

ON MODIFICATION OF THE q - L -SERIES AND ITS APPLICATIONS

HIROFUMI TSUMURA

Abstract. We slightly modify the definitions of q -Hurwitz ζ -functions and q - L -series constructed by J. Satoh. By using these modified functions, we give some relations for the ordinary Dirichlet L -series. Especially we give an elementary proof of Katsurada's formula on the values of Dirichlet L -series at positive integers.

Introduction

Satoh defined q - L -series $L_q(s, \chi)$ in [S-1], which interpolated Carlitz's q -Bernoulli numbers at non-positive integers. His result was a response to Koblitz's problem suggested in [Ko]. In fact, $L_q(s, \chi)$ could be regarded just as what Koblitz required. $L_q(s, \chi)$ was essentially defined as a sum of two q -series. This causes difficulty in studying $L_q(s, \chi)$.

In [T-3], we considered the modified q -Riemann ζ -function, which is an example of Satoh's recent result (see [S-2]). By elementary calculations of q -series, we proved the formulas for $\zeta(2k+1)$ given by Cvijović and Klinowski ([C-K]).

In the present paper, corresponding to our previous work in [T-3], we modify the definition of q - L -series. In Section 1, we consider the modified q -Hurwitz ζ -function. In Section 2, we define the modified q - L -series. By investigating their properties, we prove some relations for the values of modified q - L -series (see Lemma 7). By letting $q \rightarrow 1$ in these relations, we prove some relations between the values of ordinary Dirichlet L -series at positive integers (see Proposition 2). Furthermore we give another proof of Katsurada's recent result on the values of Dirichlet L -series at positive integers (see Proposition 3). His result was proved by using the Mellin transformation technique ([Ka]).

Received October 22, 1999.

Revised March 31, 2000.

2000 Mathematics Subject Classification: Primary 11M41, 11M06. Secondly 11B68
11M06, 11M35, 33B15.

The author would like to thank the referee for his valuable suggestions and comments.

§1. q -Hurwitz ζ -function

For $q \in \mathbf{R}$ with $0 < q < 1$, let $[z] = [z; q] = (1 - q^z)/(1 - q)$ for an indeterminate z . Note that $\lim_{q \rightarrow 1}[z] = z$. The modified q -Bernoulli numbers $\{\tilde{\beta}_n(q)\}$ can be defined by

$$F_q(t) = \sum_{n=0}^{\infty} \tilde{\beta}_n(q) \frac{t^n}{n!},$$

where $F_q(t)$ is determined as a solution of the following q -difference equation

$$F_q(t) = e^t F_q(qt) - t, \quad F_q(0) = \frac{q-1}{\log q},$$

(see [T-1]). Moreover we let $F_1(t) = t/(e^t - 1)$, and $\tilde{\beta}_n(1) = B_n$ which is the ordinary Bernoulli number. If $0 < q < 1$ then the following series representation for $F_q(t)$ holds:

$$(1.1) \quad F_q(t) = \frac{q-1}{\log q} e^{t/(1-q)} - t \sum_{n=0}^{\infty} q^n e^{[n]t},$$

(see [S-2],[T-3]). By above considerations, we can see that $F_q(t)$ is continuous as a function of (q, t) on $(0, 1] \times \{t \in \mathbf{C} \mid |t| < 2\pi\}$. As generalizations, we defined the modified q -Bernoulli polynomials by

$$F_q(q^x t) e^{[x]t} = \sum_{n=0}^{\infty} \tilde{\beta}_n(x, q) \frac{t^n}{n!}.$$

Note that

$$F_q(q^x t) e^{[x]t} = \frac{q-1}{\log q} e^{t/(1-q)} - t \sum_{n=0}^{\infty} q^{n+x} e^{[n+x]t}.$$

We define the modified q -Hurwitz ζ -function by

$$(1.2) \quad \tilde{\zeta}_q(s, x) = \frac{(1-q)^s}{(1-s)\log q} + \sum_{n=0}^{\infty} \frac{q^{n+x}}{[n+x]^s},$$

for $x > 0$. The following lemma holds (see [T-3] §4).

LEMMA 1. For $k \in \mathbf{N}$, $\tilde{\zeta}_q(1 - k, x) = -\frac{\tilde{\beta}_k(x, q)}{k}$.

It follows from (1.2) that $\tilde{\zeta}_q(s, x)$ is meromorphic in the whole complex plane and has a simple pole at $s = 1$ with residue $(q - 1)/\log q$, since $\lim_{n \rightarrow \infty} [n] = 1/(1 - q)$ if $0 < q < 1$. It is obvious that if $Re(s) > 1$ then $\lim_{q \rightarrow 1-0} \tilde{\zeta}_q(s, x) = \zeta(s, x)$ which is the ordinary Hurwitz ζ -function. More strongly we can prove the following.

LEMMA 2. $\lim_{q \rightarrow 1-0} \tilde{\zeta}_q(s, x) = \zeta(s, x)$ and $\lim_{q \rightarrow 1-0} (\partial/\partial s)\tilde{\zeta}_q(s, x) = (\partial/\partial s)\zeta(s, x)$ for any $s \in \mathbf{C}$ except for $s = 1$.

Proof. According to the well-known method (e.g. [W, Theorem 4.2]), we consider the function

$$H(s, q) = (e^{2\pi\sqrt{-1}s} - 1) \int_0^\infty t^{s-2} F_q(-q^x t) e^{-[x]t} dt,$$

for any $s \in \mathbf{C}$ and $q \in (0, 1]$. Then it follows from (1.1) that $H(s, q) = (e^{2\pi\sqrt{-1}s} - 1)\Gamma(s)\tilde{\zeta}_q(s, x)$, and $H(s, q)$ is holomorphic for any $s \in \mathbf{C}$ if $0 < q \leq 1$. We can verify that $\lim_{q \rightarrow 1-0} H(s, q) = H(s, 1)$ and $\lim_{q \rightarrow 1-0} (\partial/\partial s)H(s, q) = (\partial/\partial s)H(s, 1)$. Thus we have the assertion.

If $0 < q < 1$ then, by (1.2), we have

$$(1.3) \quad \frac{\partial}{\partial s} \tilde{\zeta}_q(s, x) = \frac{(1 - q)^s \{\log(1 - q) + 1\}}{(1 - s)^2 \log q} - \sum_{n=0}^\infty \frac{q^{n+x} \log[n + x]}{[n + x]^s}.$$

Let

$$(1.4) \quad a(q) = \frac{\partial}{\partial s} \tilde{\zeta}_q(0, 1) = \frac{\log(1 - q) + 1}{\log q} - \sum_{m=1}^\infty q^m \log[m].$$

By Lemma 2, we have

$$(1.5) \quad \lim_{q \rightarrow 1-0} a(q) = \lim_{q \rightarrow 1-0} \frac{\partial}{\partial s} \tilde{\zeta}_q(0, 1) = \frac{\partial}{\partial s} \zeta(0, 1) = -\frac{1}{2} \log(2\pi).$$

Let $b(q) = \exp(-a(q))$. Then $\lim_{q \rightarrow 1-0} b(q) = \sqrt{2\pi}$. By combining (1.4) and (1.5), we get the following relation which can be regarded as a q -representation for the divergent formula $\prod_{m \geq 1} m = \infty! = \sqrt{2\pi}$ given by Riemann.

PROPOSITION 1. $\lim_{q \rightarrow 1-0} e^{-\frac{\log(1-q)+1}{\log q}} \prod_{m=1}^{\infty} [m]^{q^m} = \sqrt{2\pi}.$

§2. *q-L-series*

For a primitive Dirichlet character χ with conductor f , we define the modified *q-L-series* by

$$(2.1) \quad \tilde{L}_q(s, \chi) = \sum_{a=1}^f \chi(a)[f]^{-s} \tilde{\zeta}_{q^f} \left(s, \frac{a}{f} \right).$$

We can verify that

$$\begin{aligned} \tilde{L}_q(s, \chi) &= \sum_{a=1}^f \chi(a)[f]^{-s} \left\{ \frac{(1-q^f)^s}{(1-s)\log q^f} + \sum_{n=0}^{\infty} \frac{q^{f(n+a/f)}}{[n+a/f, q^f]^s} \right\} \\ &= \frac{(1-q)^s}{f(1-s)\log q} \sum_{a=1}^f \chi(a) + \sum_{n=1}^{\infty} \frac{\chi(n)q^n}{[n]^s}. \end{aligned}$$

So we have

$$(2.2) \quad \tilde{L}_q(s, \chi) = \begin{cases} \sum_{n=1}^{\infty} \frac{\chi(n)q^n}{[n]^s} & (\chi \neq 1) \\ \frac{(1-q)^s}{(1-s)\log q} + \sum_{n=1}^{\infty} \frac{q^n}{[n]^s} & (\chi = 1) \end{cases}$$

In fact, $\tilde{L}_q(s, 1)$ coincides with the *q-series* $\tilde{Z}_q(s)$ defined in [T-3], which can be regarded as a *q-analogue* of the Riemann ζ -function. Note that if $\chi \neq 1$ then $\tilde{L}_q(s, \chi)$ is holomorphic in the whole complex plane.

Now we define the generalized *q-Bernoulli numbers* by

$$(2.3) \quad \tilde{\beta}_{k,\chi}(q) = [f]^{k-1} \sum_{a=1}^f \chi(a) \tilde{\beta}_k \left(\frac{a}{f}, q^f \right),$$

for $k \geq 0$. Note that $\lim_{q \rightarrow 1} \tilde{\beta}_{k,\chi}(q) = B_{k,\chi}$ which is the generalized Bernoulli number defined by

$$(2.4) \quad \sum_{a=1}^f \frac{\chi(a)te^{at}}{e^{ft} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}.$$

By (2.1),(2.3) and Lemma 1, we have the following.

LEMMA 3. For $k \in \mathbf{N}$, $\tilde{L}_q(1 - k, \chi) = -\frac{\tilde{\beta}_{k,\chi}(q)}{k}$.

From now on, we assume that $\chi \neq 1$. Let

$$H_q(t, \chi) = -t \sum_{n=1}^{\infty} \chi(n) q^n e^{[n]t},$$

for $q \in \mathbf{R}$ with $0 < q < 1$. It follows from the definition of $F_q(t)$ and (2.2) that $H_q(t, \chi)$ is the generating function of $\{\tilde{\beta}_{k,\chi}(q)\}$, and is holomorphic in the whole complex plane. For the sake of convenience, let $H_1(t, \chi)$ be the function in the left-hand side of (2.4).

We can see that poles of $H_1(t, \chi)$ are $\{2\pi\sqrt{-1}l/f + 2n\pi\sqrt{-1} \mid n \in \mathbf{Z}, l = 0, 1, \dots, f - 1\}$. So we let

$$\begin{aligned} (2.5) \quad h(t, f) &= \prod_{l=1}^f (t - 2\pi\sqrt{-1}l/f)(t + 2\pi\sqrt{-1}l/f) \\ &= \prod_{l=1}^f (t^2 + 4\pi^2 l^2 / f^2) = \sum_{l=0}^f C_l(f) t^{2l}, \end{aligned}$$

and let $I_q(t, \chi) = H_q(t, \chi)h(t, f)$ for any q with $0 < q \leq 1$. Then we see that $I_q(t, \chi)$ is holomorphic on $|t| \leq 2\pi$. Let

$$(2.6) \quad I_q(t, \chi) = \sum_{n=0}^{\infty} A_n(q, \chi) \frac{t^n}{n!}.$$

Then we have the following.

LEMMA 4. Let r and d be real numbers with $0 < r < 2\pi$ and $0 < d < 1$. Then there exists a constant $R(r, d) > 0$ such that $|A_k(q, \chi)/k!| \leq R(r, d)/r^k$ for $k \geq 0$, if $d \leq q \leq 1$.

Proof. Let C_r be a circle around O of radius r in the complex plane. By the consideration in §1, we can see that $I_q(t, \chi)$ is continuous as a function of (q, t) on the compact set $[d, 1] \times C_r$. So we let $R(r, d) = \text{Max}|I_q(t, \chi)|$ on $[d, 1] \times C_r$. By the fact that

$$\frac{A_k(q, \chi)}{k!} = \frac{1}{2\pi\sqrt{-1}} \int_{C_r} I_q(t, \chi) t^{-k-1} dt,$$

we get the proof of Lemma.

Now we consider the following permutation and combination function:

$$P(X, k) = \prod_{j=0}^{k-1} (X - j), \quad \binom{X}{k} = \frac{P(X, k)}{k!},$$

for any $k \in \mathbf{Z}$ with $k \geq 0$. Formally we let $P(0, 0) = 1$. If $m \in \mathbf{Z}$ with $0 \leq m < k$, then $P(m, k) = 0$. By considering the binomial expansions of both sides of $(1 + t)^{X+Y} = (1 + t)^X(1 + t)^Y$, we get the following.

LEMMA 5. $\binom{X + Y}{k} = \sum_{j=0}^k \binom{X}{k - j} \binom{Y}{j}$, namely $P(X + Y, k) = \sum_{j=0}^k \binom{k}{j} P(X, k - j)P(Y, j)$.

By Lemma 3 and using the above notations, we have

$$\begin{aligned} I_q(t, \chi) &= \sum_{l=0}^f C_l(f) \sum_{n=0}^{\infty} \tilde{\beta}_{n, \chi}(q) \frac{t^{n+2l}}{n!} \\ &= \sum_{l=0}^f C_l(f) \sum_{N \geq 2l} P(N, 2l) \tilde{\beta}_{N-2l, \chi}(q) \frac{t^N}{N!} \\ &= - \sum_{N=0}^{\infty} \left(\sum_{l=0}^f C_l(f) P(N, 2l + 1) \tilde{L}_q(1 - N + 2l, \chi) \right) \frac{t^N}{N!}. \end{aligned}$$

Thus we have the following.

LEMMA 6. For $N \in \mathbf{Z}$ with $N \geq 0$,

$$\begin{aligned} A_N(q, \chi) &= \sum_{l=0}^f C_l(f) P(N, 2l) \tilde{\beta}_{N-2l, \chi}(q) \\ &= - \sum_{l=0}^f C_l(f) P(N, 2l + 1) \tilde{L}_q(1 - N + 2l, \chi). \end{aligned}$$

Remark. Since $B_{2k+1, \chi} = 0$ if $\chi(-1) = 1$ and $B_{2k, \chi} = 0$ if $\chi(-1) = -1$ (e.g. [W] Chap.4), we have $\lim_{q \rightarrow 1} A_{2k+1}(q, \chi) = 0$ if $\chi(-1) = 1$, and $\lim_{q \rightarrow 1} A_{2k}(q, \chi) = 0$ if $\chi(-1) = -1$, for $k \geq 0$.

LEMMA 7. For $m \in \mathbf{N}$ and $\theta \in \mathbf{R}$ with $|\theta| \leq 2\pi$,

$$\begin{aligned}
 (1) \quad & \sum_{d=0}^f P(2m+2d-1, 2d) \sum_{l=d}^f \binom{2l+1}{2d} C_l(f) (-1)^{l-d} \theta^{2(l-d)+1} \\
 & \qquad \qquad \qquad \times \sum_{n=1}^{\infty} \frac{\chi(n)q^n}{[n]^{2m+2d}} \cos([n]\theta) \\
 & - \sum_{d=0}^f P(2m+2d, 2d+1) \sum_{l=d}^f \binom{2l+1}{2d+1} C_l(f) (-1)^{l-d} \theta^{2(l-d)} \\
 & \qquad \qquad \qquad \times \sum_{n=1}^{\infty} \frac{\chi(n)q^n}{[n]^{2m+2d+1}} \sin([n]\theta) \\
 = & \sum_{k=0}^{m-1} \frac{(-1)^{k+1} \theta^{2k+1}}{(2k+1)!} \\
 & \times \sum_{l=0}^f C_l(f) P(2m-2k+2l-1, 2l+1) \tilde{L}_q(2m-2k+2l, \chi) \\
 & + (-1)^m \theta^{2m} \sum_{n=0}^{\infty} \frac{1}{P(2n+2m+1, 2m)} \frac{(-1)^{n+1} \theta^{2n+1}}{(2n+1)!} A_{2n+1}(q, \chi). \\
 (2) \quad & \sum_{d=0}^f P(2m+2d-1, 2d) \sum_{l=d}^f \binom{2l+1}{2d} C_l(f) (-1)^{l-d+1} \theta^{2(l-d)+1} \\
 & \qquad \qquad \qquad \times \sum_{n=1}^{\infty} \frac{\chi(n)q^n}{[n]^{2m+2d}} \sin([n]\theta) \\
 & + \sum_{d=0}^f P(2m+2d, 2d+1) \sum_{l=d}^f \binom{2l+1}{2d+1} C_l(f) (-1)^{l-d+1} \theta^{2(l-d)} \\
 & \qquad \qquad \qquad \times \sum_{n=1}^{\infty} \frac{\chi(n)q^n}{[n]^{2m+2d+1}} \cos([n]\theta) \\
 = & \sum_{k=0}^{m-1} \frac{(-1)^{k+1} \theta^{2k}}{(2k)!} \\
 & \times \sum_{l=0}^f C_l(f) P(2m-2k+2l, 2l+1) \tilde{L}_q(2m-2k+2l+1, \chi) \\
 & + (-1)^{m-1} \theta^{2m} \sum_{n=0}^{\infty} \frac{1}{P(2n+2m, 2m+1)} \frac{(-1)^n \theta^{2n}}{(2n)!} A_{2n}(q, \chi).
 \end{aligned}$$

Proof. We only give the proof of (1). The proof of (2) is given in just

the same manner as that of (1). For simplicity, we denote C_l instead of $C_l(f)$. Let

$$J_q(\theta, \chi, m) = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} \times \left\{ \sum_{l=0}^f C_l P(2k-2m+1, 2l+1) \tilde{L}_q(1-(2k-2m+1)+2l, \chi) \right\}.$$

By Lemma 5, we have

$$\begin{aligned} & J_q(\theta, \chi, m) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} \sum_{l=0}^f C_l \\ &\quad \times \sum_{u=0}^{2l+1} \binom{2l+1}{u} P(2k+1, 2l+1-u) P(-2m, u) \tilde{L}_q(-2k+2m+2l, \chi) \\ &= \sum_{l=0}^f C_l \sum_{u=0}^{2l+1} \binom{2l+1}{u} P(-2m, u) \\ &\quad \times \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} P(2k+1, 2l+1-u) \tilde{L}_q(-2k+2m+2l, \chi) \\ &= \sum_{l=0}^f C_l \sum_{d=0}^l \binom{2l+1}{2d} P(-2m, 2d) \\ &\quad \times \sum_{k=l-d}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} P(2k+1, 2l+1-2d) \tilde{L}_q(-2k+2m+2l, \chi) \\ &\quad \quad \quad + \sum_{l=0}^f C_l \sum_{d=0}^l \binom{2l+1}{2d+1} P(-2m, 2d+1) \\ &\quad \times \sum_{k=l-d}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} P(2k+1, 2l-2d) \tilde{L}_q(-2k+2m+2l, \chi). \end{aligned}$$

Since $0 < q < 1$, we can easily verify that

$$(2.7) \quad \sum_{n=1}^{\infty} \frac{\chi(n) q^n}{[n]^s} \cos([n]\theta) = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} \tilde{L}_q(s-2k, \chi),$$

and

$$(2.8) \quad \sum_{n=1}^{\infty} \frac{\chi(n)q^n}{[n]^s} \sin([n]\theta) = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} \tilde{L}_q(s-2k-1, \chi).$$

By noticing that $P(-N, e) = (-1)^e P(N+e-1, e)$, and letting $n = k-l+d$, we have

$$\begin{aligned} J_q(\theta, \chi, m) &= \sum_{d=0}^f P(2m+2d-1, 2d) \sum_{l=0}^f \binom{2l+1}{2d} C_l (-1)^{l-d} \theta^{2(l-d)+1} \\ &\quad \times \sum_{n=1}^{\infty} \frac{\chi(n)q^n}{[n]^{2m+2d}} \cos([n]\theta) \\ &\quad + \sum_{d=0}^f P(2m+2d, 2d+1) \sum_{l=0}^f \binom{2l+1}{2d+1} C_l (-1)^{l-d} \theta^{2(l-d)} \\ &\quad \times \sum_{n=1}^{\infty} \frac{\chi(n)q^n}{[n]^{2m+2d+1}} \sin([n]\theta). \end{aligned}$$

On the other hand, by Lemma 6, we have

$$\begin{aligned} J_q(\theta, \chi, m) &= \sum_{k=0}^{m-1} \frac{(-1)^{k+1} \theta^{2k+1}}{(2k+1)!} \sum_{l=0}^f C_l P(2k-2m+1, 2l+1) \tilde{L}_q(2m-2k+2l, \chi) \\ &\quad + \sum_{k=m}^{\infty} \frac{(-1)^{k+1} \theta^{2k+1}}{(2k+1)!} A_{2k-2m+1}(q, \chi). \end{aligned}$$

Thus we have the proof of (1).

By letting $q \rightarrow 1$ with respect to the equations in Lemma 7, we get some relations for the values of ordinary Dirichlet L -series at positive integers.

PROPOSITION 2. *Let $m \in \mathbf{N}$ and $C_l(f) \in \mathbf{R}$ defined by (2.5).*

(1) *If $\chi(-1) = 1$ and $\chi \neq 1$, then*

$$\sum_{d=0}^f P(2m+2d-1, 2d)$$

$$\begin{aligned} & \times \sum_{l=d}^f \binom{2l+1}{2d} C_l(f) (-1)^{l-d} (2\pi)^{2(l-d)+1} L(2m+2d, \chi) \\ &= \sum_{k=0}^{m-1} \frac{(-1)^{k+1} (2\pi)^{2k+1}}{(2k+1)!} \\ & \quad \times \sum_{l=0}^f C_l(f) P(2m-2k+2l-1, 2l+1) L(2m-2k+2l, \chi). \end{aligned}$$

(2) If $\chi(-1) = -1$, then

$$\begin{aligned} & \sum_{d=0}^f P(2m+2d, 2d+1) \\ & \quad \times \sum_{l=d}^f \binom{2l+1}{2d+1} C_l(f) (-1)^{l-d+1} (2\pi)^{2(l-d)} L(2m+2d+1, \chi) \\ &= \sum_{k=0}^{m-1} \frac{(-1)^{k+1} (2\pi)^{2k}}{(2k)!} \\ & \quad \times \sum_{l=0}^f C_l(f) P(2m-2k+2l, 2l+1) L(2m-2k+2l+1, \chi). \end{aligned}$$

Proof. By Lemma 4, we can see that both sides of the equations in (1) and (2) of Lemma 7 are uniformly convergent with respect to $q \in (0, 1]$, if $\theta = 2\pi$. So we can let $q \rightarrow 1$. By Remark after Lemma 6, we get the proof.

In [Ka], Katsurada recently proved the following series representations for the values of $L(s, \chi)$ at positive integers by using the Mellin transformation technique. In the rest of this section, we give another proof of Katsurada's result by using the same method as above.

PROPOSITION 3. ([Ka, Theorem 3]) *Let n be a positive integer, x be a real number with $|x| \leq 1$ and $\tau(\chi) = \sum_{a=1}^f \chi(a) \exp(2\pi\sqrt{-1}a/f)$ be the Gauss sum.*

(1) *If $\chi(-1) = 1$ and $\chi \neq 1$, then*

$$nL(2n+1, \chi) - n \sum_{l=1}^{\infty} \frac{\chi(l) \cos(2\pi lx/f)}{l^{2n+1}} - \frac{\pi x}{f} \sum_{l=1}^{\infty} \frac{\chi(l) \sin(2\pi lx/f)}{l^{2n}}$$

$$= (-1)^n \left(\frac{2\pi x}{f}\right)^{2n} \left\{ \sum_{k=1}^{n-1} \frac{(-1)^{k-1} k L(2k+1, \chi)}{(2n-2k)! (2\pi x/f)^{2k}} + \frac{\tau(\chi)}{f} \sum_{k=1}^{\infty} \frac{(2k)! L(2k, \bar{\chi})}{(2n+2k)!} x^{2k} \right\};$$

(2) If $\chi(-1) = -1$, then

$$\begin{aligned} L(2n, \chi) &= \sum_{l=1}^{\infty} \frac{\chi(l) \cos(2\pi l x/f)}{l^{2n}} \\ &= (-1)^n \left(\frac{2\pi x}{f}\right)^{2n-1} \left\{ \sum_{k=1}^{n-1} \frac{(-1)^{k-1} L(2k, \chi)}{(2n-2k)! (2\pi x/f)^{2k-1}} + \frac{2\sqrt{-1}\tau(\chi)}{f} \sum_{k=0}^{\infty} \frac{(2k)! L(2k+1, \bar{\chi})}{(2n+2k)!} x^{2k+1} \right\}. \end{aligned}$$

Proof. Suppose that $\chi(-1) = 1$ and $\chi \neq 1$, $q \in \mathbf{R}$ with $0 < q < 1$, and $\theta \in \mathbf{R}$ with $|\theta| < 2\pi/f$. By (2.7), (2.8) and Lemma 3, we have

$$\begin{aligned} (3.1) \quad n \sum_{l=1}^{\infty} \frac{\chi(l) q^l \cos([l]\theta)}{[l]^{2n+1}} - \frac{\theta}{2} \sum_{l=1}^{\infty} \frac{\chi(l) q^l \sin([l]\theta)}{[l]^{2n}} \\ = n \tilde{L}_q(2n+1, \chi) + \sum_{j=1}^{n-1} \frac{(-1)^j \theta^{2j}}{(2j)!} (n-j) \tilde{L}_q(2n+1-2j, \chi) \\ + \frac{1}{2} \sum_{j=n+1}^{\infty} \frac{(-1)^j \theta^{2j}}{(2j)!} \tilde{\beta}_{2j-2n, \chi}(q). \end{aligned}$$

By the definition of $\tilde{\beta}_{n, \chi}(q)$ and the same reason as that in the proof of Proposition 2, we can see that both sides of (3.1) are uniformly convergent with respect to $q \in (0, 1]$ if $|\theta| < 2\pi/f$. Hence we can let $q \rightarrow 1$ in both sides of (3.1). By using the well-known relation

$$B_{2j, \chi} = \frac{2(-1)^{j+1} \tau(\chi)}{f} \left(\frac{f}{2\pi}\right)^{2j} (2j)! L(2j, \bar{\chi}),$$

we have

$$n \sum_{l=1}^{\infty} \frac{\chi(l) \cos(l\theta)}{l^{2n+1}} - \frac{\theta}{2} \sum_{l=1}^{\infty} \frac{\chi(l) \sin(l\theta)}{l^{2n}}$$

$$\begin{aligned}
 &= nL(2n + 1, \chi) + (-1)^n \theta^{2n} \left\{ \sum_{k=1}^{n-1} \frac{(-1)^k k L(2k + 1, \chi)}{(2n - 2k)! \theta^{2k}} \right. \\
 &\quad \left. - \frac{\tau(\chi)}{f} \sum_{m=1}^{\infty} \frac{(-1)^m \theta^{2m} (2m)!}{(2m + 2n)!} \left(\frac{f}{2\pi} \right)^{2m} L(2m, \bar{\chi}) \right\}.
 \end{aligned}$$

By putting $\theta = (2\pi x/f)$, we get the proof of (1).

Suppose that $\chi(-1) = -1$. By (2.7) and Lemma 3, we have

$$\begin{aligned}
 &\sum_{l=1}^{\infty} \frac{\chi(l) q^l \cos([l]\theta)}{[l]^{2n}} \\
 &= \tilde{L}_q(2n, \chi) + \sum_{j=1}^{n-1} \frac{(-1)^j \theta^{2j}}{(2j)!} \tilde{L}_q(2n - 2j, \chi) \\
 &\quad + \sum_{j=n}^{\infty} \frac{(-1)^j \theta^{2j}}{(2j)!} \left(-\frac{\tilde{\beta}_{2j-2n+1, \chi}(q)}{2j - 2n + 1} \right).
 \end{aligned}$$

By letting $q \rightarrow 1$, putting $\theta = 2\pi x/f$ and by using the relation

$$B_{2j+1, \chi} = \frac{2(-1)^j \sqrt{-1} \tau(\chi)}{f} \left(\frac{f}{2\pi} \right)^{2j+1} (2j)! L(2j + 1, \bar{\chi}),$$

we get the proof of (2). Thus we have the assertion.

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Department of Management
Tokyo Metropolitan College
Azuma-cho, Akishima-shi
Tokyo 196-8540, Japan
tsumura@tmca.ac.jp