge \cdots \wedge dz_n.$$

Since $\varphi_2 = h\varphi_1$, we have $B_i = hA_i$ for $i = 0, \ldots, n$. Since $\pi^*\varphi_j$ are d-closed, we have

$$\sum_{i=0}^{n}(-1)^i \frac{\partial A_i}{\partial z_i} = 0 \quad \text{and} \quad \sum_{i=0}^{n}(-1)^i \frac{\partial B_i}{\partial z_i} = 0.$$

Now

$$\begin{aligned}\frac{\partial h}{\partial z_0}A_0 &= \frac{\partial B_0}{\partial z_0} - h\frac{\partial A_0}{\partial z_0} = \frac{\partial B_0}{\partial z_0} + h\sum_{i=1}^{n}(-1)^i\frac{\partial A_i}{\partial z_i}\\ &= \frac{\partial B_0}{\partial z_0} + \sum_{i=1}^{n}(-1)^i\frac{\partial B_i}{\partial z_i} = 0.\end{aligned}$$

Note that $\partial h/\partial z_0 \neq 0$. Hence we get $A_0|F = 0$.

(ii) Let F' be a general fiber of f_h such that $F' \not\subset \mathrm{Z}(\pi^*\varphi_1)$. Suppose that $h^{n-1,0}(F') = 0$. Then by 2.5 we get $\iota_{F'}^*(\pi^*\varphi_1) \neq 0$. On the other hand, by (i), we have $\iota_{F'}^*(\pi^*\varphi_1) = 0$. This is a contradiction. ∎

The following lemma plays an important role in the proof of the Theorem 1.1.

LEMMA 2.8. *Let $f\colon X \to B$ be a fiber space of relative dimension $n \geq 2$, and F a general fiber of f. Assume that $\operatorname{rk}\mathcal{H}_X^n = 1$ (where $\mathcal{H}_X^n$ is as in Section 1), and $h^{n-1,0}(F) = 0$. Then $h^0(\Omega_X^n(-F)) = 0$.*

Proof. Consider the exact sequence

$$0 \longrightarrow H^0(\Omega_X^n(-F)) \longrightarrow H^0(\Omega_X^n) \xrightarrow{r} H^0(\Omega_X^n|_F).$$

Note that for any $\varphi \in H^0(\Omega_X^n)$, $\varphi \in \operatorname{Ker} r$ if and only if $\mathrm{Z}(\varphi) \supset F$. Since F is a general fiber of f, we have $\operatorname{Im} r \neq 0$ if $h^{n,0}(X) > 0$. We choose and fix a section $\varphi_0 \in H^0(\Omega_X^n)$ such that $r(\varphi_0) \neq 0$. Now it's enough to prove that $\operatorname{Ker} r = 0$. Otherwise, let $0 \neq \varphi_1 \in \operatorname{Ker} r$. Then $\mathrm{Z}(\varphi_1) \supset F$. Since $\operatorname{rk} \mathcal{H}_X^n = 1$, $\varphi_1 \wedge \varphi_0 = 0$. So there exists a rational function h on X such that $\varphi_1 = h\varphi_0$. Since $\mathrm{Z}(\varphi_0) \not\supset F$ by the choice of φ_0, h vanishes on F.

Let $f_h \colon X \to C$ be the fiber space induced by the rational map

$$(1 : h) \colon X \dashrightarrow \mathbb{P}^1.$$

By 2.7, $h^{n-1,0}(F_h) > 0$, where F_h is a smooth fiber of f_h. This implies f and f_h are different fibrations of X since $h^{n-1,0}(F) = 0$ by the assumption. So $f_h|_F \colon F \to C$ is surjective. Since h vanishes on F and is constant on the fibers of f_h, we get that h vanishes on X. This is a contradiction. □

The following proposition is a special case of 7.2.1 in [9].

Proposition 2.9. *Let $f \colon X \to Y$ be a morphism from a $(n+1)$-fold to a smooth projective n-fold. Suppose that, over a Zariski open set U of X, $\varphi \in H^0(X, \Omega_X^n)$ can be writen locally around each point $p \in U$ as $\varphi = \alpha f^*(\omega)$, where $\alpha \in \mathcal{O}_{p,X}$ and $\omega \in \Omega^n_{f(p),Y}$. Then $\varphi = \alpha f^*(\omega')$ for some $\omega' \in H^0(Y, \Omega_Y^n)$.*

2.10. Proof of Theorem 1.1

We prove that if $\operatorname{rk} \mathcal{H}_X^n = 1$ and $h^{n,0}(X) = p_g(F)$, then f is birationally trivial; the converse is clear since $\operatorname{rk} \mathcal{H}_X^n$ is a birational invariant of X (note that $\operatorname{rk} \mathcal{H}_X^n$ equals to the greatest integer i such that $\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_i \neq 0$ in $H^0(\bigwedge^i \Omega_X^n)$ for some $\varphi_1, \ldots, \varphi_i \in H^0(\Omega_X^n)$).

Let $\varphi_0, \varphi_1, \ldots, \varphi_m$ ($m = h^{n,0}(X) - 1$) be a basis of $H^0(\Omega_X^n)$. Since $\operatorname{rk} \mathcal{H}_X^n = 1$, there are non-constant rational functions h_i on X such that $\varphi_i = h_i \varphi_0$ for $i = 1, \ldots, m$. Consider the rational map

$$\Phi = (1 : h_1 : h_2 : \cdots : h_m) \colon X \dashrightarrow \mathbb{P}^m.$$

By Bogomolov's theorem [4], $\dim(\operatorname{Im} \Phi) \leq n$.

Let F be a general fiber of f, and ι the embedding of F in X. Since $h^{n-1,0}(F) = 0$, by 2.5,

$$\operatorname{Ker}(\iota^* \colon H^0(\Omega_X^n) \to H^0(\Omega_F^n)) \simeq H^0(\Omega_X^n(-F)).$$

By Lemma 2.8, $h^0(\Omega_X^n(-F)) = 0$. So $\iota^* \colon H^0(\Omega_X^n) \to H^0(\Omega_F^n)$ is an embedding, hence an isomorphism by the assumption $h^{n,0}(X) = p_g(F)$. This implies that $h_i|_F$, the restricion of h_i on F, are non-constant rational functions on F, and

$$\Phi|_F = (1 : h_1|_F : h_2|_F : \cdots : h_m|_F) \colon X \dashrightarrow \mathbb{P}^m$$

is nothing but the canonical map ϕ_F of F. Since ϕ_F is generically finite by assumption, we get $\dim(\operatorname{Im}\Phi) \geq \dim(\operatorname{Im}(\Phi|_F)) = n$. So $\operatorname{Im}\Phi = \operatorname{Im}(\Phi|_F)$ is a variety of dimension n. This implies that f has constant moduli if ϕ_F is birational. Now we show that if $\deg \phi_F$ is prime and $p_g(\operatorname{Im}\phi_F) = 0$, f also has constant moduli.

Consider the following commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\Phi'} & Y \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle s} \\ X & \overset{\Phi}{\dashrightarrow} & \operatorname{Im}\Phi, \end{array}$$

where π is the blowing up of the locus of indeterminacy of the rational map Φ and Φ' is the Stein factorization of $\Phi \circ \pi$. Taking the desingularisation of Y instead of Y, we can assume that Y is smooth.

Claim. $p_g(Y) = h^{n,0}(X)$.

Proof of the Claim. The case when $\dim X = 3$ is proved in [8, p. 861]; the general case can be similarly verified. Indeed, it's enough to verify that $\pi^*\varphi_i$ ($i = 0, \ldots, m$) are pull-backs of holomorphic n-forms on Y. Since $\operatorname{Im}\Phi$ has dimension n in $\mathbb{P}^m$, we may assume, after changing coordinates, that $z_i = Z_i/Z_0$ for $i = 1, \ldots, n$, forms a local coordinate system at a generic point $p \in \operatorname{Im}\Phi$, where $Z_0, \ldots, Z_m$ are homogeneous coordinates of $\mathbb{P}^m$. Consider the compositions g_i of $s \circ \Phi'$ with the projection

$$p_i \colon \operatorname{Im}\Phi \longrightarrow \mathbb{P}^1, \quad (1 : h_1(x) : h_2(x) : \cdots : h_m(x)) \longmapsto (1 : h_i(x)).$$

By blowing up if necessary, we may assume that all g_i $(i = 1, \dots, n)$ are morphisms. Let

$$U = X' \setminus \bigcup_{i=1}^{n} \{\text{singular fibers of } g_i\}.$$

Let $i_1 \colon F_1 \subset X'$ be the inclusion of a smooth fiber of g_1. Then by 2.7, we have $\iota_1^*(\pi^*\varphi_i) = 0$ for $i = 0$ and 1. Since $\varphi_i = h_i\varphi_0$, we get $\iota_1^*(\pi^*\varphi_i) = 0$ for all i. It implies that around $x \in U$,

$$\pi^*\varphi_i = \alpha_{1i} g_1^*(dz_1) \wedge \tau_{1i},$$

where $\alpha_{1i} \in \mathcal{O}_{x,X'}$ and $\tau_{1i} \in \Omega^{n-1}_{x,X'}$. Similarly, we have

$$\pi^*\varphi_i = \alpha_{2i} g_2^*(dz_2) \wedge \tau_{2i} = \cdots = \alpha_{ni} g_n^*(dz_n) \wedge \tau_{ni},$$

where $\alpha_{2i}, \dots, \alpha_{ni}$ are in $\mathcal{O}_{x,X'}$ and $\tau_{2i}, \dots, \tau_{ni}$ are in $\Omega^{n-1}_{x,X'}$. This shows that around $x \in U$,

$$\begin{aligned}\pi^*\varphi_i &= \alpha g_1^*(dz_1) \wedge g_2^*(dz_2) \wedge \cdots \wedge g_n^*(dz_n)\\ &= \alpha \Phi'^*((p_1 \circ s)^*(dz_1) \wedge (p_2 \circ s)^*(dz_2) \wedge \cdots \wedge (p_n \circ s)^*(dz_n))\end{aligned}$$

for some $\alpha \in \mathcal{O}_{x,X'}$. Now by Proposition 2.9, we have that $\pi^*\varphi_i$ are pull-backs of holomorphic n-forms on Y. ∎

Now we continue to prove the Theorem 1.1. Let F' be the strict transform of F under π. We have the following commutative diagram

$$\begin{array}{ccc} F' & \xrightarrow{\Phi'|_{F'}} & Y \\ \Big\downarrow{\scriptstyle \pi|_{F'}} & & \Big\downarrow{\scriptstyle s} \\ F & \overset{\Phi|_F = \phi_F}{\dashrightarrow} & \operatorname{Im}\Phi. \end{array}$$

If $\deg \phi_F$ is prime and $p_g(\operatorname{Im}\phi_F) = 0$, then $\deg s \neq 1$ since $p_g(Y) \neq p_g(\operatorname{Im}\phi_F)$. So $\deg(\Phi'|_{F'}) = 1$, and we have that f has constant moduli.

By 2.2, X is birationally equivalent to $(F \times \widetilde{B})/G$, where $\widetilde{B}$ and G are in 2.2. We claim that $|G| = 1$. In fact, from

$$h^{n,0}(F \times \widetilde{B}) = p_g(F) = h^{n,0}(X) = \dim H^0(\Omega^n_{F \times B'})^G,$$

we get $H^0(\Omega^n_F)^G = H^0(\Omega^n_F)$. So G induces identity on $\operatorname{Im}\phi_F$. This implies ϕ_F factors through $F \to F/G \to \operatorname{Im}\phi_F$. So we have $|G| = 1$ under the condition that either ϕ_F of F is birational, or ϕ_F is generically finite of degree being a prime number and $p_g(\operatorname{Im}\phi_F) = 0$. ∎

Remark 2.11. We give some remarks about the conditions on F in Theorem 1.1.

(1) If we only assume that F is of general type, the question may be too general to have a positive answer. But I failed to find an example of a birationally trivial fiber space which has a birationally non-trivial smooth deformation.

(2) If $h^{n-1,0}(F) \neq 0$, the existence of non-zero global $(n-1)$-forms on F makes the case more complicated (compare 2.5). Fortunately, since varieties with $h^{n-1}(\mathcal{O}_F) = h^{n-1,0}(F) > 0$ are special in the class of n dimensional varieties of general type, this is not a strong condition.

(3) Some typical examples of n-folds of general type with vanishing $h^{n-1,0}$: (a) regular surfaces of general type when $n = 2$, (b) smooth complete intersections in a projective space, (c) cyclic coverings of $\mathbb{CP}^n$ branched along a smooth divisor, and (d) products of varieties satisfying certain numerical conditions; e.g., let $F = Y \times S$, where Y (resp. S) is a smooth projective $(n-2)$-fold (resp. surface) of general type satisfying one of the following conditions: (i) $p_g(S) = 0$, (ii) $q(S) = 0$ and $h^{n-3,0}(Y) = 0$, or (iii) $p_g(Y) = 0$ and $h^{n-3,0}(Y) = 0$.

(4) We note that, if the canonical map ϕ_F of F is generically finite, then we have either $p_g(\operatorname{Im}\phi_F) = 0$ or $p_g(\operatorname{Im}\phi_F) = p_g(F)$ (cf. [1, Theorem 3.1]). The following example shows that the condition on ϕ_F can not be weaken.

EXAMPLE 2.12. Let S be a (smooth projective) regular surface. Assume that $\phi_S\colon S \to \operatorname{Im}\phi_S$ is generically finite of degree 2 and $p_g(S) = p_g(\operatorname{Im}\phi_S)$. (See [1, Proposition 3.6] for examples of such surfaces.) Let σ be the involution of S corresponding to ϕ_S. Let $\widetilde{B}$ be a smooth curve with an involution τ such that $\widetilde{B} \to B\colon = \widetilde{B}/\tau$ is étale. Take $X = (S \times \widetilde{B})/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on $S \times \widetilde{B}$ by $(s, \tilde{b}) \to (\sigma(s), \tau(\tilde{b}))$. It's easy to check that $\operatorname{rk}\mathcal{H}^2_X = 1$ and $h^{2,0}(X) = p_g(S)$. But the fiber space $f\colon X \to B$, which is induced by the projection $S \times \widetilde{B} \to \widetilde{B}$, is not birationally trivial.

§3. Miscellaneous results

Let F be a projective manifold with trivial canonical sheaf. An automorphism σ of F is said symplectic, if σ induces trivial action on $H^0(\omega_F)$, where ω_F is the canonical sheaf of F.

THEOREM 3.1. *Let $f\colon X \to B$ be a fiber space of relative dimension n over a curve B, and F a general fiber of f. Assume that F is a projective manifold with trivial canonical sheaf and that $h^{n-1,0}(F) = 0$ (e.g.,* *an algebraic $K3$ surface and its higher dimensional analogue, a projective Calabi-Yau manifold, etc.). Then $h^{n,0}(X) \leq 1$, and $h^{n,0}(X) = 1$ if and only if either f is birationally trivial, or f is birationally isomorphic to $(F \times \widetilde{B})/G \to \widetilde{B}/G$, where G is a finite group acting on F and $\widetilde{B}$ such that the action of G on F is symplectic and $\widetilde{B}/G \simeq B$.*

Proof. $h^{n,0}(X) \leq 1$ follows by 2.3. Now we assume that $h^{n,0}(X) = 1$. Let

$$\Sigma = \{\text{critical points of } f\} \cup \{p \in B \mid f^*p \subset \mathrm{Z}(\varphi)\},$$

where φ be the unique holomorphic n-form on X up to scalar multiple. Set $B^o = B \setminus \Sigma$, $X^o = f^{-1}B^o$ and $f^o = f|_{X^o}$.

Since $p_g(F) = 1$, $\mathcal{L} := f^o_*\omega_{X^o}$ is an invertible sheaf. We have an exact sequence of sheaves

$$0 \longrightarrow (f^o)^*\mathcal{L} \longrightarrow \omega_{X^o}.$$

So $\omega_{X^o} = (f^o)^*\mathcal{L} \otimes \mathcal{O}_{X^o}(D)$ for some non-negative divisor D on X^o. From $\mathcal{O}_F = \omega_{X^o}|_F = (f^o)^*\mathcal{L} \otimes \mathcal{O}_{X^o}(D)|_F = \mathcal{O}_F(D)$, we have that D consists of fibers of f^o. Hence $\omega_{X^o} = (f^o)^*\mathcal{L}'$ for some $\mathcal{L}' \in \mathrm{pic}(B^o)$.

Since $h^{n-1,0}(F) = 0$ by the assumption, by 2.5 we have that for any fiber F of f^o, $\iota^*\varphi \neq 0$, where ι is the embedding of F in X^o. By Lemma 4.3 of [5], we get that f^o has constant moduli.

By 2.2, X is birational to $(F \times \widetilde{B})/G$, where G and $\widetilde{B}$ are in 2.2. Since

$$\dim H^0(\Omega^n_{F\times\widetilde{B}})^G = h^0(\Omega^n_X) = 1 = h^0(\Omega^n_{F\times\widetilde{B}}),$$

we have that either $|G| = 1$ or G acts trivially on $H^0(\Omega^n_F)$. This proves the "only if" part. The "if" part is clear. □

THEOREM 3.2. *Let $f\colon X \to B$ be a fiber space of relative dimension n over a curve B, and F a general fiber of f. Assume that F is an Abelian variety. Then $q(X) \leq n + g(B)$, and $q(X) = n + g(B)$ if and only if either f is birationally trivial, or f is birationally isomorphic to $(F \times \widetilde{B})/G \to \widetilde{B}/G$, where G is a finite Abelian group acting on F and $\widetilde{B}$ such that the action of G on F consists of translations of F and $\widetilde{B}/G \simeq B$.*

Proof. By the universal property of the Albanese map, we have a morphism $\alpha\colon \operatorname{Alb} X \to \operatorname{Alb} B$ such that the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\operatorname{alb}_X} & \operatorname{Alb} X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \alpha} \\ B & \xrightarrow{\operatorname{alb}_B} & \operatorname{Alb} B \end{array}$$

commutes. Note that α is a fiber bundle whose fiber A is an Abelian variety of dimension $q(X) - g(B)$. Let p be a general point of B. We have that

$$\operatorname{alb}_X|_{f^*(p)}\colon f^*(p) \longrightarrow A = \alpha^*(\operatorname{alb}_B(p))$$

is surjective since the image of $f^*(p)$ in A generates A and $f^*(p)$ itself is an Abelian variety. So $q(X) - g(B) \leq n = \dim f^*(p)$.

Now assume that $q(X) - g(B) = n$. Then f has constant moduli since there are at most coutable Abelian varieties isogenous to a given Abelian variety. By 2.2, X is birational to $(F \times \widetilde{B})/G$, where G and $\widetilde{B}$ are in 2.2. Since

$$\dim H^0(\Omega^1_{F\times\widetilde{B}})^G = h^0(\Omega^1_X) = n + g(B) = h^0(\Omega^1_{F\times\widetilde{B}}),$$

we have that G acts trivially on $H^0(\Omega^1_F)$. If there is an element $\sigma \in G$ such that σ has a fixed point, say $p \in F$, then σ acts trivially on the tangent space $\mathrm{T}_p F$, since σ acts trivially on $H^0(\Omega^1_F)$. This implies $\sigma = 1$. So we have either $|G| = 1$ or G consists of translations of F. This proves the "only if" part. The converse is clear. ∎

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