Y. Ando
Nagoya Math. J.
Vol. 164 (2001), 147–184

ON HOLOMORPHIC MAPS WITH ONLY FOLD SINGULARITIES

YOSHIFUMI ANDO[1]

Dedicated to Professor Takuo Fukuda on his sixtieth birthday

Abstract. Let $f : N \to P$ be a holomorphic map between n-dimensional complex manifolds which has only fold singularities. Such a map is called a holomorphic fold map. In the complex 2-jet space $J^2(n, n; \mathbf{C})$, let Ω^{10} denote the space consisting of all 2-jets of regular map germs and fold map germs. In this paper we prove that Ω^{10} is homotopy equivalent to $\mathrm{SU}(n+1)$. By using this result we prove that if the tangent bundles TN and TP are equipped with $\mathrm{SU}(n)$-structures in addition, then a holomorphic fold map f canonically determines the homotopy class of an $\mathrm{SU}(n+1)$-bundle map of $TN \oplus \theta_N$ to $TP \oplus \theta_P$, where θ_N and θ_P are the trivial line bundles.

Introduction

Let N and P be complex manifolds of dimension n. We shall say that a holomorphic map germ of (N, x) to (P, y) has a fold singularity at x if it is written as $(z_1, \dots, z_{n-1}, z_n) \mapsto (z_1, \dots, z_{n-1}, z_n^2)$ under suitable local coordinate systems near x and y. Such a germ will be called a fold map germ. A holomorphic map $f : N \to P$ will be called a holomorphic fold map if f has only fold singularities.

Let $J^k(n, n; \mathbf{C})$ ($J^k(n, n)$ for short) denote the k-jet space of all k-jets of holomorphic map germs $(\mathbf{C}^n, \mathbf{0}) \to (\mathbf{C}^n, \mathbf{0})$. We consider the subspace Ω^1 of $J^1(n, n)$ consisting of all 1-jets whose kernel rank is either 0 or 1, and the subspace Ω^{10} of $J^2(n, n)$ consisting of all 2-jets of regular germs and fold map germs. The purpose of this paper is to determine their homotopy types. Let $J^2(N, P; \mathbf{C})$ ($J^2(N, P)$ for short) denote the complex 2-jet space, which is the total space of a fibre bundle over $N \times P$ and $\Omega^{10}(N, P; \mathbf{C})$ ($\Omega^{10}(N, P)$

Received December 1, 1998.
Revised February 22, 2000.
2000 Mathematics Subject Classification: Primary 58K15; Secondary 58A20, 58C10.
[1]This research was partially supported by Grant-in-Aid for Scientific Research (No.11640081), Ministry of Education, Science and Culture, Japan.

for short) denote its subbundle associated with Ω^{10}. The the 2-jet extension j^2f of a holomorphic fold map $f : N \to P$ is a section of $\Omega^{10}(N,P)$ over N. The homotopy type of Ω^{10} will be important in the study of the space consisting of all holomorphic fold maps. This paper is partially the complex version of [A1] and [A2], although the arguments are quite different and more complicated except for Sections 1 and 2.

Let S^{2k-1}, D_r^{2k} and $\mathbf{CP}^{k-1}$ denote the unit sphere of dimension $2k-1$ in $\mathbf{C}^k$, the disk of radius r and of dimension $2k$ in $\mathbf{C}^k$, and the complex projective space of dimension $k-1$ respectively. Let $\mathrm{U}(k)$ and $\mathrm{SU}(k)$ denote the unitary group and the special unitary group of degree k respectively. Now we explain the homotopy types of Ω^1 and Ω^{10}. Let I_a $(a \in \mathbf{R})$ be the diagonal $n \times n$-matrix (n-matrix for short) with diagonal components $(1,\dots,1,e^{\sqrt{-1}a})$. Let $\mathbf{v}$ be a point of $\mathbf{CP}^{n-1}$ represented by a vector $\mathbf{s} = {}^t(s_1,s_2,\dots,s_n)$ of S^{2n-1}. Then we define the n-matrix $G(\mathbf{v},e^{\sqrt{-1}\theta})$ by

$$G(\mathbf{v},e^{\sqrt{-1}\theta}) = I_\theta(E_n + (e^{\sqrt{-1}\theta}-1)(s_i\bar{s}_j)),$$

where E_n is the unit matrix of rank n and $(s_i\bar{s}_j)$ is the n-matrix with (i,j) component given by $s_i\bar{s}_j$. It will be shown that $G(\mathbf{v},e^{\sqrt{-1}\theta})$ lies in $\mathrm{SU}(n)$ (see (3.3)). Let $\mathrm{OC}(\mathbf{CP}^{n-1})$ denote the open cone over $\mathbf{CP}^{n-1}$, that is, the quotient space $\mathbf{CP}^{n-1}\times[0,1)/\mathbf{CP}^{n-1}\times 0$. Then we define the homeomorphism

$$g : \mathbf{CP}^{n-1}\times \mathrm{Int}(D^2_{1/2}\setminus\{\mathbf{0}\})\times \mathrm{SU}(n) \longrightarrow \mathbf{CP}^{n-1}\times(\sqrt{3}/2,1)\times S^1\times\mathrm{SU}(n)$$

by $g(\mathbf{v},be^{\sqrt{-1}\theta},U) = (\mathbf{v},(1-b^2)^{1/2},e^{\sqrt{-1}\theta},G(\mathbf{v},e^{\sqrt{-1}\theta})U)$. We make the new space $\mathbf{CP}^{n-1}\times\mathrm{Int}\,D^2_{1/2}\times\mathrm{SU}(n)\cup_g\mathrm{OC}(\mathbf{CP}^{n-1})\times S^1\times\mathrm{SU}(n)$ by pasting the two subspaces by g.

We consider the two actions of $\mathrm{SU}(n)\times\mathrm{SU}(n)$: one on $J^2(n,n)$ through the source and target spaces $(\mathbf{C}^n,\mathbf{0})$, and the other on $\mathrm{SU}(n+1)$ through $\mathrm{SU}(n)\times(1)$ from the right and left hand sides. The main theorem of the present paper is the following.

THEOREM 1. (1) *There exists a topological embedding of* $\mathbf{CP}^{n-1}\times\mathrm{Int}\,D^2_{1/2}\times\mathrm{SU}(n)\cup_g\mathrm{OC}(\mathbf{CP}^{n-1})\times S^1\times\mathrm{SU}(n)$ *into* Ω^1 *whose image is a deformation retract of* Ω^1 $(n\ge 2)$.

(2) *There exists an equivariant topological embedding* $i_n : \mathrm{SU}(n+1)\to\Omega^{10}$ *with respect to the actions of* $\mathrm{SU}(n)\times\mathrm{SU}(n)$ *whose image is a deformation retract of* Ω^{10} $(n\ge 1)$.

An n-dimensional complex vector bundle with structure group $\mathrm{SU}(n)$ will be called an $\mathrm{SU}(n)$-vector bundle. Let M be a complex manifold of dimension n. In this paper, an $\mathrm{SU}(n)$-structure of TM refers to a reduction (E', φ) of the structure group $\mathrm{GL}(n, \mathbf{C})$ of the tangent bundle TM to $\mathrm{SU}(n)$, where E' is an $\mathrm{SU}(n)$-vector bundle over M and $\varphi : TM \to E'$ is a bundle map. Then (E', φ) induces a homotopy class of a classifying map of E', $M \to B_{\mathrm{SU}(n)}$. It is well known that TM has an $\mathrm{SU}(n)$-structure if and only if the first Chern class of M vanishes.

Let $L^2(n)$ be the group of all 2-jets of biholomorphic map germs $(\mathbf{C}^n, \mathbf{0}) \to (\mathbf{C}^n, \mathbf{0})$. The structure group of the fibre bundle $\pi_N \times \pi_P : J^2(N, P) \to N \times P$ with fibre $J^2(n, n)$ is $L^2(n) \times L^2(n)$. Since $\mathrm{GL}(n, \mathbf{C})$ is naturally a subgroup of $L^2(n)$ and the quotient space $L^2(n)/\,\mathrm{GL}(n; \mathbf{C})$ is contractible, the structure group $L^2(n) \times L^2(n)$ of the fibre bundle $\pi_N \times \pi_P : J^2(N, P) \to N \times P$ is reduced to $\mathrm{GL}(n; \mathbf{C}) \times \mathrm{GL}(n; \mathbf{C})$. If TN and TP have $\mathrm{SU}(n)$-structures (E, φ_N) and (F, φ_P) respectively, then the structure group of $J^2(N, P)$ is, furthermore, reduced from $\mathrm{GL}(n; \mathbf{C}) \times \mathrm{GL}(n; \mathbf{C})$ to $\mathrm{SU}(n) \times \mathrm{SU}(n)$. Moreover, we have the subbundle $\mathrm{SU}(E \oplus \theta_N, F \oplus \theta_P)$ of $\mathrm{Hom}(E \oplus \theta_N, F \oplus \theta_P)$ associated with $\mathrm{SU}(n+1)$, where θ_N and θ_P are the trivial complex line bundles over N and P respectively. We will prove in Section 7 that there exists a fibre map $i(N, P) : \mathrm{SU}(E \oplus \theta_N, F \oplus \theta_P) \to \Omega^{10}(N, P)$ associated with the equivariant embedding $i_n : \mathrm{SU}(n+1) \to \Omega^{10}$ in Theorem 1 (2). The $\mathrm{SU}(n)$-vector bundles E and F not only have hermitian metrics, but also enable us to consider the determinant on each fibre of a bundle map of E to F. A bundle map of E to F will be called an $\mathrm{SU}(n)$-bundle map if it preserves norms and the determinant on each fibre is equal to 1. The following theorem will be proved in Section 7.

THEOREM 2. *Let N and P be complex manifolds of dimension n with $\mathrm{SU}(n)$-structures (E, φ_N) and (F, φ_P) respectively. Then we have the following.*

(1) *The map $i(N, P) : \mathrm{SU}(E \oplus \theta_N, F \oplus \theta_P) \to \Omega^{10}(N, P)$ is a fibre homotopy equivalence.*

(2) *If there exists a holomorphic fold map f of N into P, then $j^2 f$ determines the homotopy class of an $\mathrm{SU}(n+1)$-bundle map of $E \oplus \theta_N$ to $F \oplus \theta_P$ covering f through $i(N, P)$.*

The set of all continuous sections of $\mathrm{SU}(E \oplus \theta_N, F \oplus \theta_P)$ over N corresponds bijectively to that of all $\mathrm{SU}(n+1)$-bundle maps of $E \oplus \theta_N$ to $F \oplus \theta_P$.

For a holomorphic fold map f, the section $j^2 f : N \to \Omega^{10}(N, P)$ determines the homotopy class of the section $i(N, P)^{-1} \circ j^2 f$ of $\mathrm{SU}(E \oplus \theta_N, F \oplus \theta_P)$, where $i(N, P)^{-1}$ is the homotopy inverse of $i(N, P)$. This gives the homotopy class of an $\mathrm{SU}(n+1)$-bundle map $\widetilde{f} : E \oplus \theta_N \to F \oplus \theta_P$ covering f in Theorem 2 (2). Since $\widetilde{f}$ is reduced to an $\mathrm{SU}(n)$-bundle map of E to F by the obstruction theory, we have the following corollary.

COROLLARY 3. *Let N and P be complex manifolds of dimension n whose first Chern classes vanish. If there is a holomorphic fold map $f : N \to P$, then there exists a bundle map of TN to TP covering f.*

The assertion in the C^∞-category corresponding to Theorem 2 is described in [A2, Corollary 2] and Corollary 3 can be compared with the results [E, 3.8, 3.9 and 3.10 Theorem] and [Sa, Lemma 3.1] in the C^∞-category.

In Section 2 we will prepare lemmas in linear algebra. Let Σ^1 denote the subspace of $J^1(n, n)$ consisting of all 1-jets with kernel rank 1. We will prove in Section 3 that Σ^1 is homotopy equivalent to $\mathbf{CP}^{n-1} \times \mathrm{SU}(n)$ (Theorems 3.1 and 3.7). It is known that the normal bundle of Σ^1 in Ω^1 is the trivial complex line bundle $\mathrm{Hom}(\mathbf{K}, \mathbf{Q})$, where $\mathbf{K}$ is the kernel bundle and $\mathbf{Q}$ is the cokernel bundle of the first derivative over Σ^1. Therefore the tubular neighbourhood of Σ^1 is homotopy equivalent to $\mathbf{CP}^{n-1} \times D^2_{1/2} \times \mathrm{SU}(n)$. We will study how $\partial(\mathbf{CP}^{n-1} \times D^2_{1/2} \times \mathrm{SU}(n))$ is pasted to $\mathrm{U}(n) \cong S^1 \times \mathrm{SU}(n)$ ($\cong$ here refers to a homeomorphism) to prove Theorem 1 (1).

Let Σ^{10} denote the subspace of $J^2(n, n)$ consisting of all 2-jets of fold map germs. In Section 5 we will see that the fibre bundle Σ^{10} over Σ^1 is homotopy equivalent to the canonical S^1-bundle $S^{2n-1} \times \mathrm{SU}(n)$ over $\mathbf{CP}^{n-1} \times \mathrm{SU}(n)$ and hence the tubular neighbourhood of Σ^{10} in Ω^{10} is homotopy equivalent to $S^{2n-1} \times D^2_{1/2} \times \mathrm{SU}(n)$. The tubular neighbourhood of $U(n) \cong S^1 \times \mathrm{SU}(n))$ in Ω^{10} is homotopy equivalent to $D^{2n}_1 \times S^1 \times \mathrm{SU}(n)$. Then we will see that the pasting map of $\partial(S^{2n-1} \times D^2_{1/2} \times \mathrm{SU}(n))$ to $\partial(D^{2n}_1 \times S^1 \times \mathrm{SU}(n))$ is induced from g by considering the S^1-bundle above and that the pasted space becomes the total space of a fibre bundle over S^{2n+1} with fibre $\mathrm{SU}(n)$. We will prove in Section 5 that there exists a bundle map from this space to $\mathrm{SU}(n+1)$ by constructing in Section 4 a special bundle structure of the fibre bundle $\mathrm{SU}(n+1)$ over $\mathrm{SU}(n+1)/\mathrm{SU}(n) \times \mathrm{SU}(1) \cong S^{2n+1}$.

Next we will specify the embedding of $\mathrm{SU}(n+1)$ into Ω^{10} of Theorem 1 (2) in Section 5 and prove in Section 6 that it is equivariant with respect to the actions of $\mathrm{SU}(n) \times \mathrm{SU}(n)$. In Section 7 we will prove Theorem 2 and give certain examples of holomorphic fold maps.

§1. Notations

Let $\mathbf{C}^n$ denote the n-dimensional complex number space consisting of all column vectors of n complex numbers. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ denote the canonical basis of $\mathbf{C}^n$ with $\mathbf{e}_i = {}^t(0, \dots, 0, \overset{i}{1}, 0, \dots, 0)$. The hermitian inner product of vectors $\mathbf{v}$, $\mathbf{w}$ is denoted by $(\mathbf{v}, \mathbf{w})$ and the norm of $\mathbf{v}$ is denoted by $\|\mathbf{v}\|$. In this paper a linear map $\mathbf{C}^n \to \mathbf{C}^n$ or a quadratic form on $\mathbf{C}^n$ is identified with an n-matrix or an n-symmetric matrix respectively.

The details and further results of this section can be found in [Bo] and [L] although we work in the complex category. The space of all homomorphisms of a vector space V into a vector space W over $\mathbf{C}$ will be denoted by $\mathrm{Hom}(V, W)$. The basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ induces the identifications of $J^1(n, n)$ with $\mathrm{Hom}(\mathbf{C}^n, \mathbf{C}^n)$ and of $J^2(n, n)$ with $\mathrm{Hom}(\mathbf{C}^n, \mathbf{C}^n) \oplus \mathrm{Hom}(\mathbf{C}^n \bigcirc \mathbf{C}^n, \mathbf{C}^n)$, where $\mathbf{C}^n \bigcirc \mathbf{C}^n$ is the 2-fold symmetric product of $\mathbf{C}^n$. Let Σ^i denote the subspace of $J^1(n, n)$ consisting of all homomorphisms $\alpha : \mathbf{C}^n \to \mathbf{C}^n$ with kernel rank i $(0 \le i \le n)$. We usually denote an element of $J^2(n, n)$ as (α, β) for $\alpha : \mathbf{C}^n \to \mathbf{C}^n$ and $\beta : \mathbf{C}^n \bigcirc \mathbf{C}^n \to \mathbf{C}^n$. Consider the composition of the restriction $\beta \,|\, \mathrm{Ker}(\alpha) \bigcirc \mathrm{Ker}(\alpha)$ and the natural projection of $\mathbf{C}^n$ onto $\mathrm{Cok}(\alpha)$. It induces a new homomorphism of $\mathrm{Ker}(\alpha)$ into $\mathrm{Hom}(\mathrm{Ker}(\alpha), \mathrm{Cok}(\alpha))$ denoted by $\widetilde{\beta}$. Let Σ^{ij} be the subspace consisting of all elements (α, β) such that α and $\widetilde{\beta}$ are of kernel ranks i and j respectively. The notation Σ^i is often used for $\Sigma^i \times \mathrm{Hom}(\mathbf{C}^n \bigcirc \mathbf{C}^n, \mathbf{C}^n)$ if there is no confusion.

The space Ω^1 denotes the union $\Sigma^0 \cup \Sigma^1$ in $J^1(n, n)$ and Ω^{10} denotes the union $\Sigma^0 \cup \Sigma^{10}$ in $J^2(n, n)$. Both spaces are open subsets. We say that a 2-jet of Σ^{10} or its singularity at the origin is of fold type.

In this paper maps are basically continuous, but may be holomorphic or C^∞-differentiable if so stated.

§2. Lemmas

In this section we will discuss several results proved by elementary arguments in linear algebra in the complex category. The diagonal matrix with diagonal components $\mathbf{a} = (a_1, \dots, a_n)$ will be denoted by $\Delta(\mathbf{a})$. In particular, $\Delta(1,\dots,1,e^{\sqrt{-1}a})$ of rank n is written as I_a. For an n-matrix A, ${}^t\bar{A}$ is denoted by A^*.

LEMMA 2.1. *Let A be an n-matrix. Then A is decomposed as $S\Delta(\mathbf{d})T$, where S and T are unitary matrices and $d_1,\dots,d_n$ are nonnegative real numbers such that* (1) *$d_1^2,\dots,d_n^2$ are the eigen-values of A^*A and* (2) *$d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$.*

Proof. The hermitian and nonnegative definite matrix A^*A is diagonalized by a unitary matrix U as

$$U^*(A^*A)U = \Delta(d_1^2,\dots,d_n^2).$$

Set $U^*AU = (\mathbf{a}_1,\dots,\mathbf{a}_n)$. Then $(\mathbf{a}_i,\mathbf{a}_j) = 0$ for $i \neq j$ and $(\mathbf{a}_i,\mathbf{a}_i) = d_i^2$. When $\mathbf{a}_i \neq \mathbf{0}$, set $\mathbf{f}_i = \mathbf{a}_i/\|\mathbf{a}_i\|$. Then we can find an orthonormal basis $\mathbf{f}_1,\dots,\mathbf{f}_n$ by choosing $\mathbf{f}_j$ for j with $\mathbf{a}_j = \mathbf{0}$ appropriately. It follows that

$$U^*AU = (\mathbf{f}_1,\dots,\mathbf{f}_n)\Delta(\|\mathbf{a}_1\|,\dots,\|\mathbf{a}_n\|).$$

This proves (1).

We can prove that in the decomposition of A two values d_i and d_j are exchanged by using the matrix $P_{ij} = (p_{ij})$ such that $p_{kk} = 1$ when k is equal to neither i nor j and that $p_{ij} = p_{ji} = 1$ and $p_{st} = 0$ otherwise. This follows from $A = SP_{ij}P_{ij}\Delta(\mathbf{d})P_{ij}P_{ij}T$ and $P_{ij}\Delta(\mathbf{d})P_{ij} = \Delta(d_1,\dots,d_j,\dots d_i,\dots,d_n)$. □

If $d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$ holds, then we say in this paper that the diagonal components $\mathbf{d} = (d_1,\dots,d_n)$ is decreasing. Let A_j $(j = 1,\dots,s)$ be square i_j-matrices. The new matrix

$$\begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_s \end{pmatrix}$$

will be denoted by $A_1 \dot{+} \cdots \dot{+} A_s$. Let E_j denote the unit matrix of rank j.

The following lemma is a key tool of this paper.

LEMMA 2.2. *Let $\mathbf{v}$ and $\mathbf{w}$ be decreasing diagonal components. Suppose that $S\Delta(\mathbf{v})T = \Delta(\mathbf{w})$ for S and T of $\mathrm{U}(n)$. Then*

(1) $\mathbf{v} = \mathbf{w}$. *Hence* $\Delta(\mathbf{v})$ $(= \Delta(\mathbf{w}))$ *is written as*

$$a_1 E_{i_1} \dot{+} a_2 E_{i_2} \dot{+} \cdots \dot{+} a_s E_{i_s},$$

where $a_1, \ldots, a_s$ *are all distinct and* $n = i_1 + \cdots + i_s$.

(2) S *and* T *are also matrices of the forms*

$$S = S_1 \dot{+} \cdots \dot{+} S_s \quad \text{and} \quad T = T_1 \dot{+} \cdots \dot{+} T_s$$

respectively, where both S_j *and* T_j *are of ranks* i_j $(j = 1, \ldots, s)$.

(3) *If* a_j *is not zero, then* $S_j T_j = E_{i_j}$.

Proof. We shall prove the lemma by comparing the components of $S\Delta(\mathbf{v})$ and $\Delta(\mathbf{w})T^*$. Set $S = (s_{ij})$ and $T^* = (t_{ij})$. Then we have

$$\begin{pmatrix} v_1 s_{11} & \ldots & v_n s_{1n} \\ \vdots & & \vdots \\ v_1 s_{n1} & \ldots & v_n s_{nn} \end{pmatrix} = \begin{pmatrix} w_1 t_{11} & \ldots & w_n t_{1n} \\ \vdots & & \vdots \\ w_1 t_{n1} & \ldots & w_n t_{nn} \end{pmatrix}.$$

By comparing the components of p-th rows and q-th columns of the matrices above, we obtain the following inequalities.

$$\begin{aligned} |v_1|^2 &\geq |v_1 s_{p1}|^2 + |v_2 s_{p2}|^2 + \cdots + |v_n s_{pn}|^2 \\ &= |w_p t_{p1}|^2 + |w_p t_{p2}|^2 + \cdots + |w_p t_{pn}|^2 \\ &= |w_p|^2, \end{aligned} \tag{2.3.1}$$

$$\begin{aligned} |w_1|^2 &\geq |w_1 t_{1q}|^2 + |w_2 t_{2q}|^2 + \cdots + |w_n t_{nq}|^2 \\ &= |v_q s_{1q}|^2 + |v_q s_{2q}|^2 + \cdots + |v_q s_{nq}|^2 \\ &= |v_q|^2. \end{aligned} \tag{2.3.2}$$

Setting $p = q = 1$, we have $v_1 = w_1$.

Now we prove the lemma by induction on n. Assume that the assertion is true for dimensions less than n. If $v_n = 0$ or $w_n = 0$, then the number of i's with $v_i = 0$ coincides with that of j's with $w_j = 0$. Let i_s denote this number. By the unitarity of S and T it follows that $s_{pq} = t_{pq} = 0$ when only one of p and q is smaller than $n - i_s + 1$ and the other is not. So let $a_s = 0$, S_s and T_s

denote i_s-matrices (s_{pq}) and (t_{pq}), where $n-i_s+1 \le p,q \le n$, respectively. Therefore the assertion for n follows from the induction hypothesis.

Next assume that v_i and w_j are not zero for all i and j. Suppose that

$$v_1 = v_2 = \cdots = v_i > v_{i+1} \quad \text{and} \quad w_1 = w_2 = \cdots = w_i > w_{j+1}$$

Then we can prove that $i = j$ and $s_{pq} = t_{pq} = 0$ when only one of p and q is smaller than $i+1$ and the other is not. In fact, if $p \le j$, then (2.3.1) implies $|v_1|^2 \ge |w_p|^2 = |w_1|^2 = |v_1|^2$ and so

$$\begin{aligned} |v_1|^2 &= |v_1 s_{p1}|^2 + |v_2 s_{p2}|^2 + \cdots + |v_n s_{pn}|^2 \\ &= |v_1|^2(|s_{p1}|^2 + |s_{p2}|^2 + \cdots + |s_{pn}|^2). \end{aligned}$$

This equality together with $v_i > v_{i+1}$ shows that

$$s_{p,i+1} = \cdots = s_{pn} = 0 \quad \text{for} \quad p \le j.$$

If $q \le i$, then (2.3.2) again implies $|w_1|^2 \ge |v_q|^2 = |v_1|^2 = |w_1|^2$ and so

$$\begin{aligned} |w_1|^2 &= |w_1 t_{1q}|^2 + |w_2 t_{2q}|^2 + \cdots + |w_n t_{nq}|^2 \\ &= |w_1|^2(|t_{1q}|^2 + |t_{2q}|^2 + \cdots + |t_{nq}|^2). \end{aligned}$$

Similarly we obtain that

$$t_{j+1,q} = \cdots = t_{nq} = 0 \quad \text{for} \quad q \le i.$$

Since the first j row vectors of S and the first i column vectors of T^* are linearly independent, we have $i = j$, which becomes i_1. The assertions (2) and (3) for S_1 and T_1 also follow from the unitarity of S and T. Therefore the lemma follows from the induction on n, since the case of $n = 1$ is trivial. ∎

The following lemma is a subtle version of Lemma 2.2 and its proof is technically the same.

LEMMA 2.3. *Let* $\mathbf{v}$ *be decreasing diagonal components given in Lemma 2.2. For two sequences* $\{S^k\}$ *and* $\{T^k\}$ *of* $\mathrm{U}(n)$ *and a sequence of decreasing diagonal components* $\{\mathbf{d}^k\}$*, suppose that the sequence* $\{S^k\Delta(\mathbf{d}^k)T^k\}$ *converges to* $\Delta(\mathbf{v})$*. Then*

(1) $\{\mathbf{d}^k\}$ *converges to* $\mathbf{v}$*,*

(2) *If a pair* (p, q) *of numbers does not satisfy*

$$i_1 + i_2 + \cdots + i_j < p, q \leq i_1 + i_2 + \cdots + i_{j+1}$$

for any number j *with* $0 \leq j < s$, *then every sequence* $\{s^k_{pq}\}$ (*resp.* $\{t^k_{pq}\}$) *made of* (p, q) *components of* S^k (*resp.* T^k) *converges to zero.*

(3) *Let* $\delta(S^k)$ (*resp.* $\delta(T^k)$) *denote the new matrix made from* S^k (*resp.* T^k) *by replacing every* (p, q) *component described in* (2) *with zero. Thus* $\delta(S^k)$ *and* $\delta(T^k)$ *have the natural decompositions* $\delta(S^k)_1 \dot{+} \cdots \dot{+} \delta(S^k)_s$ *and* $\delta(T^k)_1 \dot{+} \cdots \dot{+} \delta(T^k)_s$ *respectively. Then for any number* j *with* $a_j \neq 0$, *the sequence* $\{\delta(S^k)_j \delta(T^k)_j\}$ *converges to* E_{i_j}.

Proof. (1) The set of eigen values changes continuously with respect to matrices ([W, Appendix V.4]). By considering the eigen values of $(S^k \Delta(\mathbf{d}^k) T^k)^* (S^k \Delta(\mathbf{d}^k) T^k)$ we know that $\{\mathbf{d}^k\}$ converges to $\mathbf{v}$.

(2) Let $(\|A\| = \sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2)^{1/2}$ be the norm of a matrix $A = (a_{ij})$. It is clear that $\|SA\| = \|A\| = \|AS\|$ for S in $U(n)$. Set $\mathbf{d}^k = (d^k_1, \ldots, d^k_n)$. We may suppose that v_1 is not zero. By the assumption and (1), given any positive real number ε, there is a number l such that if $k > l$, then we have

$$\|S^k \Delta(\mathbf{d}^k) T^k - \Delta(\mathbf{v})\| < \varepsilon \quad \text{or} \quad \|S^k \Delta(\mathbf{d}^k) - \Delta(\mathbf{v})(T^k)^*\| < \varepsilon$$

and

$$|d^k_i - v_i| < \varepsilon \quad \text{for} \quad 1 \leq i \leq n.$$

Set $S^k = (s^k_{pq})$ and $(T^k)^* = (t^k_{pq})$. Take a number p with $p \leq i_1$. Then we have $v_p = v_1 \neq 0$ and

$$|d^k_q s^k_{pq} - v_p t^k_{pq}| < \varepsilon \quad \text{for} \quad 1 \leq q \leq n.$$

It yields

$$|(d^k_q / v_p) s^k_{pq} - t^k_{pq}| < \varepsilon / v_p$$

and so

$$|t^k_{pq}| < |(d^k_q / v_p) s^k_{pq}| + \varepsilon / v_p.$$

Hence, we have

$$1 = \sum_{q=1}^n |t^k_{pq}|^2 < \sum_{q=1}^n (|d^k_q / v_p) s^k_{pq}| + \varepsilon / v_p)^2$$

$$
\begin{aligned}
&\le \sum_{q=1}^{n} (|(v_q/v_1)s_{pq}^k| + |(d_q^k - v_q)/v_1)s_{pq}^k| + \varepsilon/v_1)^2 \\
&< \sum_{q=1}^{n} (|(v_q/v_1)s_{pq}^k| + 2\varepsilon/v_1)^2 \\
&= \sum_{q=1}^{n} (v_q/v_1)^2 |s_{pq}^k|^2 + (4\varepsilon/v_1)\Big(\sum_{q=1}^{n} |(v_q/v_1)s_{pq}^k|\Big) + 4n\varepsilon^2/{v_1}^2 \\
&\le \sum_{q=1}^{i_1} |s_{pq}^k|^2 + \sum_{q=i_1+1}^{n} (v_q/v_1)^2 |s_{pq}^k|^2 + 4n\varepsilon/v_1 + 4n\varepsilon^2/{v_1}^2 \\
&= 1 + \sum_{q=i_1+1}^{n} (-1 + (v_q/v_1)^2)|s_{pq}^k|^2 + 4n\varepsilon/v_1 + 4n\varepsilon^2/{v_1}^2.
\end{aligned}
$$

This implies

$$
\sum_{q=i_1+1}^{n} (1 - (v_q/v_1)^2)|s_{pq}^k|^2 < 4n\varepsilon/v_1 + 4n\varepsilon^2/{v_1}^2.
$$

Since ε can be any positive real number and $|v_q/v_1|$ is not bigger than $|v_{i_1+1}/v_1| < 1$ for $q > i_1$, $\{s_{pq}^k\}$ converges to 0 for $p \le i_1$ and $q > i_1$ when $k \to \infty$. Similarly s_{pq}^k converges to 0 for such numbers p and q. This fact also holds for T. Hence (2) is proved by induction on n.

(3) It follows from (2) that

$$
\begin{aligned}
\Delta(\mathbf{v}) &= \lim_{k\to\infty} S^k \Delta(\mathbf{d}^k) T^k \\
&= \lim_{k\to\infty} \delta(S^k)\Delta(\mathbf{v})\delta(T^k) \\
&= \lim_{k\to\infty} \Delta(\mathbf{v})\delta(S^k)\delta(T^k).
\end{aligned}
$$

Since $\mathbf{v}$ is decreasing, $\delta(S^k)_j\delta(T^k)_j$ must converge to E_{i_j} for those numbers j with $a_j \neq 0$. ∎

§3. Homotopy type of Ω^1

In this section we shall study the homology types of Ω^1 and Σ^1 in $\mathrm{Hom}(\mathbf{C}^n, \mathbf{C}^n)$ for $n \ge 2$. Let Ω^1_s (resp. Σ^1_s) denote the space consisting of all matrices $A = (a_{ij})$ such that $A \in \Omega^1$ (resp. $A \in \Sigma^1$) and $\|A\| = 1$. Clearly it is a deformation retract of Ω^1 (resp. Σ^1). Hence, we study their homotopy types.

Let Δ denote the contractible space consisting of all decreasing diagonal components $\mathbf{d}$ such that $d_{n-1} > 0$ and $\sum_{i=1}^{n} d_i^2 = 1$. In Δ we consider the subspace consisting of all special diagonal components of the form $\mathbf{d}_{ab} = (a/\sqrt{n-1}, \dots, a/\sqrt{n-1}, b/\sqrt{n})$, where a and b satisfy $a^2 + (b^2/n) = 1$ and $a/\sqrt{n-1} \geq b/\sqrt{n}$. Note that unless $b = 1$, we have $a/\sqrt{n-1} > b/\sqrt{n}$. For a subset B of $[0,1]$ we define Δ_B to be the subset of Δ consisting of all diagonal components $\mathbf{d}_{ab}$ with $b \in B$.

Lemma 2.1 is a motivation for defining the surjection

$$\mathcal{H} : \mathrm{SU}(n) \times \Delta \times S^1 \times \mathrm{SU}(n) \longrightarrow \Omega_s^1$$

by $\mathcal{H}(S, \mathbf{d}, e^{\sqrt{-1}\,\theta}, U) = S\Delta(\mathbf{d})I_{-\theta}U$. Here note that given decreasing diagonal components $\mathbf{d} = (d_1, \dots, d_n)$, $S\Delta(\mathbf{d})I_{-\theta}U \in \Omega_s^1$ if and only if $\mathbf{d} \in \Delta$. We denote the image $\mathcal{H}(\mathrm{SU}(n) \times \Delta_B \times S^1 \times \mathrm{SU}(n))$ by $K(B)$.

THEOREM 3.1. *Let $n \geq 2$. There exists a deformation retraction of Ω_s^1 to $K([0,1])$ whose restriction to Σ_s^1 induces a deformation retraction of Σ_s^1 to $K(\{0\})$.*

Proof. If $n = 2$, then it is clear that Ω_s^1 coincides with $K([0,1])$ and that Σ_s^1 coincides with $K(\{0\})$. Thus we may assume that $n \geq 3$. Let Δ' be the set of all diagonal components $\mathbf{f} = (f_1, \dots, f_{n-2}, 0, 0)$ with $f_1 \geq f_2 \geq \cdots \geq f_{n-2} \geq 0$ and $\sum_{i=1}^{n-2} {f_i}^2 = 1$. First we shall prove that Δ is identified with the space $(\Delta' * \Delta_{[0,1]}) \setminus \Delta'$, where $\Delta' * \Delta_{[0,1]}$ is the join of Δ' and $\Delta_{[0,1]}$ taken on the unit sphere S^{n-1}.

For $\mathbf{d} = (d_1, \dots, d_n)$ of $\Delta \setminus \Delta_{[0,1]}$, there exist uniquely determined $\mathbf{f}$, $\mathbf{d}_{ab}$ and t with $1 > t > 0$ such that if we set $\mathbf{d}' = t\mathbf{f} + (1-t)\mathbf{d}_{ab}$, then $\mathbf{d} = \mathbf{d}'/\|\mathbf{d}'\|$. In fact, let $t\mathbf{f} + (1-t)\mathbf{d}_{ab} = c(s\mathbf{f}' + (1-s)\mathbf{d}_{a'b'})$ with $c > 0$. Then

$$(1-t)a = c(1-s)a', \quad (1-t)b = c(1-s)b',$$

$$(1-t)^2(a^2 + (b^2/n)) = c^2(1-s)^2(a'^2 + (b'^2/n)).$$

This yields $1 - t = c(1-s)$. Hence $a = a'$ and $b = b'$. So we have $t\mathbf{f} = cs\mathbf{f}'$ and $t = cs$. Thus we obtain that $t = s$, $a = a'$, $b = b'$, $\mathbf{f} = \mathbf{f}'$ and $c = 1$.

Next we show the existence of $\mathbf{f}$, $\mathbf{d}_{ab}$ and t. By using the equation $\mathbf{d} = \mathbf{d}'/\|\mathbf{d}'\|$, we obtain

$$\|\mathbf{d}'\|d_{n-1} = (1-t)a/\sqrt{n-1}, \quad \|\mathbf{d}'\|d_n = (1-t)b/\sqrt{n}$$

and

$$\|\mathbf{d}'\|^2((n-1)d_{n-1}^2+d_n^2)=(1-t)^2(a^2+(b^2/n))=(1-t)^2.$$

For simplicity, set $u=((n-1)d_{n-1}^2+d_n^2)^{1/2}>0$. It must be that $a=\sqrt{n-1}\,d_{n-1}/u$ and $b=\sqrt{n}\,d_n/u$ with $a^2+(b^2/n)=1$ and $a/\sqrt{n-1}\geq b/\sqrt{n}$, and that $\mathbf{f}$ and t satisfy the equation

$$\mathbf{d}=(1/\|\mathbf{d}'\|)t\mathbf{f}+((1-t)/\|\mathbf{d}'\|)\mathbf{d}_{ab}=(ut/(1-t))\mathbf{f}+u\mathbf{d}_{ab}.$$

Therefore, for $\mathbf{d}$ of $\Delta\setminus\Delta_{[0,1]}$ we define a and b as above, and $\mathbf{f}$ and t so that they satisfy $\mathbf{f}=(\mathbf{d}-u\mathbf{d}_{ab})/\|\mathbf{d}-u\mathbf{d}_{ab}\|$ and $ut/(1-t)=\|\mathbf{d}-u\mathbf{d}_{ab}\|$. By definition, it is easy to see that $f_{n-1}=f_n=0$, $\|\mathbf{d}-u\mathbf{d}_{ab}\|>0$ and $0<t<1$.

In the following we represent $\mathbf{d}$ in Δ as $(t\mathbf{f}+(1-t)\mathbf{d}_{ab})/\|t\mathbf{f}+(1-t)\mathbf{d}_{ab}\|$, where $\mathbf{d}\in\Delta_{[0,1]}$ if and only if $t=0$. Now we define the deformation retraction r_λ of Δ to $\Delta_{[0,1]}$ with $r=\mathrm{id}_\Delta$ by

$$r_\lambda(\mathbf{d})=((1-\lambda)(t\mathbf{f}+(1-t)\mathbf{d}_{ab})+\lambda\mathbf{d}_{ab})/\|(1-\lambda)(t\mathbf{f}+(1-t)\mathbf{d}_{ab})+\lambda\mathbf{d}_{ab}\|.$$

It has the property that if $d_i=d_j$, then the i-th and the j-th components of $r_\lambda(\mathbf{d})$ denoted by d_i^λ and d_j^λ respectively coincide with each other. In fact, for the case $i\leq j\leq n-1$ this follows from $\mathbf{f}=(\mathbf{d}-u\mathbf{d}_{ab})/\|\mathbf{d}-u\mathbf{d}_{ab}\|$ and for the case $i\leq n-1$ and $j=n$, we have $d_i=d_{i+1}=\cdots=d_n$ and so $a/\sqrt{n-1}=b/\sqrt{n}$. This yields $f_i=f_{i+1}=\cdots=f_n$ and so $d_i^\lambda=d_{i+1}^\lambda=\cdots=d_n^\lambda$.

Now we define the deformation retraction R_λ of Ω_s^1 to $K([0,1])$, whose restriction of Σ_s^1 induces a deformation retraction of Σ_s^1 to $K(\{0\})$. We always consider the representation of a matrix A of Ω_s^1 as $A=S\Delta(\mathbf{d})I_{-\theta}U$, where S, $T\in\mathrm{SU}(n)$. Then define R_λ by $R_\lambda(A)=S\Delta(r_\lambda(\mathbf{d}))I_{-\theta}U$. This is well defined and continuous as is seen below. Let $A=S'\Delta(\mathbf{d})I_{-\theta}U'$. If $d_i=d_j$, then $d_i^\lambda=d_j^\lambda$. Furthermore, the matrices $(S')^*S$ and $I_{-\theta}U(U')^*I_\theta$ belong to $\mathrm{SU}(n)$ and satisfy the properties stated in Lemma 2.2, since $(S')^*S\Delta(\mathbf{d})I_{-\theta}U(U')^*I_\theta=\Delta(\mathbf{d})$. Hence, it follows that $(S')^*S\Delta(r_\lambda(\mathbf{d}))\times I_{-\theta}U(U')^*I_\theta=\Delta(r_\lambda(\mathbf{d}))$. This implies that $R_\lambda(A)$ does not depend on the choice of S and U. It is easy to see that $R_\lambda(A)$ keeps Σ_s^1 and that R_1 maps Σ_s^1 onto $K(\{0\})$.

For the proof of continuity, take a sequence $\{A^k\}$ of Ω_s^1 with representation $A^k=S^k\Delta(\mathbf{d}^k)I_{-\theta_k}U^k$ as in Lemma 2.3 and a sequence $\{\lambda_m\}$ such that $\lim_{k\to\infty} A^k = A$ and $\lim_{m\to\infty}\lambda_m = \lambda$. Then $\{\mathbf{d}^k\}$ converges to $\mathbf{d}$ by Lemma 2.3 (1). Since

$$\lim_{k\to\infty} S^* S^k \Delta(\mathbf{d}^k) I_{-\theta_k} U^k U^* I_\theta - \Delta(\mathbf{d}), \tag{3.1.1}$$

it follows that S^*S^k and $I_{-\theta_k}U^kU^*I_\theta$ satisfy the properties of Lemma 2.3, which induce $\delta(S^*S^k)$ and $\delta(I_{-\theta_k}U^kU^*I_\theta)$. Therefore, we have

$$\begin{aligned}
&\lim_{k\to\infty,\, m\to\infty} S^*S^k\Delta(r_{\lambda_m}(\mathbf{d}^k))I_{-\theta_k}U^kU^*I_\theta \\
&\quad= \lim_{k\to\infty,\, m\to\infty} \delta(S^*S^k)\Delta(r_{\lambda_m}(\mathbf{d}^k))\delta(I_{-\theta_k}U^kU^*I_\theta) \\
&\quad= \lim_{k\to\infty,\, m\to\infty} \Delta(r_{\lambda_m}(\mathbf{d}^k))\delta(S^*S^k)\delta(I_{-\theta_k}U^kU^*I_\theta) \\
&\quad= \Delta(r_\lambda(\mathbf{d})).
\end{aligned} \tag{3.1.2}$$

Thus (3.1.2) proves that $\lim_{k\to\infty,\, m\to\infty} R_{\lambda_m}(A^k) = R_\lambda(A)$. □

In the following we shall prove that $K([0,1])$ is the space stated in Theorem 1 (1) in Introduction.

We begin by proving that the restriction of $\mathcal{H}$ to $\mathrm{SU}(n)\times\Delta_{(0,1)}\times S^1\times\mathrm{SU}(n)$ onto $K((0,1))$ is a fibre bundle. Let $\mathcal{H}(S,\mathbf{d}_{ab},e^{\sqrt{-1}\,\theta},U) = \mathcal{H}(S',\mathbf{d}_{ab},e^{\sqrt{-1}\,\theta},U')$. Then $(S')^*S\Delta(\mathbf{d}_{ab})I_{-\theta}U(U')^*I_\theta = \Delta(\mathbf{d}_{ab})$. By we have that $(S')^*S$ and $I_{-\theta}U(U')^*I_\theta$ have the decompositions $S_1 \dot{+} (z_1)$ and $U_1 \dot{+} (z_2)$ respectively with $S_1U_1 = E_{n-1}$ and $z_1z_2 = 1$. Hence we have $(S')^*SI_{-\theta}U(U')^*I_\theta = E_n$, that is, $SI_{-\theta}U = S'I_{-\theta}U'$ and $S\mathbf{e}_n = S'(S_1 \dot{+} (z_1))\mathbf{e}_n = z_1S'\mathbf{e}_n$, where $\mathbf{e}_n = {}^t(0,\ldots,0,1)$. This observation enables us to define the surjections,

$$P : \mathrm{SU}(n)\times\Delta_{(0,1)}\times S^1\times\mathrm{SU}(n) \longrightarrow \mathbf{CP}^{n-1}\times\Delta_{(0,1)}\times S^1\times\mathrm{SU}(n),$$
$$H : \mathbf{CP}^{n-1}\times\Delta_{(0,1)}\times S^1\times\mathrm{SU}(n) \longrightarrow K((0,1))$$

by $P(S,\mathbf{d}_{ab},e^{\sqrt{-1}\,\theta},U) = ([S\mathbf{e}_n],\mathbf{d}_{ab},e^{\sqrt{-1}\,\theta},I_\theta SI_{-\theta}U)$ and $\mathcal{H}\,|\,\mathrm{SU}(n)\times\Delta_{(0,1)}\times S^1\times\mathrm{SU}(n) = H\circ P$, where $[*]$ refers to the element of $\mathbf{CP}^{n-1}$ represented by $*$. The precise description of H is as follows. Let $\mathbf{v}$ be an element of $\mathbf{CP}^{n-1}$ represented by a vector $\mathbf{s}$ with length 1. Find a matrix S of $\mathrm{SU}(n)$ with $S\mathbf{e}_n = \mathbf{s}$ (this notation will be often used below without stating it explicitly). Then we know that

$$H(\mathbf{v},\mathbf{d}_{ab},e^{\sqrt{-1}\,\theta},U) = S\Delta(\mathbf{d}_{ab})S^*I_{-\theta}U. \tag{3.2}$$

In fact, it does not depend on the choice of $\mathbf{s}$ and S, because a direct calculation shows

$$S\Delta(x,\dots,x,y)S^* = xE_n + (y-x)(s_i\bar{s}_j). \tag{3.3}$$

and we have

$$\begin{aligned} H\circ P(S,\mathbf{d}_{ab},e^{\sqrt{-1}\,\theta},U) &= H(\mathbf{v},\mathbf{d}_{ab},e^{\sqrt{-1}\,\theta},I_\theta SI_{-\theta}U)\\ &= S\Delta(\mathbf{d}_{ab})S^*I_{-\theta}(I_\theta SI_{-\theta}U)\\ &= S\Delta(\mathbf{d}_{ab})I_{-\theta}U\\ &= \mathcal{H}(S,\mathbf{d}_{ab},e^{\sqrt{-1}\,\theta},U). \end{aligned}$$

Here we note that H is naturally extended to the continuous surjection $\widetilde{H}$ of $\mathrm{OC}(\mathbf{CP}^{n-1})\times S^1\times\mathrm{SU}(n)$ onto $K((0,1])$ by setting $\widetilde{H}(*,e^{\sqrt{-1}\,\theta},U) = (1/\sqrt{n})I_{-\theta}U$, where $*$ is the cone point, since we have

$$\begin{aligned} \lim_{b\to1} H(\mathbf{v},\mathbf{d}_{ab},e^{\sqrt{-1}\,\theta},U) &= \lim_{b\to1} S\Delta(\mathbf{d}_{ab})S^*I_{-\theta}U\\ &= S(1/\sqrt{n})E_nS^*I_{-\theta}U = (1/\sqrt{n})I_{-\theta}U, \end{aligned}$$

which does not depend on the vector $\mathbf{v}$. Here note that the point $(\mathbf{v},\mathbf{d}_{ab})$ corresponds to the point $(\mathbf{v},(1-b^2)^{1/2})$ in $\mathrm{OC}(\mathbf{CP}^{n-1}) = \mathbf{CP}^{n-1}\times [0,1)/\mathbf{CP}^{n-1}\times 0$.

We define the other map

$$P_\Sigma : \mathrm{SU}(n)\times\Delta_{(0,1/2)}\times S^1\times\mathrm{SU}(n) \longrightarrow \mathbf{CP}^{n-1}\times\Delta_{(0,1/2)}\times S^1\times\mathrm{SU}(n)$$

by $P_\Sigma(S,\mathbf{d}_{ab},e^{\sqrt{-1}\,\theta},U) = ([S\mathbf{e}_n],\mathbf{d}_{ab},e^{\sqrt{-1}\,\theta},\mathrm{SU})$. This map induces the surjection

$$H_\Sigma : \mathbf{CP}^{n-1}\times\Delta_{(0,1/2)}\times S^1\times\mathrm{SU}(n) \longrightarrow K((0,1/2))$$

defined by

$$H_\Sigma(\mathbf{v},\mathbf{d}_{ab},e^{\sqrt{-1}\,\theta},U) = S\Delta(\mathbf{d}_{ab})I_{-\theta}S^*U \tag{3.4}$$

so that $\mathcal{H}\,|\,\mathrm{SU}(n)\times\Delta_{(0,1/2)}\times S^1\times\mathrm{SU}(n) = H_\Sigma\circ P_\Sigma$, where S is a matrix of $\mathrm{SU}(n)$ with $[S\mathbf{e}_n] = \mathbf{v}$. In fact, this map is well defined, since

$S\Delta(\mathbf{d}_{ab})I_{-\theta}S^*U = S\Delta(\mathbf{d}_{ab})S^*SI_{-\theta}S^*U$, and $S\Delta(\mathbf{d}_{ab})S^*$ and $SI_{-\theta}S^*$ depend only on $\mathbf{v}$ by (3.3). Then we have

$$\begin{aligned} H_\Sigma \circ P_\Sigma(S, \mathbf{d}_{ab}, e^{\sqrt{-1}\theta}, U) &= H_\Sigma([S\mathbf{e}_n], \mathbf{d}_{ab}, e^{\sqrt{-1}\theta}, \mathrm{SU}) \\ &= S\Delta(\mathbf{d}_{ab})I_{-\theta}U \\ &= \mathcal{H}(S, \mathbf{d}_{ab}, e^{\sqrt{-1}\theta}, U). \end{aligned}$$

Now H_Σ is naturally extended to the continuous surjection

$$\widetilde{H}_\Sigma : \mathbf{CP}^{n-1} \times \mathrm{Int}\, D^2_{1/2} \times \mathrm{SU}(n) \longrightarrow K([0,1/2))$$

defined by $\widetilde{H}_\Sigma(\mathbf{v}, be^{\sqrt{-1}\theta}, U) = H_\Sigma(\mathbf{v}, \mathbf{d}_{ab}, e^{\sqrt{-1}\theta}, U)$ for $0 < b < 1/2$ and $\widetilde{H}_\Sigma(\mathbf{v}, \mathbf{0}, U) = S\Delta(1/\sqrt{n-1}, \ldots, 1/\sqrt{n-1}, 0)S^*U$, since we have

$$\begin{aligned} \lim_{b\to 0} H_\Sigma(\mathbf{v}, \mathbf{d}_{ab}, e^{\sqrt{-1}\theta}, U) &= \lim_{b\to 0} S\Delta(\mathbf{d}_{ab})I_{-\theta}S^*U \\ &= S\Delta(1/\sqrt{n-1}, \ldots, 1/\sqrt{n-1}, 0)S^*U. \end{aligned}$$

LEMMA 3.5. (1) *The map* $\widetilde{H} : \mathrm{OC}(\mathbf{CP}^{n-1}) \times S^1 \times \mathrm{SU}(n) \to K((0,1])$ *is a continuous bijection.*

(2) *The map* $\widetilde{H}_\Sigma : \mathbf{CP}^{n-1} \times \mathrm{Int}\, D^2_{1/2} \times \mathrm{SU}(n) \to K([0,1/2))$ *is a continuous bijection.*

Proof. (1) Let A be a matrix of $K((0,1])$, which is represented as $S\Delta(\mathbf{d}_{ab})I_{-\theta}U$ with $S, U \in \mathrm{SU}(n)$. We show that the inverse H_1 of $\widetilde{H}$ is given by

$$\begin{aligned} H_1(A) &= ([S\mathbf{e}_n], \mathbf{d}_{ab}, e^{\sqrt{-1}\theta}, I_\theta SI_{-\theta}U) \qquad \text{for} \quad 0 < b < 1, \\ H_1(A) &= (*, e^{\sqrt{-1}\theta}, I_\theta SI_{-\theta}U) \qquad \text{for} \quad b = 1. \end{aligned}$$

First we see that H_1 is well defined. By Lemma 2.1, $\mathbf{d}_{ab}$ is determined by A. Let $S'\Delta(\mathbf{d}_{ab})I_{-\theta}U'$ be another representation. Then it follows from Lemma 2.2 that $SI_{-\theta}U = S'I_{-\theta}U'$, and $[S\mathbf{e}_n] = [S'\mathbf{e}_n]$ for $0 < b < 1$. Let us see that it is actually the inverse of $\widetilde{H}$. In fact, for $0 < b < 1$, we have

$$\begin{aligned} \widetilde{H} \circ H_1(A) &= \widetilde{H}([S\mathbf{e}_n], \mathbf{d}_{ab}, e^{\sqrt{-1}\theta}, I_\theta SI_{-\theta}U) \\ &= S\Delta(\mathbf{d}_{ab})S^*I_{-\theta}I_\theta SI_{-\theta}U \\ &= A, \end{aligned}$$

and for $b = 1$, we have

$$\begin{aligned}\widetilde{H} \circ H_1(A) &= \widetilde{H}(*, e^{\sqrt{-1}\theta}, I_\theta S I_{-\theta} U) \\ &= (1/\sqrt{n}) I_{-\theta} I_\theta S I_{-\theta} U \\ &= S(1/\sqrt{n}) E_n I_{-\theta} U \\ &= A.\end{aligned}$$

We have, inversely, for $0 < b < 1$

$$\begin{aligned}H_1 \circ \widetilde{H}(\mathbf{v}, \mathbf{d}_{ab}, e^{\sqrt{-1}\theta}, U) &= H_1(S\Delta(\mathbf{d}_{ab}) S^* I_{-\theta} U) \\ &= H_1(S\Delta(\mathbf{d}_{ab}) I_{-\theta} I_\theta S^* I_{-\theta} U) \\ &= ([S\mathbf{e}_n], \mathbf{d}_{ab}, e^{\sqrt{-1}\theta}, I_\theta S I_{-\theta} I_\theta S^* I_{-\theta} U) \\ &= ([S\mathbf{e}_n], \mathbf{d}_{ab}, e^{\sqrt{-1}\theta}, U).\end{aligned}$$

Similarly, for $b = 1$, we see that $H_1 \circ \widetilde{H}(*, e^{\sqrt{-1}\theta}, U) = (*, e^{\sqrt{-1}\theta}, U)$.

(2) A matrix A of $K([0, 1/2))$ is represented as $S\Delta(\mathbf{d}_{ab}) I_{-\theta} U$ as above and the inverse $(H_\Sigma)_1$ of $\widetilde{H}_\Sigma$ is given by

$$(H_\Sigma)_1(A) = ([S\mathbf{e}_n]), be^{\sqrt{-1}\theta}, \mathrm{SU}).$$

It follows from Lemma 2.2 that this is well defined. In fact, let $A = S'\Delta(\mathbf{d}_{ab}) I_{-\theta'} U'$ be another representation of A. Then we have $S^* S' \Delta(\mathbf{d}_{ab}) I_{-\theta'} U' U^* I_\theta = \Delta(\mathbf{d}_{ab})$. We can represent as $S^* S' = S_1 \dot{+} (z_1)$ and $I_{-\theta'} U' U^* I_\theta = U_1 \dot{+} (z_2)$, that is, $U' U^* = U_1 \dot{+} (z_2)$ with $S_1 U_1 = E_{n-1}$, and $z_1 z_2 = 1$ by Lemma 2.2 for $b > 0$ and by $z_1 \det S_1 = z_2 \det S_2 = 1$ for $b = 0$. Hence, we have $S^* S' U' U^* = E_n$ and so $SU = S'U'$. $(H_\Sigma)_1$ is actually the inverse of $\widetilde{H}_\Sigma$, since we have

$$\begin{aligned}\widetilde{H}_\Sigma \circ (H_\Sigma)_1(A) &= \widetilde{H}_\Sigma([S\mathbf{e}_n], be^{\sqrt{-1}\theta}, SU) \\ &= S\Delta(\mathbf{d}_{ab}) I_{-\theta} S^* SU \\ &= A\end{aligned}$$

and

$$\begin{aligned}(H_\Sigma)_1 \circ \widetilde{H}_\Sigma(\mathbf{v}, be^{\sqrt{-1}\theta}, U) &= (H_\Sigma)_1(S\Delta(\mathbf{d}_{ab}) I_{-\theta} S^* U) \\ &= ([S\mathbf{e}_n], be^{\sqrt{-1}\theta}, U).\end{aligned}$$

□

Consequently we have two bijections of $\mathbf{CP}^{n-1} \times \Delta_{(0,1/2)} \times S^1 \times \mathrm{SU}(n)$ onto $K((0,1/2))$ by H and H_Σ. Here recall the matrix $G(\mathbf{v}, e^{\sqrt{-1}\,\theta}) = I_\theta(E_n + (e^{-\sqrt{-1}\,\theta} - 1)(s_i\bar{s}_j))$ in Introduction, which is equal to $I_\theta S I_{-\theta} S^*$ by (3.3) for all S with $[S\mathbf{e}_n] = \mathbf{v}$. Let us determine the map $H^{-1} \circ H_\Sigma$ by using (3.2), (3.4) and Lemma 3.5. We have

$$(3.6)\quad \begin{aligned} H^{-1} \circ H_\Sigma(\mathbf{v}, \mathbf{d}_{ab}, e^{\sqrt{-1}\,\theta}, U) &= H^{-1}(S\Delta(\mathbf{d}_{ab})I_{-\theta}S^*U) \\ &= (\mathbf{v}, \mathbf{d}_{ab}, e^{\sqrt{-1}\,\theta}, I_\theta S I_{-\theta} S^* U) \\ &= (\mathbf{v}, \mathbf{d}_{ab}, e^{\sqrt{-1}\,\theta}, G(\mathbf{v}, e^{\sqrt{-1}\,\theta})U). \end{aligned}$$

It is easy to see that $H^{-1} \circ H_\Sigma \,|\, \mathbf{CP}^{n-1} \times \Delta_{(0,1/2)} \times S^1 \times \mathrm{SU}(n)$ is a homeomorphism.

Theorem 3.7. *Let $n \geq 2$. Under the notation in Introduction, the space $K([0,1])$ is homeomorphic to $\mathbf{CP}^{n-1} \times \mathrm{Int}\, D^2_{1/2} \times \mathrm{SU}(n) \cup_g \mathrm{OC}(\mathbf{CP}^{n-1}) \times S^1 \times \mathrm{SU}(n)$ and the space $K(\{0\})$ is homeomorphic to $\mathbf{CP}^{n-1} \times \{\mathbf{0}\} \times \mathrm{SU}(n)$.*

Proof. We define the map $j_n : \mathbf{CP}^{n-1} \times \mathrm{Int}\, D^2_{1/2} \times \mathrm{SU}(n) \cup_g \mathrm{OC}(\mathbf{CP}^{n-1}) \times S^1 \times \mathrm{SU}(n) \to K([0,1])$ by $j_n(\mathbf{v}, be^{\sqrt{-1}\,\theta}, U) = \widetilde{H}_\Sigma(\mathbf{v}, be^{\sqrt{-1}\,\theta}, U)$ for $0 \leq b < 1/2$ and $j_n(\mathbf{v}, (1-b^2)^{1/2}, e^{\sqrt{-1}\,\theta}, U) = \widetilde{H}(\mathbf{v}, \mathbf{d}_{ab}, e^{\sqrt{-1}\,\theta}, U)$ for $0 < b \leq 1$. It follows from Lemma 3.5 and (3.6) that j_n is well defined and is a continuous bijection. Since $\mathbf{CP}^{n-1} \times \mathrm{Int}\, D^2_{1/2} \times \mathrm{SU}(n) \cup_g \mathrm{OC}(\mathbf{CP}^{n-1}) \times S^1 \times \mathrm{SU}(n)$ is compact, we have that j_n is a homeomorphism. Furthermore, j_n maps $\mathbf{CP}^{n-1} \times \mathbf{0} \times \mathrm{SU}(n)$ onto $K(\{0\})$. ∎

Proof of Theorem 1 (1). The assertion follows from Theorems 3.1 and 3.7. ∎

Remark 3.8. Let $\mathbf{v} = [S\mathbf{e}_n]$ as above. The kernel of $\widetilde{H}_\Sigma(\mathbf{v}, \mathbf{0}, U)$ is generated by $U^*S\mathbf{e}_n$ and the orthogonal complement of its image is generated by $S\mathbf{e}_n$.

§4. Structure of the fibre bundle $\mathrm{SU}(n+1)$ over $\mathrm{SU}(n+1)/\mathrm{SU}(n)$

In this section let $n \geq 1$. In contrast to the canonical basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of $\mathbf{C}^n$, we write the canonical basis of $\mathbf{C}^{n+1}$ by $\{\mathbf{e}'_1, \ldots, \mathbf{e}'_{n+1}\}$. Let E_{n+1} be the unit matrix of rank $n+1$. We shall consider the fibre bundle $\pi : \mathrm{SU}(n+1) \to \mathrm{SU}(n+1)/\mathrm{SU}(n) \times (1) \cong S^{2n+1}$ and specify its structure. In this paper a point of S^{2n+1} will be written as $\mathbf{z} = {}^t(x_1, \ldots, x_n, z_{n+1})$ with $\mathbf{x} = {}^t(x_1, \ldots, x_n) \in \mathbf{C}^n$ and $z_{n+1} = be^{\sqrt{-1}\,\theta}$. Let $S_{\mathcal{R}}$ and S_Σ be the subsets of S^{2n+1} consisting of all points z such that $0 < b \leq 1$ and $0 \leq b < 1/2$ respectively.

For a point $\mathbf{z}$ of $S_{\mathcal{R}}$ with $0 < b \leq 1$, we define the matrix $r(\mathbf{z})$ of $\mathrm{SU}(n+1)$ so that

$$r(\mathbf{z})(\mathbf{e}'_{n+1}) = e^{-\sqrt{-1}\,\theta}\mathbf{z}, \tag{4.1-(i)}$$

$$r(\mathbf{z})(\mathbf{z} - z_{n+1}\mathbf{e}'_{n+1}) = b\mathbf{z} - e^{\sqrt{-1}\,\theta}\mathbf{e}'_{n+1}, \tag{4.1-(ii)}$$

(4.1-(iii)) if $0 < b < 1$, then $r(\mathbf{z})$ is the identity on the orthogonal complement of the subspace generated by the two vectors $\mathbf{e}'_{n+1}$ and $\mathbf{z} - z_{n+1}\,\mathbf{e}'_{n+1}$ over $\mathbf{C}$ and if $b = 1$, then $r(z) = E_{n+1}$.

For a point $\mathbf{z}$ of S_Σ with $0 \leq b < 1/2$, we define the matrix $r_\Sigma(\mathbf{z})$ of $\mathrm{SU}(n+1)$ so that

$$r_\Sigma(\mathbf{z})(\mathbf{e}'_{n+1}) = \mathbf{z}, \tag{4.2-(i)}$$

$$r_\Sigma(\mathbf{z})(\mathbf{z} - z_{n+1}\mathbf{e}'_{n+1}) = \bar{z}_{n+1}\mathbf{z} - \mathbf{e}'_{n+1}, \tag{4.2-(ii)}$$

(4.2-(iii)) $r_\Sigma(\mathbf{z})$ is the identity on the orthogonal complement of the subspace generated by the two vectors $\mathbf{e}'_{n+1}$ and $\mathbf{z} - z_{n+1}\mathbf{e}'_{n+1}$ over $\mathbf{C}$.

The explicit formulas of the matrices $r(\mathbf{z})$ and $r_\Sigma(\mathbf{z})$ are as follows:

$$r(\mathbf{z}) = \begin{pmatrix} R(\mathbf{z}) & e^{-\sqrt{-1}\,\theta}\mathbf{x} \\ -e^{\sqrt{-1}\,\theta}({}^t\bar{\mathbf{x}}) & b \end{pmatrix} \text{ and } r_\Sigma(\mathbf{z}) = \begin{pmatrix} R_\Sigma(\mathbf{x}) & \mathbf{x} \\ -{}^t\bar{\mathbf{x}} & be^{\sqrt{-1}\,\theta} \end{pmatrix},$$

where the (i,j) components of $R(\mathbf{z})$ and $R_\Sigma(\mathbf{z})$ are $\delta_{ij} - x_i\bar{x}_j/(1+b)$ and $\delta_{ij} - x_i\bar{x}_j((1 - be^{-\sqrt{-1}\,\theta})/(1-b^2))$ respectively.

Lemma 4.3. *The determinants of* $r(\mathbf{z})$ *and* $r_\Sigma(\mathbf{z})$ *are equal to* 1.

Proof. First we show $\det(r(\mathbf{z})) = 1$. For $b \neq 1$, let $\mathbf{f}_1, \dots, \mathbf{f}_{n-1}$ denote vectors such that $(\mathbf{f}_1, \dots, \mathbf{f}_{n-1}, \mathbf{e}'_{n+1}, (\mathbf{z} - z_{n+1}\mathbf{e}'_{n+1})/\|x\|)$ is an orthonormal basis. Then by definition we have

$$\begin{aligned} &r(\mathbf{z})(\mathbf{f}_1, \dots, \mathbf{f}_{n-1}, \mathbf{e}'_{n+1}, (\mathbf{z} - z_{n+1}\mathbf{e}'_{n+1})/\|\mathbf{x}\|) \\ &\qquad = (\mathbf{f}_1, \dots, \mathbf{f}_{n-1}, e^{-\sqrt{-1}\theta}\mathbf{z}, (b\mathbf{z} - e^{\sqrt{-1}\theta}\mathbf{e}'_{n+1})/\|\mathbf{x}\|). \end{aligned}$$

Hence,

$$\begin{aligned} &\|\mathbf{x}\|^2 \det(r(\mathbf{z})) \\ &= \det((\mathbf{f}_1, \dots, \mathbf{f}_{n-1}, \mathbf{e}'_{n+1}, \mathbf{z} - z_{n+1}\mathbf{e}'_{n+1})^* \\ &\qquad \times (\mathbf{f}_1, \dots, \mathbf{f}_{n-1}, e^{-\sqrt{-1}\theta}\mathbf{z}, b\mathbf{z} - e^{\sqrt{-1}\theta}\mathbf{e}'_{n+1})) \\ &= \det(E_{n-1} \dot{+} ({}^t(\overline{\mathbf{e}'_{n+1}, \mathbf{z} - z_{n+1}\mathbf{e}'_{n+1}})(e^{-\sqrt{-1}\theta}\mathbf{z}, b\mathbf{z} - e^{\sqrt{-1}\theta}\mathbf{e}'_{n+1}))) \\ &= \det({}^t(\overline{\mathbf{e}'_{n+1}, \mathbf{z} - z_{n+1}\mathbf{e}'_{n+1}})(e^{-\sqrt{-1}\theta}\mathbf{z}, b\mathbf{z} - e^{\sqrt{-1}\theta}\mathbf{e}'_{n+1})) \\ &= \det \begin{pmatrix} b & (-\|\mathbf{x}\|^2)e^{\sqrt{-1}\theta} \\ \|\mathbf{x}\|^2 e^{-\sqrt{-1}\theta} & b\|\mathbf{x}\|^2 \end{pmatrix} \\ &= \|\mathbf{x}\|^2(b^2 + \|\mathbf{x}\|^2) \\ &= \|\mathbf{x}\|^2. \end{aligned}$$

Next let us show that $\det(r_\Sigma(\mathbf{z})) = 1$. Take an orthonormal basis $(\mathbf{f}_1, \dots, \mathbf{f}_{n-1}, \mathbf{e}'_{n+1}, (\mathbf{z} - z_{n+1}\mathbf{e}'_{n+1})/\|\mathbf{x}\|)$. Then by definition we have

$$\begin{aligned} &r_\Sigma(\mathbf{z})(\mathbf{f}_1, \dots, \mathbf{f}_{n-1}, \mathbf{e}'_{n+1}, (\mathbf{z} - z_{n+1}\mathbf{e}'_{n+1})/\|\mathbf{x}\|) \\ &\qquad = (\mathbf{f}_1, \dots, \mathbf{f}_{n-1}, \mathbf{z}, (\bar{z}_{n+1}\mathbf{z} - \mathbf{e}'_{n+1})/\|\mathbf{x}\|). \end{aligned}$$

Hence,

$$\begin{aligned} &\|\mathbf{x}\|^2 \det(r_\Sigma(\mathbf{z})) \\ &= \det((\mathbf{f}_1, \dots, \mathbf{f}_{n-1}, \mathbf{e}'_{n+1}, \mathbf{z} - z_{n+1}\mathbf{e}'_{n+1})^* \\ &\qquad \times (\mathbf{f}_1, \dots, \mathbf{f}_{n-1}, \mathbf{z}, \bar{z}_{n+1}\mathbf{z} - \mathbf{e}'_{n+1})) \\ &= \det(E_{n-1} \dot{+} ({}^t(\overline{\mathbf{e}'_{n+1}, \mathbf{z} - z_{n+1}\mathbf{e}'_{n+1}})(\mathbf{z}, \bar{z}_{n+1}\mathbf{z} - \mathbf{e}'_{n+1}))) \end{aligned}$$

$$= \det({}^t\overline{(\mathbf{e}'_{n+1}, \mathbf{z} - z_{n+1}\mathbf{e}'_{n+1})}(\mathbf{z}, \bar{z}_{n+1}\mathbf{z} - \mathbf{e}'_{n+1}))$$

$$= \det\begin{pmatrix} be^{\sqrt{-1}\,\theta} & -\|\mathbf{x}\|^2 \\ \|\mathbf{x}\|^2 & be^{-\sqrt{-1}\,\theta}\|\mathbf{x}\|^2 \end{pmatrix}$$

$$= \|\mathbf{x}\|^2.$$

∎

LEMMA 4.4. *For a point* $\mathbf{z} = {}^t(x_1, \dots, x_n), be^{\sqrt{-1}\,\theta})$ *of* S^{2n+1} *with* $0 < b < 1/2$, *set* $\mathbf{s} = \mathbf{x}/\|\mathbf{x}\|$ *and let* S *be a matrix of* $\mathrm{SU}(n)$ *with* $S\mathbf{e}_n = \mathbf{s}$. *Then we have*

$$r(\mathbf{z})^{-1} r_\Sigma(\mathbf{z}) = SI_{-\theta}S^* \dot{+} (e^{\sqrt{-1}\,\theta}).$$

Proof. Let T be the matrix $(S \dot{+} (1))^* r(\mathbf{z})^{-1} r_\Sigma(\mathbf{z})(S \dot{+} (1))(I_\theta \dot{+} (1))$. Then we have

(1) $T(\mathbf{e}'_{n+1}) = e^{\sqrt{-1}\,\theta}\mathbf{e}'_{n+1}$,

(2)
$$\begin{aligned} T(\|\mathbf{x}\|\mathbf{e}'_n) &= (S \dot{+} (1))^* r(\mathbf{z})^{-1} r_\Sigma(\mathbf{z})(S \dot{+} (1))(\|\mathbf{x}\|e^{\sqrt{-1}\,\theta}\mathbf{e}'_n) \\ &= (S \dot{+} (1))^* r(\mathbf{z})^{-1} r_\Sigma(\mathbf{z})(e^{\sqrt{-1}\,\theta}(\mathbf{z} - z_{n+1}\mathbf{e}'_{n+1})) \\ &= (S \dot{+} (1))^* r(\mathbf{z})^{-1}(b\mathbf{z} - e^{\sqrt{-1}\,\theta}\mathbf{e}'_{n+1}) \\ &= (S \dot{+} (1))^*(\mathbf{z} - z_{n+1}\mathbf{e}'_{n+1}) \\ &= \|\mathbf{x}\|\mathbf{e}'_n. \end{aligned}$$

Since $(S \dot{+} (1))\mathbf{e}'_i$ $(i = 1, \dots, n-1)$ belong to the orthogonal complement of the space generated by $\mathbf{e}'_{n+1}$ and $\mathbf{z} - z_{n+1}\mathbf{e}'_{n+1}$, we obtain by (4.1-(iii)) and (4.2-(iii) that

(3) $T\mathbf{e}'_i = \mathbf{e}'_i \quad (i = 1, \dots, n-1)$.

Therefore, it follows that $T = E_n \dot{+} (e^{\sqrt{-1}\,\theta})$. ∎

For a matrix $M \in \mathrm{SU}(n+1)$, let $M\mathbf{e}'_{n+1}$ be written as $\mathbf{z} = {}^t(x_1, \dots, x_n, z_{n+1})$ with $\mathbf{x}(M) = {}^t(x_1, \dots, x_n)$ and $z_{n+1} = be^{\sqrt{-1}\,\theta}$. If $0 < b \le 1$, then $r(\mathbf{z})^{-1}M\mathbf{e}'_{n+1} = r(\mathbf{z})^{-1}\mathbf{z} = e^{\sqrt{-1}\,\theta}\mathbf{e}'_{n+1}$ and $r(\mathbf{z})^{-1}M$ is written as $I_{-\theta}U(M) \dot{+} (e^{\sqrt{-1}\,\theta})$ by some matrix $U(M)$ of $\mathrm{SU}(n)$. If $0 \le b < 1/2$, then $r_\Sigma(\mathbf{z})^{-1}M\mathbf{e}'_{n+1} = r_\Sigma(\mathbf{z})^{-1}\mathbf{z} = \mathbf{e}'_{n+1}$ and $r_\Sigma(\mathbf{z})^{-1}M$ is written as $U_\Sigma(M) \dot{+}$

(1) by some matrix $U_\Sigma(M)$ of $\mathrm{SU}(n)$. If $\|x(M)\|$ is not 0, then set $\mathbf{s}(M) = \mathbf{x}(M)/\|\mathbf{x}(M)\|$. We define the trivializations

$$t_{\mathcal{R}} : \pi^{-1}(S_{\mathcal{R}}) \longrightarrow \mathrm{Int}\, D_1^{2n} \times S^1 \times \mathrm{SU}(n) \qquad \text{and} \tag{4.5}$$
$$t_\Sigma : \pi^{-1}(S_\Sigma) \longrightarrow S^{2n-1} \times \mathrm{Int}\, D_{1/2}^2 \times \mathrm{SU}(n)$$

of $\pi^{-1}(S_{\mathcal{R}})$ and $\pi^{-1}(S_\Sigma)$ by $t_{\mathcal{R}}(M) = (x(M), e^{\sqrt{-1}\theta}, U(M))$ and $t_\Sigma(M) = (\mathbf{s}(M), be^{\sqrt{-1}\theta}, U_\Sigma(M))$ respectively. It is not difficult to see that they are really trivializations. From now on, when a vector $\mathbf{s}$ representing $[\mathbf{s}]$ is specified, the matrix $I_\theta S I_{-\theta} S^*$ is denoted by $G(\mathbf{s}, e^{\sqrt{-1}\theta})$ in place of $G([\mathbf{s}], e^{\sqrt{-1}\theta})$.

Proposition 4.6. *If $0 < b < 1/2$ then we have*

$$t_{\mathcal{R}} \circ {t_\Sigma}^{-1}(\mathbf{s}, be^{\sqrt{-1}\theta}, U_\Sigma) = ((1-b^2)^{1/2}\mathbf{s}, e^{\sqrt{-1}\theta}, G(\mathbf{s}, e^{\sqrt{-1}\theta})U_\Sigma).$$

Proof. There exists a matrix M of $\mathrm{SU}(n+1)$ such that $\mathbf{s} = \mathbf{s}(M)$, $U_\Sigma = U_\Sigma(M)$ and ${}^t\mathbf{z} = {}^t(M\mathbf{e}'_{n+1}) = ({}^t\mathbf{x}(M), be^{\sqrt{-1}\theta})$. By definition, we have ${t_\Sigma}^{-1}(\mathbf{s}(M), be^{\sqrt{-1}\theta}, U_\Sigma(M)) = M = r_\Sigma(\mathbf{z})(U_\Sigma(M) \dot{+} (1))$. Again by definition of $U(M)$, we have

$$I_{-\theta}U(M) \dot{+} (e^{\sqrt{-1}\theta}) = r(\mathbf{z})^{-1}M = r(\mathbf{z})^{-1}r_\Sigma(\mathbf{z})(U_\Sigma(M) \dot{+} (1))$$

and so

$$U(M) \dot{+} (1) = (I_\theta \dot{+} (e^{-\sqrt{-1}\theta}))r(\mathbf{z})^{-1}r_\Sigma(\mathbf{z})(U_\Sigma(M) \dot{+} (1)).$$

By Lemma 4.4 this is equal to

$$\begin{aligned}
(I_\theta \dot{+} (e^{-\sqrt{-1}\theta}))&(SI_{-\theta}S^* \dot{+} (e^{\sqrt{-1}\theta}))(U_\Sigma(M) \dot{+} (1)) \\
&= (I_\theta S I_{-\theta} S^* U_\Sigma(M)) \dot{+} (1) \\
&= G(\mathbf{s}(M), e^{\sqrt{-1}\theta})U_\Sigma(M) \dot{+} (1).
\end{aligned}$$

Hence, $t_{\mathcal{R}}(M) = (\mathbf{x}(M), e^{\sqrt{-1}\theta}, G(\mathbf{s}(M), e^{\sqrt{-1}\theta})U_\Sigma(M))$ with $\mathbf{x}(M) = (1-b^2)^{1/2}\mathbf{s}(M)$. ∎

Let $\widetilde{g}$ be the diffeomorphism

$$\widetilde{g} : S^{2n-1} \times \mathrm{Int}(D_{1/2}^2 \setminus \{0\}) \times \mathrm{SU}(n) \longrightarrow \mathrm{Int}(D_1^{2n} \setminus D_{\sqrt{3}/2}^{2n}) \times S^1 \times \mathrm{SU}(n) \tag{4.7}$$

defined by $\widetilde{g}(\mathbf{s}, be^{\sqrt{-1}\,\theta}, U_\Sigma) = ((1-b^2)^{1/2}\mathbf{s}, e^{\sqrt{-1}\,\theta}, G(\mathbf{s}, e^{\sqrt{-1}\,\theta})U_\Sigma)$ $(0 < b < 1/2)$. Let $S^{2n-1} \times \operatorname{Int} D^2_{1/2} \times \mathrm{SU}(n) \cup_{\widetilde{g}} \operatorname{Int} D^{2n}_1 \times S^1 \times \mathrm{SU}(n)$ denote the space obtained by pasting the two spaces written above by $\widetilde{g}$. Then we can define the diffeomorphism $k : \mathrm{SU}(n+1) \to S^{2n-1} \times \operatorname{Int} D^2_{1/2} \times \mathrm{SU}(n) \cup_{\widetilde{g}} \operatorname{Int} D^{2n}_1 \times S^1 \times \mathrm{SU}(n)$ by

$$k(M) = \begin{cases} (\mathbf{x}(M), e^{\sqrt{-1}\,\theta}, U(M)) & \text{for} \quad 0 < b \leq 1, \\ (\mathbf{s}(M), be^{\sqrt{-1}\,\theta}, U_\Sigma(M)) & \text{for} \quad 0 \leq b < 1/2. \end{cases} \tag{4.8}$$

The map $\pi' : S^{2n-1} \times \operatorname{Int} D^2_{1/2} \times \mathrm{SU}(n) \cup_{\widetilde{g}} \operatorname{Int} D^{2n}_1 \times S^1 \times \mathrm{SU}(n) \to S^{2n+1}$ defined by $\pi'(\mathbf{x}, e^{\sqrt{-1}\,\theta}, U) = (\mathbf{x}, (1 - \|\mathbf{x}\|^2)^{1/2} e^{\sqrt{-1}\,\theta})$ for $0 < b \leq 1$ and $\pi'(\mathbf{s}, be^{\sqrt{-1}\,\theta}, U_\Sigma) = ((1-b^2)^{1/2}\mathbf{s}, be^{\sqrt{-1}\,\theta})$ for $0 \leq b < 1/2$ becomes a principal bundle with fibre $\mathrm{SU}(n)$. Then the following proposition follows from the arguments above.

Proposition 4.9. *Let $n \geq 1$. The map k above gives a C^∞ bundle map of the principal bundle $\pi : \mathrm{SU}(n+1) \to \mathrm{SU}(n+1)/\mathrm{SU}(n) \times (1) \cong S^{2n+1}$ to the principal bundle $\pi' : S^{2n-1} \times \operatorname{Int} D^2_{1/2} \times \mathrm{SU}(n) \cup_{\widetilde{g}} \operatorname{Int} D^{2n}_1 \times S^1 \times \mathrm{SU}(n) \to S^{2n+1}$.*

§5. Homotopy type of Ω^{10}

We first review the homotopy type of Σ^{10} in the context of Sections 3 and 4. Let π^2_1 be the canonical forgetting map of $J^2(n,n)$ onto $J^1(n,n)$. Now we see what fibre bundle the restriction $\pi^2_1 \,|\, \Sigma^{10} : \Sigma^{10} \to \Sigma^1$ is. When $(\pi^2_1)^{-1}(\Sigma^1)$ is identified with $\Sigma^1 \times \operatorname{Hom}(\mathbf{C}^n \circ \mathbf{C}^n, \mathbf{C}^n)$, we have two line bundles $\mathbf{K}$ and $\mathbf{Q}$ over Σ^1 defined by

$$\mathbf{K} = \{(\alpha, \mathbf{k}) \,|\, \alpha \in \Sigma^1,\ \mathbf{k} \in \operatorname{Ker} \alpha\}$$

and

$$\mathbf{Q} = \{(\alpha, \mathbf{v}) \,|\, \alpha \in \Sigma^1,\ \mathbf{v} \in \operatorname{Cok} \alpha\}$$

respectively. Then we have the following exact sequence of vector bundles over Σ^1 :

$$0 \longrightarrow \mathbf{K} \longrightarrow \Sigma^1 \times \mathbf{C}^n \xrightarrow{h} \Sigma^1 \times \mathbf{C}^n \longrightarrow \mathbf{Q} \longrightarrow 0,$$

where h is the fibrewise homomorphism defined by $h(\alpha, \mathbf{x}) = (\alpha, \alpha(\mathbf{x}))$. Consider the map $C : \Sigma^1 \to \mathbf{CP}^{n-1}$defined as $C(\alpha)$ being the line orthogonal to $\operatorname{Im}(\alpha)$ in $\mathbf{C}^n$. Then $C_1(\mathbf{K}) = C_1(\mathbf{Q}) = C^*(c_1)$, where c_1 is the first Chern class of the canonical line bundle over $\mathbf{CP}^{n-1}$. It is known that the normal bundle of Σ^1 in $J^1(n,n)$ is equivalent to $\mathrm{Hom}(\mathbf{K},\mathbf{Q})$ (see [L, p.11, 2. Proof of Proposition 2] and [Bo, p.50, Lemma 7.13 and Theorem 7.14]). Since $C_1(\mathrm{Hom}(\mathbf{K},\mathbf{Q})) = C_1(\mathbf{Q}) - C_1(\mathbf{K}) = 0$, this normal bundle is trivial. Restricting the map $\widetilde{H}_\Sigma$ to $\mathbf{CP}^{n-1} \times \mathbf{0} \times \mathrm{SU}(n)$ in Section 3, we have an embedding of $\mathbf{CP}^{n-1} \times \mathbf{0} \times \mathrm{SU}(n)$ into Σ^1 inducing a homotopy equivalence. The composition of C and $\widetilde{H}_\Sigma \,|\, \mathbf{CP}^{n-1} \times \mathbf{0} \times \mathrm{SU}(n)$ coincides with the canonical projection of $\mathbf{CP}^{n-1}\times\mathbf{0}\times\mathrm{SU}(n)$ onto $\mathbf{CP}^{n-1}$, since $C \circ \widetilde{H}_\Sigma(\mathbf{v},\mathbf{0},U_\Sigma) = C(S\Delta(\mathbf{d}_{10})S^*U_\Sigma) = [S\mathbf{e}_n] = \mathbf{v}$. This implies $(\widetilde{H}_\Sigma \,|\, \mathbf{CP}^{n-1} \times \mathbf{0} \times \mathrm{SU}(n))^*(C_1(\mathbf{Q})) = c_1 \times 1$. We define the fibrewise homomorphism r of $\Sigma^1 \times \mathrm{Hom}(\mathbf{C}^n \bigcirc \mathbf{C}^n, \mathbf{C}^n)$ onto $\mathrm{Hom}(\mathbf{K} \bigcirc \mathbf{K}, \mathbf{Q})$ over Σ^1 by $r(\alpha,\beta) = \mathrm{pr} \circ \beta \,|\, \mathrm{Ker}(\alpha) \bigcirc \mathrm{Ker}(\alpha)$, where pr denotes the projection of $\mathbf{C}^n$ onto $\mathrm{Cok}(\alpha)$. Let $\mathfrak{R}$ be the subspace of $\mathrm{Hom}(\mathbf{K} \bigcirc \mathbf{K}, \mathbf{Q})$ consisting of all isomorphisms. By the definition of Σ^{10} we know that Σ^{10} coincides with $r^{-1}(\mathfrak{R})$. Since $C_1(\mathrm{Hom}(\mathbf{K} \bigcirc \mathbf{K}, \mathbf{Q})) = -2C_1(\mathbf{K}) + C_1(\mathbf{Q}) = -C_1(\mathbf{K})$, $\mathrm{Hom}(\mathbf{K}\bigcirc\mathbf{K},\mathbf{Q})$ is equivalent to $\mathrm{Hom}(\mathbf{K},\mathbf{C})$ as vector bundles, and there is an orientation reversing bundle map between the associated sphere bundles $S(\mathrm{Hom}(\mathbf{K},\mathbf{C}))$ and $S(\mathbf{K})$. Hence the fibre bundle Σ^{10} over Σ^1 is homotopy equivalent to the S^1-bundle $S^{2n-1} \times \mathbf{0} \times \mathrm{SU}(n)$ over $\mathbf{CP}^{n-1} \times \mathbf{0} \times \mathrm{SU}(n)$ induced from the S^1-bundle of S^{2n-1} over $\mathbf{CP}^{n-1}$ associated with c_1 of $H^2(\mathbf{CP}^{n-1};\mathbf{Z})$. Furthermore Σ^{10} has $S^{2n-1} \times \mathbf{0} \times \mathrm{SU}(n)$ as its deformation retract.

In Ω^{10}, $\Sigma^0 \times \mathrm{Hom}(\mathbf{C}^n\bigcirc, \mathbf{C}^n, \mathbf{C}^n)$ over Σ^0 has a contractible fibre. Hence by the arguments above, Ω^{10} has, as its deformation retract, the subspace which is the total space of the above S^1-bundle over $\mathbf{CP}^{n-1} \times \mathrm{Int}\, D^2_{1/2} \times \mathrm{SU}(n) \cup_g \mathrm{OC}(\mathbf{CP}^{n-1}) \times S^1 \times \mathrm{SU}(n)$ except for over $\{*\} \times S^1 \times \mathrm{SU}(n)$ with $*$ being the cone point of $\mathrm{OC}(\mathbf{CP}^{n-1})$. It is nothing but $S^{2n-1} \times \mathrm{Int}\, D^2_{1/2} \times \mathrm{SU}(n) \cup_{\widetilde{g}} \mathrm{Int}\, D^{2n}_1 \times S^1 \times \mathrm{SU}(n)$. Hence it follows from Proposition 4.9 that Ω^{10} is homotopy equivalent to $\mathrm{SU}(n+1)$. This is an intuitive proof of Theorem 1 (2).

Now we shall specify the embedding

$$h : S^{2n-1} \times \mathrm{Int}\, D^2_{1/2} \times \mathrm{SU}(n) \bigcup_{\widetilde{g}} \mathrm{Int}\, D^{2n}_1 \times S^1 \times \mathrm{SU}(n) \longrightarrow \Omega^{10}.$$

For a point $(\mathbf{x}, e^{\sqrt{-1}\,\theta}, U)$ of $\mathrm{Int}\, D^{2n}_1 \times S^1 \times \mathrm{SU}(n)$, we define the map $\beta(\mathbf{x}, e^{\sqrt{-1}\,\theta}, U) : \mathbf{C}^n \bigcirc \mathbf{C}^n \to \mathbf{C}^n$ by

(5.1) $\beta(\mathbf{x}, e^{\sqrt{-1}\,\theta}, U)(\mathbf{a},\mathbf{b})$

$$= \{{}^t\mathbf{a}{}^t(G(\mathbf{s}, e^{\sqrt{-1}\,\theta})^*U)\bar{S}\Delta(0,\ldots,0,\|\mathbf{x}\|)S^*(G(\mathbf{s}, e^{\sqrt{-1}\,\theta})^*U)\mathbf{b}\}\mathbf{s}$$

$$\text{for} \quad \mathbf{x} \neq \mathbf{0} \qquad \text{and}$$

$$\beta(\mathbf{0}, e^{\sqrt{-1}\,\theta}, U)(\mathbf{a}, \mathbf{b}) = \mathbf{0},$$

where if $\|\mathbf{x}\| \neq 0$, then $\mathbf{s} = \mathbf{x}/\|\mathbf{x}\|$ and $S\mathbf{e}_n = \mathbf{s}$. The matrix

$${}^t(G(\mathbf{s}, e^{\sqrt{-1}\,\theta})^*U)\bar{S}\Delta(0,\ldots,0,\|\mathbf{x}\|)S^*(G(\mathbf{s}, e^{\sqrt{-1}\,\theta})^*U)$$

is equal to

$$\begin{aligned}&{}^tUI_{-\theta}\bar{S}I_\theta{}^tS\bar{S}\Delta(0,\ldots,0,\|\mathbf{x}\|)S^*SI_\theta S^*I_{-\theta}U\\ &\qquad = {}^tUI_{-\theta}\bar{S}I_\theta\Delta(0,\ldots,0,\|\mathbf{x}\|)I_\theta S^*I_{-\theta}U\\ &\qquad = {}^tUI_{-\theta}(e^{2\sqrt{-1}\,\theta}\|\mathbf{x}\|(\bar{s}_i\bar{s}_j))I_{-\theta}U.\end{aligned}$$

For a point $(\mathbf{s}, be^{\sqrt{-1}\,\theta}, U_\Sigma)$ of $S^{2n-1} \times \operatorname{Int} D^2_{1/2} \times \mathrm{SU}(n)$, we define the map $\beta_\Sigma(\mathbf{s}, be^{\sqrt{-1}\,\theta}, U_\Sigma) : \mathbf{C}^n \bigcirc \mathbf{C}^n \to \mathbf{C}^n$ by

$$(5.2)\quad \beta_\Sigma(\mathbf{s}, be^{\sqrt{-1}\,\theta}, U_\Sigma)(\mathbf{a}, \mathbf{b}) = \{{}^t\mathbf{a}{}^tU_\Sigma\bar{S}\Delta(0,\ldots,0,(1-b^2)^{1/2})S^*U_\Sigma\mathbf{b}\}\mathbf{s},$$

which is equal to

$$\{{}^t\mathbf{a}{}^tU_\Sigma((1-b^2)^{1/2}(\bar{s}_i\bar{s}_j))U_\Sigma\mathbf{b}\}\mathbf{s}.$$

If $0 < b < 1/2$, then we have that $\beta(\mathbf{x}, e^{\sqrt{-1}\,\theta}, U) = \beta_\Sigma(\mathbf{s}, be^{\sqrt{-1}\,\theta}, U_\Sigma)$, since $U = G(\mathbf{s}, e^{\sqrt{-1}\,\theta})U_\Sigma = I_\theta SI_{-\theta}S^*U_\Sigma$, where $\|\mathbf{x}\| = (1-b^2)^{1/2}$ by definition. Hence, β and β_Σ define the well-defined map of $S^{2n-1} \times \operatorname{Int} D^2_{1/2} \times \mathrm{SU}(n) \cup_{\tilde{g}} \operatorname{Int} D^{2n}_1 \times S^1 \times \mathrm{SU}(n)$ to $\mathrm{Hom}(\mathbf{C}^n \bigcirc \mathbf{C}^n, \mathbf{C}^n)$.

The motivation for the definition above is the facts that when $b = 0$, we have $\widetilde{H}_\Sigma([\mathbf{s}], \mathbf{0}, U_\Sigma) = S\Delta(\mathbf{d}_{10})I_{-\theta}S^*U_\Sigma = S\Delta(\mathbf{d}_{10})S^*U_\Sigma$ and that its kernel vector is $U_\Sigma{}^*S\mathbf{e}_n$ and its cokernel vector is $\mathbf{s}$. Hence, if $b = 0$, then we should have that $\beta_\Sigma(\mathbf{s}, \mathbf{0}, U_\Sigma)(U_\Sigma{}^*S\mathbf{e}_n, U_\Sigma{}^*S\mathbf{e}_n) = \mathbf{s}$. If $b = 1$, then $\widetilde{H}(\mathbf{0}, e^{\sqrt{-1}\,\theta}, U) = (1/\sqrt{n})I_{-\theta}U$ and we must require $\beta(\mathbf{0}, e^{\sqrt{-1}\,\theta}, U)$ to be the null-homomorphism.

From now on, we often use the notation $\widetilde{H}(\mathbf{x}, e^{\sqrt{-1}\,\theta}, U)$ (resp. $\widetilde{H}(\mathbf{0}, e^{\sqrt{-1}\,\theta}, U)$) in place of $H([\mathbf{s}], \mathbf{d}_{ab}, e^{\sqrt{-1}\,\theta}, U)$ (resp. $(1/\sqrt{n})I_{-\theta}U$) for $0 < b < 1$ (resp. $b = 1$) and $\widetilde{H}_\Sigma(\mathbf{s}, be^{\sqrt{-1}\,\theta}, U_\Sigma)$ in place of $\widetilde{H}_\Sigma([\mathbf{s}], be^{\sqrt{-1}\,\theta}, U_\Sigma)$ for simplicity, when a vector $\mathbf{x}$ or $\mathbf{s}$ representing $[\mathbf{x}]$ or $[\mathbf{s}]$ is specified respectively. Then the map h is defined by

$$
\begin{aligned}
(5.3)\quad & (h \,|\, \operatorname{Int} D_1^{2n} \times S^1 \times \mathrm{SU}(n))(\mathbf{x}, e^{\sqrt{-1}\theta}, U) \\
&\qquad = (\widetilde{H}(\mathbf{x}, e^{\sqrt{-1}\theta}, U), \beta(\mathbf{x}, e^{\sqrt{-1}\theta}, U)) \qquad (0 < b \leq 1), \\
& (h \,|\, S^{2n-1} \times \operatorname{Int} D_{1/2}^2 \times \mathrm{SU}(n))(\mathbf{s}, be^{\sqrt{-1}\theta}, U_\Sigma) \\
&\qquad = (\widetilde{H}_\Sigma(\mathbf{s}, be^{\sqrt{-1}\theta}, U_\Sigma), \beta_\Sigma(\mathbf{s}, be^{\sqrt{-1}\theta}, U_\Sigma)) \quad (0 \leq b < 1/2).
\end{aligned}
$$

We have the following proposition by the definition of h together with the observation above.

PROPOSITION 5.4. *The map h is a topological embedding ($n \geq 2$).*

We define the topological embedding $i_n : \mathrm{SU}(n+1) \to \Omega^{10}$ as follows.

$$
\begin{aligned}
& \text{For } n = 1, i_n(M) = (be^{\sqrt{-1}\theta}, \bar{\mathbf{x}}), \\
& \text{for } n \geq 2, \\
& i_n(M) = h \circ k(M) \\
& = \begin{cases} (\widetilde{H}(\mathbf{x}(M), e^{\sqrt{-1}\theta}, U(M)), \beta(\mathbf{x}(M), e^{\sqrt{-1}\theta}, U(M))) \\ \hfill (0 < b \leq 1) \\ (\widetilde{H}_\Sigma(\mathbf{s}(M), be^{\sqrt{-1}\theta}, U_\Sigma(M)), \beta_\Sigma(\mathbf{s}(M), be^{\sqrt{-1}\theta}, U_\Sigma(M))) \\ \hfill (0 \leq b < 1/2). \end{cases}
\end{aligned}
$$

THEOREM 5.5. *The map i_n is a topological embedding and the image of i_n is a deformation retract of Ω^{10}.*

Proof. We only need to prove the second assertion. The case $n = 1$ is easy to prove. Hence, we assume $n \geq 2$. By Proposition 4.9 and the definition of i_n, the image of i_n coincides with that of h. By Theorem 3.1, it is enough to construct a deformation retraction of $(\pi_1^2 \,|\, \Omega^{10})^{-1}(K([0,1]))$ to the image of h. We identify an element β of $\mathrm{Hom}(\mathbf{C}^n \bigcirc \mathbf{C}^n, \mathbf{C}^n)$ with the n-tuple $(B_1, \ldots, B_n)$ of symmetric n-matrices. Then the norm $\|\beta\|$ is defined to be $\sum_{i=1}^n \|B_i\|$.

We first consider the homotopy h_λ of $(\pi_1^2 \,|\, \Omega^{10})^{-1}(K([0,1]))$ defined as follows. For an element (α, β) of $(\pi_1^2 \,|\, \Omega^{10})^{-1}(K(\{b\}))$, we set

$$h_\lambda(\alpha, \beta)$$

$$= \begin{cases} (\alpha, ((1-\lambda)+\lambda(1-b^2)^{1/2})(\|\beta\| - 2(1-b^2)^{1/2})(\beta/\|\beta\|) \\ \qquad +2(1-b^2)^{1/2}(\beta/\|\beta\|)) & \text{if } \|\beta\| \geq 2(1-b^2)^{1/2} \text{ and } \|\beta\| \neq 0, \\ (\alpha,\beta) & \text{if } \|\beta\| \leq 2(1-b^2)^{1/2}. \end{cases}$$

It is easy to see that the image of h_1 coincides with $(\pi_1^2 \,|\, \Omega^{10})^{-1}(K([0,1)))\cup K(\{1\}) \times \{\mathbf{0}\}$.

Next we construct a deformation retraction R_λ of $(\pi_1^2 \,|\, \Omega^{10})^{-1}(K([0,1))) \cup K(\{1\}) \times \{\mathbf{0}\}$ to the image of h. Take an element (α,β) of $(\pi_1^2 \,|\, \Omega^{10})^{-1}(K(\{b\}))$ such that α is written as $\widetilde{H}([\mathbf{x}], \mathbf{d}_{ab}, e^{\sqrt{-1}\,\theta}, U)$ with $\|\mathbf{x}\| = (1-b^2)^{1/2}$ for $0 < b < 1$ or $\widetilde{H}_\Sigma([\mathbf{s}], be^{\sqrt{-1}\,\theta}, U_\Sigma)$ for $0 \leq b < 1/2$. Let $\widetilde{K}_\alpha$ be the subspace generated by $U^*G(\mathbf{s}, e^{\sqrt{-1}\,\theta})\mathbf{x}$ for $0 < b < 1$ and the subspace generated by $U_\Sigma^*\mathbf{s}$ for $0 \leq b < 1/2$. Let $\widetilde{Q}_\alpha$ be the subspace generated by $\mathbf{x}$ or $\mathbf{s}$ for $0 \leq b < 1$. Let $\widetilde{\mathbf{K}}$ and $\widetilde{\mathbf{Q}}$ be the complex line bundles over $K([0,1))$ defined by $\widetilde{\mathbf{K}}_{(\alpha,\beta)} = \widetilde{\mathbf{K}}_\alpha$ and $\widetilde{\mathbf{Q}}_{(\alpha,\beta)} = \widetilde{\mathbf{Q}}_\alpha$ respectively. By definition, we have $\mathbf{K}\,|_{K(\{0\})} = \mathbf{K}$ and $\widetilde{\mathbf{Q}}\,|_{K(\{0\})}$ is identified with $\mathbf{Q}$ by Remark 3.8. Then we have a canonical isomorphism $K([0,1)) \times \mathbf{C} \to \mathrm{Hom}(\widetilde{\mathbf{K}}, \widetilde{\mathbf{Q}})$ such that $\alpha \times 1$ is mapped to the isomorphism sending $U^*G(\mathbf{s}, e^{\sqrt{-1}\,\theta})\mathbf{x}$ to $\mathbf{x}$ for $0 < b < 1$ and sending $U_\Sigma^*\mathbf{s}$ to $\mathbf{s}$ for $0 \leq b < 1/2$, which does not depend on the choice of $\mathbf{x}$ or $\mathbf{s}$ representing $[\mathbf{x}]$ or $[\mathbf{s}]$ respectively and is uniquely determined by α. Let us recall the following $\mathbf{R}$-linear bundle map of $\mathrm{Hom}(\widetilde{\mathbf{K}}, \mathbf{C})$ to $\widetilde{\mathbf{K}}$. Define the hermitian form $h_{\widetilde{\mathbf{K}}}$ on $\widetilde{\mathbf{K}}$ by $h_{\widetilde{\mathbf{K}}}(z_1\mathbf{v}, z_2\mathbf{v}) = z_1\bar{z}_2\|\mathbf{v}\|^2 = z_1\bar{z}_2$, where $\mathbf{v}$ is any vector of length 1 in $\widetilde{\mathbf{K}}_\alpha$. Then we have the orientation reversing bundle map over $\mathbf{R}$, $B_h : \widetilde{\mathbf{K}} \to \mathrm{Hom}(\widetilde{\mathbf{K}}, \mathbf{C})$ defined by $B_h(z\mathbf{v}) = h_{\widetilde{\mathbf{K}}}(\ \ , z\mathbf{v})$, where we note that $h_{\widetilde{\mathbf{K}}}(\ \ , z\mathbf{v})$ is a $\mathbf{C}$-homomorphism. Then we have $B_h(z\mathbf{v}) = \bar{z}B_h(\mathbf{v})$. These observations induce the map

$$\Psi : \mathrm{Hom}(\widetilde{\mathbf{K}} \bigcirc \widetilde{\mathbf{K}}, \widetilde{\mathbf{Q}}) \cong \mathrm{Hom}(\widetilde{\mathbf{K}}, \mathrm{Hom}(\widetilde{\mathbf{K}}, \widetilde{\mathbf{Q}})) \cong \mathrm{Hom}(\widetilde{\mathbf{K}}, \mathbf{C}) \xrightarrow{B_h^{-1}} \widetilde{\mathbf{K}}.$$

For a non-zero vector $\mathbf{x}$ of $\mathbf{C}^n$, let $\mathrm{pr}(\mathbf{x})$ denote the orthogonal projection of $\mathbf{C}^n$ onto the subspace of dimension 1 generated by $\mathbf{x}$ over $\mathbf{C}$. Since the element (α,β) induces the map $\mathrm{pr}(\mathbf{x}) \circ \beta \,|\, \widetilde{\mathbf{K}}_\alpha \bigcirc \widetilde{\mathbf{K}}_\alpha : \widetilde{\mathbf{K}}_\alpha \bigcirc \widetilde{\mathbf{K}}_\alpha \to \widetilde{\mathbf{Q}}$, Ψ determines the vector $\Psi(\mathrm{pr}(\mathbf{x}) \circ \beta \,|\, \widetilde{\mathbf{K}}_\alpha \bigcirc \widetilde{\mathbf{K}}_\alpha)$ in $\widetilde{\mathbf{K}}_\alpha$. This is written as $u(\alpha,\beta)\mathbf{k}$ by some real number $u(\alpha,\beta) \geq 0$ and some vector $\mathbf{k}$ with length 1 such that $[\mathbf{k}] = [U^*G(\mathbf{s}, e^{\sqrt{-1}\,\theta})\mathbf{x}]$ for $0 < b < 1$ and $[\mathbf{k}] = [U_\Sigma^*\mathbf{s}]$ for $0 \leq b < 1/2$. We note that $\mathbf{k}$ is determined only when $u(\alpha,\beta) > 0$. Let $\mathbf{s}(\alpha,\beta)$ denote $G(\mathbf{s}, e^{\sqrt{-1}\,\theta})^*U\mathbf{k}$ for $0 < b < 1$ and $U_\Sigma\mathbf{k}$ for $0 \leq b < 1/2$. If $u(\alpha,\beta) > 0$, then we have that

$$\mathrm{pr}(\mathbf{s}(\alpha,\beta))(\beta(\mathbf{k},\mathbf{k})) = u(\alpha,\beta)\mathbf{s}(\alpha,\beta) \qquad \text{for} \quad 0 \leq b < 1.$$

Here set $\bar{u}(\alpha, \beta) = u(\alpha, \beta)/(b^2 + u(\alpha, \beta)^2)^{1/2}$ for $0 \le b < 1$, where $b^2 + u(\alpha, \beta)^2$ never vanishes. Now we set $\mathbf{x}(\alpha, \beta) = (1 - b^2)^{1/2}\mathbf{s}(\alpha, \beta)$. If $u(\alpha, \beta) = 0$, then $\mathbf{x}(\alpha, \beta)$ or $\mathbf{s}(\alpha, \beta)$ represents any vector of length $(1-b^2)^{1/2}$ or 1 in $\widetilde{Q}_\alpha$ respectively. Furthermore, we set $\mathbf{y}(\alpha, \beta) = u(\alpha, \beta)\mathbf{s}(\alpha, \beta)$, which is always defined. The motivation for this notation is the fact that

$$\beta(\mathbf{x}, e^{\sqrt{-1}\,\theta}, U)(U^* G(\mathbf{s}, e^{\sqrt{-1}\,\theta}) S\mathbf{e}_n, U^* G(\mathbf{s}, e^{\sqrt{-1}\,\theta}) S\mathbf{e}_n) = \|\mathbf{x}\|\mathbf{s} \qquad \text{for } 0 < b < 1,$$

$$\beta_\Sigma(\mathbf{s}, be^{\sqrt{-1}\,\theta}, U_\Sigma)(U_\Sigma^* S\mathbf{e}_n, U_\Sigma^* S\mathbf{e}_n) = \|\mathbf{x}\|\mathbf{s} \qquad \text{for } 0 \le b < 1/2.$$

We note that

(1) The vector $\mathbf{y}(\alpha, \beta)$ is continuous on $(\pi_1^2 \,|\, \Omega^{10})^{-1}(K([0, 1))) \cup K(\{1\}) \times \{\mathbf{0}\}$,

(2) If $0 < \|\mathbf{x}(\alpha, \beta)\| < 1$, equivalently $0 < b < 1$, then $\|\mathbf{y}(\alpha, \beta)\| = u(\alpha, \beta)/(b^2 + u(\alpha, \beta)^2)^{1/2} < 1$,

(3) $u(\widetilde{H}(\mathbf{x}(\alpha, \beta), e^{\sqrt{-1}\,\theta}, U), \beta(\mathbf{x}(\alpha, \beta), e^{\sqrt{-1}\,\theta}, U)) = (1 - b^2)^{1/2}$,

(4) $u(\widetilde{H}_\Sigma(\mathbf{s}(\alpha, \beta), be^{\sqrt{-1}\,\theta}, U_\Sigma), \beta_\Sigma(\mathbf{s}(\alpha, \beta), be^{\sqrt{-1}\,\theta}, U_\Sigma)) = (1 - b^2)^{1/2}$ and

(5) Consider the case where $b^2 + u(\alpha, \beta)^2 = 1$, which is, in particular, satisfied for (α, β) in $\mathrm{Im}(h)$. Then we have $u(\alpha, \beta) = \bar{u}(\alpha, \beta)$ and $\mathbf{x}(\alpha, \beta) = \mathbf{y}(\alpha, \beta)$.

For an element (α, β) of $(\pi_1^2 \,|\, \Omega^{10})^{-1}(K(\{b\}))$ given above, we define $R_\lambda(\alpha, \beta)$ to be

$$\begin{cases} (\widetilde{H}((1-\lambda)\,\mathbf{x}(\alpha, \beta) + \lambda\mathbf{y}(\alpha, \beta), e^{\sqrt{-1}\,\theta}, U), \\ \qquad\qquad (1 - \lambda)\beta + \lambda\beta(\mathbf{y}(\alpha, \beta), e^{\sqrt{-1}\,\theta}, U)) & \text{for } 0 < b < 1, \\ (\alpha, \mathbf{0}) & \text{for } b = 1 \text{ and } \beta = \mathbf{0}, \\ (\widetilde{H}_\Sigma(\mathbf{s}(\alpha, \beta), \mathbf{0}, U_\Sigma), (1 - \lambda)\beta + \lambda\beta_\Sigma(\mathbf{s}(\alpha, \beta), \mathbf{0}, U_\Sigma)) & \text{for } b = 0, \end{cases}$$

where if $u(\alpha, \beta) = 0$, then $\widetilde{H}((1 - \lambda)\mathbf{x}(\alpha, \beta), e^{\sqrt{-1}\,\theta}, U)$ refers to $\widetilde{H}([(1 - \lambda)\mathbf{x}(\alpha, \beta)], \mathbf{d}_{a'b'}, e^{\sqrt{-1}\,\theta}, U)$ with $b' = (1 - (1 - \lambda)^2(1 - b^2))^{1/2}$.

Let us see that R_λ is well defined and continuous. Set $b_\lambda(\alpha, \beta) = \{1 - \|(1 - \lambda)\mathbf{x}(\alpha, \beta) + \lambda\mathbf{y}(\alpha, \beta)\|^2\}^{1/2}$. If $0 \le b_\lambda(\alpha, \beta) < 1/2$ and $0 \le$

$1 - \|\mathbf{y}(\alpha,\beta)\|^2 < 1/2$, then we may write $R_\lambda(\alpha,\beta)$ as a different form $(\widetilde{H}_\Sigma(\mathbf{s}(\alpha,\beta), b_\lambda(\alpha,\beta)e^{\sqrt{-1}\theta}, U_\Sigma), (1-\lambda)\beta+\lambda\beta_\Sigma(\mathbf{s}(\alpha,\beta),(1-\|\mathbf{y}(\alpha,\beta)\|^2)^{1/2}\times e^{\sqrt{-1}\theta}, U_\Sigma))$ by (3.6), (5.1) and (5.2). In particular, if $u(\alpha,\beta)=0$, then $0 < b < 1, \mathbf{y}(\alpha,\beta) = \mathbf{0}$ and $\beta(\mathbf{y}(\alpha,\beta), e^{\sqrt{-1}\theta}, U = \mathbf{0}$. If $b = 0$, then $u(\alpha,\beta) > 0$ and $(1-\lambda)\mathbf{x}(\alpha,\beta)+\lambda\mathbf{y}(\alpha,\beta) = (1-\lambda)\mathbf{x}(\alpha,\beta)+\lambda\mathbf{x}(\alpha,\beta) = \mathbf{x}(\alpha,\beta) = \mathbf{s}(\alpha,\beta)$. Therefore, R_λ is well defined and continuous.

We see that R_λ maps $(\pi_1^2 \,|\, \Omega^{10})^{-1}(K((0,1)))$ into $(\pi_1^2 \,|\, \Omega^{10})^{-1}(K((0,1))) \cup K(\{1\}) \times \{\mathbf{0}\}$. If $0 < \|\mathbf{x}(\alpha,\beta)\| < 1$, or equivalently $0 < b < 1$, then $\|(1-\lambda)\mathbf{x}(\alpha,\beta) + \lambda\mathbf{y}(\alpha,\beta)\|$ is less than 1 and is equal to 0 only when $\lambda = 1$ and $u(\alpha,\beta) = 0$. Furthermore, if $\lambda = 1$, $\bar{u}(\alpha,\beta) = 0$ and $0 < \|\mathbf{x}(\alpha,\beta)\| < 1$, then $R_1(\alpha,\beta) = (\widetilde{H}(\mathbf{0}, e^{\sqrt{-1}\theta}, U), \mathbf{0})$, since $\beta(\mathbf{0}, e^{\sqrt{-1}\theta}, U) = \mathbf{0}$.

We see that R_λ maps $(\pi_1^2 \,|\, \Omega^{10})^{-1}(K(\{0\}))$ into $(\pi_1^2 \,|\, \Omega^{10})^{-1}(K(\{0\}))$. By definition, we have that $\mathrm{pr}(\mathbf{s}(\alpha,\beta)))((1-\lambda)\beta + \lambda\beta_\Sigma(\mathbf{s}(\alpha,\beta), \mathbf{0}, U_\Sigma))(U_\Sigma^*\mathbf{s}(\alpha,\beta), U_\Sigma^*\mathbf{s}(\alpha,\beta)) = ((1-\lambda)u(\alpha,\beta) + \lambda)\mathbf{s}(\alpha,\beta)$. Since $(\alpha,\beta) \in \Sigma^{10}$, we have $u(\alpha,\beta) > 0$ and so $(1-\lambda)u(\alpha,\beta) + \lambda > 0$.

By definition, the image of R_1 is contained in $\mathrm{Im}(h)$. It is easy to see that $R_0 = \mathrm{id}$. It follows from (3), (4) and (5) that $R_\lambda \,|\, \mathrm{Im}(h)$ is constantly equal to $\mathrm{id}_{\mathrm{Im}(h)}$. ∎

§6. SU$(n) \times$ SU(n) action

In this section the unit vector $\mathbf{e}'_{n+1}$ of $\mathbf{C}^{n+1}$ in Section 4 is written as $\mathbf{e}_{n+1}$ to avoid confusion. We consider the following action of $\mathrm{SU}(n) \times \mathrm{SU}(n)$ on $J^2(n,n)$. An element (O', O^*) of $\mathrm{SU}(n) \times \mathrm{SU}(n)$ acts on each element (α,β) of $J^2(n,n)$ by

$$((O', O^*) \cdot (\alpha,\beta))(\mathbf{a},\mathbf{b},\mathbf{c}) = (O'\alpha(O\mathbf{a}), O'\beta(O\mathbf{b}, O\mathbf{c}))$$

and also acts on each element M of $\mathrm{SU}(n+1)$ by

$$(O', O^*) \cdot M = (O' \dot{+} (1))M(O \dot{+} (1)).$$

Note that Ω^{10} is invariant with respect to this action. We will prove that i_n is equivariant with respect to these actions of $\mathrm{SU}(n) \times \mathrm{SU}(n)$. Its proof needs a complicated observation about the embedding i_n. First we prepare two lemmas.

Lemma 6.1. *Let $M\mathbf{e}_{n+1}$ be written as $\mathbf{z} = {}^t(x_1, \ldots, x_n, z_{n+1})$ with $z_{n+1} = be^{\sqrt{-1}\theta}$ as above. Let $\mathbf{w}$ be $(O' \dot{+} (1))\mathbf{z}$ for an element O' of* SU(n). *Then we have*

(1) $r(\mathbf{w})^{-1}(O' \dot{+} (1)) = (O' \dot{+} (1))r(\mathbf{z})^{-1}$ *for* $0 < b \leq 1$,

(2) $r_\Sigma(\mathbf{w})^{-1}(O' \dot{+} (1)) = (O' \dot{+} (1))r_\Sigma(\mathbf{z})^{-1}$ *for* $0 \leq b < 1/2$,

Proof. (1) It is enough to prove $(O' \dot{+} (1))r(\mathbf{z})(O' \dot{+} (1))^* = r(\mathbf{w})$. By the property (4.1) of $r(\mathbf{w})$ we have

$$\begin{aligned} &r(\mathbf{w})(\mathbf{e}_{n+1}) = e^{-\sqrt{-1}\theta}\mathbf{w} = e^{-\sqrt{-1}\theta}(O' \dot{+} (1))\mathbf{z}, \\ &r(\mathbf{w})(\mathbf{w} - be^{\sqrt{-1}\theta}\mathbf{e}_{n+1}) = b\mathbf{w} - e^{\sqrt{-1}\theta}\mathbf{e}_{n+1} \\ &\qquad = b(O' \dot{+} (1))\mathbf{z} - e^{\sqrt{-1}\theta}\mathbf{e}_{n+1} \quad \text{and} \\ &r(\mathbf{w})\mathbf{f} = \mathbf{f} \text{ if } \mathbf{f} \text{ is orthogonal to } \mathbf{e}_{n+1} \text{ and } \mathbf{w} - be^{\sqrt{-1}\theta}\mathbf{e}_{n+1}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &(O' \dot{+} (1))r(\mathbf{z})(O' \dot{+} (1))^*(\mathbf{e}_{n+1}) \\ &\qquad = (O' \dot{+} (1))r(\mathbf{z})(\mathbf{e}_{n+1}) \\ &\qquad = (O' \dot{+} (1))e^{-\sqrt{-1}\theta}\mathbf{z} \\ &\qquad = e^{-\sqrt{-1}\theta}(O' \dot{+} (1))\mathbf{z}, \\ &(O' \dot{+} (1))r(\mathbf{z})(O' \dot{+} (1))^*(\mathbf{w} - be^{\sqrt{-1}\theta}\mathbf{e}_{n+1}) \\ &\qquad = (O' \dot{+} (1))r(\mathbf{z})(O' \dot{+} (1))^*(O' \dot{+} (1))(\mathbf{z} - z_{n+1}\mathbf{e}_{n+1}) \\ &\qquad = (O' \dot{+} (1))r(\mathbf{z})(\mathbf{z} - z_{n+1}\mathbf{e}_{n+1}) \\ &\qquad = (O' \dot{+} (1))(b\mathbf{z} - e^{\sqrt{-1}\theta}\mathbf{e}_{n+1}) \\ &\qquad = b(O' \dot{+} (1))\mathbf{z} - e^{\sqrt{-1}\theta}\mathbf{e}_{n+1}. \end{aligned}$$

Since $\mathbf{f}$ satisfies $(\mathbf{f}, \mathbf{e}_{n+1}) = (\mathbf{f}, (O' \dot{+} (1))(\mathbf{z} - z_{n+1}\mathbf{e}_{n+1})) = 0$, we have $((O' \dot{+} (1))^*\mathbf{f}, \mathbf{e}_{n+1}) = ((O' \dot{+} (1))^*\mathbf{f}, \mathbf{z} - z_{n+1}\mathbf{e}_{n+1}) = 0$. It follows from the property (4.1-(iii)) of $r(\mathbf{z})$ that

$$(O' \dot{+} (1))r(\mathbf{z})(O' \dot{+} (1))^*\mathbf{f} = (O' \dot{+} (1))(O' \dot{+} (1))^*\mathbf{f} = \mathbf{f}.$$

Thus we obtain

$$r(\mathbf{w}) = (O' \dot{+} (1))r(\mathbf{z})(O' \dot{+} (1))^*.$$

(2) The proof is similar. By definition we have

$$
\begin{aligned}
&r_\Sigma(\mathbf{w})(\mathbf{e}_{n+1}) = \mathbf{w} = (O' \dot{+} (1))\mathbf{z},\\
&r_\Sigma(\mathbf{w})(\mathbf{w} - be^{\sqrt{-1}\theta}\mathbf{e}_{n+1}) = be^{\sqrt{-1}\theta}\mathbf{w} - \mathbf{e}_{n+1}\\
&\qquad\qquad\qquad = be^{-\sqrt{-1}\theta}(O' \dot{+} (1))\mathbf{z} - \mathbf{e}_{n+1} \quad \text{and}\\
&r_\Sigma(\mathbf{w})\mathbf{f} = \mathbf{f} \text{ if } \mathbf{f} \text{ is orthogonal to } \mathbf{e}_{n+1} \text{ and } \mathbf{w} - be^{\sqrt{-1}\theta}\mathbf{e}_{n+1}.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&(O' \dot{+} (1))r_\Sigma(\mathbf{z})(O' \dot{+} (1))^*(\mathbf{e}_{n+1})\\
&\qquad = (O' \dot{+} (1))r_\Sigma(\mathbf{z})(\mathbf{e}_{n+1})\\
&\qquad = (O' \dot{+} (1))(\mathbf{z}),\\
&(O' \dot{+} (1))r_\Sigma(\mathbf{z})(O' \dot{+} (1))^*(O' \dot{+} (1))(\mathbf{z} - z_{n+1}\mathbf{e}_{n+1})\\
&\qquad = (O' \dot{+} (1))r_\Sigma(\mathbf{z})(\mathbf{z} - z_{n+1}\mathbf{e}_{n+1})\\
&\qquad = (O' \dot{+} (1))(\bar{z}_{n+1}\mathbf{z} - \mathbf{e}_{n+1})\\
&\qquad = \bar{z}_{n+1}(O' \dot{+} (1))\mathbf{z} - \mathbf{e}_{n+1}.
\end{aligned}
$$

Similarly we have that $((O' \dot{+} (1))^*\mathbf{f}, \mathbf{e}_{n+1}) = ((O' \dot{+} (1))^*\mathbf{f}, \mathbf{z} - z_{n+1}\mathbf{e}_{n+1}) = 0$. It follows from the property (4.2-(iii)) of $r_\Sigma(\mathbf{z})$ that

$$(O' \dot{+} (1))r_\Sigma(\mathbf{z})(O' \dot{+} (1))^*\mathbf{f} = (O' \dot{+} (1))(O' \dot{+} (1))^*\mathbf{f} = \mathbf{f}.$$

Thus we obtain

$$r_\Sigma(\mathbf{w}) = (O' \dot{+} (1))r_\Sigma(\mathbf{z})(O' \dot{+} (1))^*.$$

□

Lemma 6.2. *Set* $M' = (O' \dot{+} (1))M(O \dot{+} (1))$ *for* O *and* O' *in* $\mathrm{SU}(n)$. *Then we have*

(1) $U(M') = I_\theta O' I_{-\theta} U(M) O$ *for* $0 < b \leq 1$,

(2) $U_\Sigma(M') = O' U_\Sigma(M) O$ *for* $0 \leq b < 1/2$.

Proof. It follows from Lemma 6.1 that

$$(1) \qquad r(\mathbf{w})^{-1}M' = r(\mathbf{w})^{-1}(O' \dot{+} (1))M(O \dot{+} (1))$$

$$
\begin{aligned}
&= (O' \dot{+} (1))r(\mathbf{z})^{-1}M(O \dot{+} (1)) \\
&= (O' \dot{+} (1))(I_{-\theta}U(M) \dot{+} (e^{\sqrt{-1}\theta}))(O \dot{+} (1)) \\
&= O'I_{-\theta}U(M)O \dot{+} (e^{\sqrt{-1}\theta}) \\
&= I_{-\theta}(I_\theta O' I_{-\theta}U(M)O) \dot{+} (e^{\sqrt{-1}\theta}),
\end{aligned}
$$

$$
\begin{aligned}
(2) \qquad r_\Sigma(\mathbf{w})^{-1}M' &= r_\Sigma(\mathbf{w})^{-1}(O' \dot{+} (1))M(O \dot{+} (1)) \\
&= (O' \dot{+} (1))r_\Sigma(\mathbf{z})^{-1}M(O \dot{+} (1)) \\
&= (O' \dot{+} (1))(U_\Sigma(M) \dot{+} (1))(O \dot{+} (1)) \\
&= O'U_\Sigma(M)O \dot{+} (1).
\end{aligned}
$$

Thus (1) and (2) follow from the definitions of $U(M')$ and $U_\Sigma(M')$ respectively. ∎

We are ready to prove the following.

PROPOSITION 6.3. *The embedding i_n is equivariant with respect to the actions of* $\mathrm{SU}(n) \times \mathrm{SU}(n)$ *on* $\mathrm{SU}(n+1)$ *and* $J^2(n, n)$.

Proof. We use the notations given in the definition of i_n and let M, O', O and M' with $\mathbf{w} = M'\mathbf{e}_{n+1}$ and $\mathbf{z} = M\mathbf{e}_{n+1}$ be as above. We have that if $b < 1$, then $\mathbf{s}(M') = O'\mathbf{s}(M)$. Then we obtain the following.

If $0 < b < 1$, then

$$
\begin{aligned}
\widetilde{H}(\mathbf{x}(M'), e^{\sqrt{-1}\theta}, U(M')) &= O'S\Delta(\mathbf{d}_{ab})S^*O'^*I_{-\theta}U(M') \\
&= O'S\Delta(\mathbf{d}_{ab})S^*O'^*I_{-\theta}I_\theta O'I_{-\theta}U(M)O \\
&= O'S\Delta(\mathbf{d}_{ab})S^*I_{-\theta}U(M)O \\
&= O'\widetilde{H}(\mathbf{x}(M), e^{\sqrt{-1}\theta}, U(M))O.
\end{aligned}
$$

If $b = 1$, then

$$
\begin{aligned}
\widetilde{H}(\mathbf{0}, e^{\sqrt{-1}\theta}, U(M')) &= (1/\sqrt{n})I_{-\theta}U(M') \\
&= O'(1/\sqrt{n})I_{-\theta}U(M)O \\
&= O'\widetilde{H}(\mathbf{0}, e^{\sqrt{-1}\theta}, U(M))O.
\end{aligned}
$$

Let $0 < b < 1$. Since

$$\begin{aligned} G(\mathbf{s}(M'), e^{\sqrt{-1}\theta})^* U(M') &= O' S I_\theta S^* O'^* I_{-\theta} I_\theta O' I_{-\theta} U(M) O \\ &= O' S I_\theta S^* I_{-\theta} U(M) O, \end{aligned}$$

we have

$$\begin{aligned} &\beta(\mathbf{x}(M'), e^{\sqrt{-1}\theta}, U(M'))(\mathbf{a}, \mathbf{b}) \\ &= \{{}^t\mathbf{a}^t(G(\mathbf{s}(M'), e^{\sqrt{-1}\theta})^* U(M'))\bar{O}'\bar{S} \\ &\qquad \times \Delta(0, \dots, 0, \|\mathbf{x}(M')\| S^* O'^* G(\mathbf{s}(M'), e^{\sqrt{-1}\theta})^* U(M')\mathbf{b}\}\mathbf{s}(M') \\ &= \{{}^t\mathbf{a}^t O^t U(M) I_{-\theta} \bar{S} I_\theta {}^t S \bar{S} \\ &\qquad \times \Delta(0, \dots, 0, \|\mathbf{x}(M)\|) S^* S I_\theta S^* I_{-\theta} U(M) O \mathbf{b}\} O' \mathbf{s}(M) \\ &= \{{}^t(O\mathbf{a})^t(G(\mathbf{s}(M), e^{\sqrt{-1}\theta})^* U(M))\bar{S} \\ &\qquad \times \Delta(0, \dots, 0, \|\mathbf{x}(M)\|) S^* G(\mathbf{s}(M), e^{\sqrt{-1}\theta})^* U(M) O \mathbf{b}\} O' \mathbf{s}(M) \\ &= O' \beta(\mathbf{x}(M), e^{\sqrt{-1}\theta}, U(M))(O\mathbf{a}, O\mathbf{b}). \end{aligned}$$

This equality also holds in the case of $b = 1$.

If $0 \le b < 1/2$, then

$$\begin{aligned} \widetilde{H}_\Sigma(\mathbf{s}(M'), be^{\sqrt{-1}\theta}, U_\Sigma(M')) &= O' S \Delta(\mathbf{d}_{ab}) I_{-\theta} S^* O'^* U_\Sigma(M') \\ &= O' S \Delta(\mathbf{d}_{ab}) I_{-\theta} S^* O'^* O' U_\Sigma(M) O \\ &= O' S \Delta(\mathbf{d}_{ab}) I_{-\theta} S^* U_\Sigma(M) O \\ &= O' \widetilde{H}_\Sigma(\mathbf{s}(M), be^{\sqrt{-1}\theta}, U_\Sigma(M)) O \end{aligned}$$

and

$$\begin{aligned} &\beta_\Sigma(\mathbf{s}(M'), be^{\sqrt{-1}\theta}, U_\Sigma(M'))(\mathbf{a}, \mathbf{b}) \\ &= \{{}^t\mathbf{a}^t U_\Sigma(M') \bar{O}' \bar{S} \Delta(0, \dots, 0, \|\mathbf{x}(M')\|) S^* O'^* U_\Sigma(M') \mathbf{b}\} \mathbf{s}(M') \\ &= \{{}^t\mathbf{a}^t O^t U_\Sigma(M)^t O' \bar{O}' \bar{S} \Delta(0, \dots, 0, \|\mathbf{x}(M)\|) S^* O'^* O' U_\Sigma(M) O \mathbf{b}\} O' \mathbf{s}(M) \\ &= \{{}^t(O\mathbf{a})^t U_\Sigma(M) \bar{S} \Delta(0, \dots, 0, \|\mathbf{x}(M)\|) S^* U_\Sigma(M) O \mathbf{b}\} O' \mathbf{s}(M) \\ &= O' \beta_\Sigma(\mathbf{s}(M), be^{\sqrt{-1}\theta}, U_\Sigma(M))(O\mathbf{a}, O\mathbf{b}). \end{aligned}$$

This proves that i_n is equivariant with respect to the actions of $\mathrm{SU}(n) \times \mathrm{SU}(n)$. ∎

Proof of Theorem 1 (2). The assertion follows from Theorem 5.5 and Proposition 6.3. ∎

§7. Holomorphic fold maps

Let $J^2(N,P)$ be the complex 2-jet space of complex manifolds N and P. Let π_N and π_P be the projections mapping a jet to its source and target respectively. Let $L^2(n)$ be the group of 2-jets of all biholomorphic map germs $(\mathbf{C}^n,0) \to (\mathbf{C}^n,0)$. The map $\pi_N \times \pi_P : J^2(N,P) \to N \times P$ gives the structure of a fibre bundle with fibre $J^2(n,n)$ having the structure group $L^2(n) \times L^2(n)$. Let $\mathrm{Hom}(TN \oplus (TN \bigcirc TN), TP)$ be the vector bundle over $N \times P$ with structure group $GL(n;\mathbf{C}) \times GL(n;\mathbf{C})$, which is the union of all spaces $\mathrm{Hom}(T_xN \oplus (T_xN \bigcirc T_xN), T_yP)$ for (x,y) of $N \times P$, where $T_xN \bigcirc T_xN$ denotes the 2-fold symmetric product of T_xN. If a basis of $\mathbf{C}^n$ is fixed, then we have the canonical $\mathbf{C}$-linear isomorphism $j : J^2(n,n) \to \mathrm{Hom}(\mathbf{C}^n \oplus (\mathbf{C}^n \bigcirc \mathbf{C}^n), \mathbf{C}^n)$ by considering Taylor expansions. It is clear that j is equivariant with respect to the actions of $GL(n;\mathbf{C}) \times GL(n;\mathbf{C})$ on both spaces $J^2(n,n)$ and $\mathrm{Hom}(\mathbf{C}^n \oplus (\mathbf{C}^n \bigcirc \mathbf{C}^n), \mathbf{C}^n)$. Since $GL(n;\mathbf{C})$ is naturally a subgroup of $L^2(n)$ and the quotient space $L^2(n)/GL(n;\mathbf{C})$ is contractible, the structure group $L^2(n) \times L^2(n)$ of the fibre bundle $\pi_N \times \pi_P : J^2(N,P) \to N \times P$ is reduced to $GL(n;\mathbf{C}) \times GL(n;\mathbf{C})$. Hence it follows from [St, 12.6 Corollary] that we obtain a bundle map

$$J : J^2(N,P) \longrightarrow \mathrm{Hom}(TN \oplus (TN \bigcirc TN), TP),$$

which is uniquely determined up to homotopy.

Let $z = j^2_x f$ with $y = f(x)$ be a 2-jet in $J^2_{x,y}(N,P)$, which is the subset of $J^2(N,P)$ consisting of all 2-jets of germs of (N,x) to (P,y). Set $\mathbf{D} = \pi_N^*(TN)$ and $\mathbf{P} = \pi_P^*(TP)$. Then there is a homomorphism $d_1 : \mathbf{D} \to \mathbf{P}$ defined as follows. Let $\mathbf{D}_z$ and $\mathbf{P}_z$ be the fibres of $\mathbf{D}$ and $\mathbf{P}$ over z respectively. Then $d_{1,z} : \mathbf{D}_z \to \mathbf{P}_z$ refers to $df : T_xN \to T_yP$. We define $\Sigma^i(N,P)$ to be the set of all jets z with $\dim(\mathrm{Ker}(d_{1.z})) = i$. Then we have the subbundle $\mathbf{K} = \mathrm{Ker}(d_1)$ and the cokernel bundle $\mathbf{Q} = \mathrm{Cok}(d_1)$ over $\Sigma^i(N,P)$. In [Bo, p.50, Lemma 7.13 and Theorem 7.14] (see also [Ma, §2]) the second intrinsic derivative $d_2 : \mathbf{K} \to \mathrm{Hom}(\mathbf{K},\mathbf{Q})$ has been defined by using the second derivative of z. We define $\Sigma^{10}(N,P)$ to be the set of all jets z such that $\dim(\mathrm{Ker}(d_{1,z})) = 1$ and $d_{2,z} : \mathbf{K}_z \to \mathrm{Hom}(\mathbf{K}_z,\mathbf{Q}_z)$ is an isomorphism. Let $\Omega^{10}(N,P)$ be the union of the set of all regular jets and $\Sigma^{10}(N,P)$.

There is a canonical identification of $J^k(n,n)$ with $J^k_{0,0}(\mathbf{C}^n,\mathbf{C}^n)$. In $\mathrm{Hom}(TN \oplus (TN \bigcirc TN), TP)$ we can also define $\Sigma^1(N,P)'$, $\Sigma^{10}(N,P)'$ and $\Omega^{10}(N,P)'$ associated with Σ^1, Σ^{10} and Ω^{10} in Section 1 respectively. The two constructions above associated with Σ^1, Σ^{10} and Ω^{10} correspond with each other by J. Then $\Omega^{10}(N,P)$ and $\Omega^{10}(N.P)'$ are the subbundles of $J^2(N,P)$ and $\mathrm{Hom}(TN\oplus(TN\bigcirc TN),TP)$ respectively. Then J induces a bundle map of $\Omega^{10}(N,P)$ to $\Omega^{10}(N,P)'$.

For an n-dimensional complex manifold M, let us recall that an $\mathrm{SU}(n)$-structure of TM is a reduction (E,φ) of the structure group $GL(n;\mathbf{C})$ to $\mathrm{SU}(n)$, where E is an n-dimensional $\mathrm{SU}(n)$-vector bundle over M and $\varphi : TM\to E$ is a bundle map covering id_M (see [St, 9.2]). Two $\mathrm{SU}(n)$-structures (E_1,φ_1) and (E_2,φ_2) of TM are equivalent if there exists an $\mathrm{SU}(n)$-bundle map $B: E_1\to E_2$ such that $\varphi_2 = B\circ\varphi_1$. Consider the spherical fibre space $p' : \mathrm{BSU}(n)\to\mathrm{BU}(n)$ with fibre S^1 induced from the inclusion of $\mathrm{SU}(n)$ into $\mathrm{U}(n)$. Let $c_{TM} : M\to\mathrm{BU}(n)$ be the classifying map of TM. It is well known that equivalence classes of $\mathrm{SU}(n)$-structures of TM correspond bijectively to homotopy classes of continuous maps $c: M\to\mathrm{BSU}(n)$ with $p'\circ c = c_{TM}$.

Suppose that $\mathrm{SU}(n)$-structures (E,φ_N) and (F,φ_P) of TN and TP are given respectively. Then we can define the canonical bundle map

$$\Phi : \mathrm{Hom}(TN\oplus(TN\bigcirc TN),TP)\longrightarrow\mathrm{Hom}(E\oplus(E\bigcirc E),F)$$

by using φ_N and φ_P. The map $\Phi\circ J$ induces a biholomorphic map between fibres $J^2_{x,y}(N,P)$ and $\mathrm{Hom}(E_x\oplus(E_x\bigcirc E_x),F_y)$ (however, Φ may not be biholomorphic in general). On the other hand, we have the subbundle $\mathrm{SU}(E\oplus\theta_N,F\oplus\theta_P)$ of $\mathrm{Hom}(E\oplus\theta_N,F\oplus\theta_P)$ associated with $\mathrm{SU}(n+1)$.

We shall apply the embedding $i_n : \mathrm{SU}(n+1)\to\Omega^{10}$ $(\subset\mathrm{Hom}(\mathbf{C}^n\oplus(\mathbf{C}^n\bigcirc\mathbf{C}^n),\mathbf{C}^n))$ to $\mathrm{SU}(E\oplus\theta_N,F\oplus\theta_P)$ and $\mathrm{Hom}(E\oplus(E\bigcirc E),F)$. Let $i(N,P)'$ be the map of $\mathrm{SU}(E\oplus\theta_N,F\oplus\theta_P)$ to $\Phi(\Omega^{10}(N,P)')$ associated with i_n. Then we obtain a subspace homeomorphic to $\mathrm{SU}(n+1)$ denoted by $\mathrm{SU}_{x,y}(E,F)$ in $\mathrm{Hom}(E_x\oplus(E_x\bigcirc E_x),F_y)$. This space is well defined by Proposition 6.3. The space $\mathrm{SU}(E,F)$ is defined to be the union of all spaces $\mathrm{SU}_{x,y}(E,F)$ in $\Phi(\Omega^{10}(N,P)')$, where (x,y) varies all over $N\times P$. It becomes a subbundle with structure group $\mathrm{SU}(n)\times\mathrm{SU}(n)$ coming from those of E and F. It is clear that the image of $i(N,P)'$ coincides with $\mathrm{SU}(E,F)$ and is homotopy equivalent to $\Phi(\Omega^{10}(N,P)')$ by Theorem 1 (2). Now we define the map $i(N,P)$ to be

$$(\Phi\circ J\,|\,\Omega^{10}(N,P))^{-1}\circ i(N,P)' : \mathrm{SU}(E\oplus\theta_N,F\oplus\theta_P)\longrightarrow\Omega^{10}(N,P).$$

Proof of Theorem 2. (1) The map $i(N,P)$ gives a homotopy equivalence of fibre bundles, since $\Phi \circ J \,|\, \Omega^{10}(N,P)$ is a bundle map and $i(N,P)'$ is a fibre homotopy equivalence.

(2) Let $i(N,P)^{-1} : \Omega^{10}(N,P) \to \mathrm{SU}(E \oplus \theta_N, F \oplus \theta_P)$ be the homotopy inverse of $i(N,P)$. For a holomorphic fold map f, the section $j^2 f$ determines the homotopy class of a section $i(N,P)^{-1} \circ j^2 f$ of $\mathrm{SU}(E \oplus \theta_N, F \oplus \theta_P)$. This gives the homotopy class of an $\mathrm{SU}(n+1)$-bundle map $\widetilde{f} : E \oplus \theta_N \to F \oplus \theta_P$ covering f in Theorem 2 (2). ∎

Proof of Corollary 3. Since the first Chern classes of N and P vanish, there exist $\mathrm{SU}(n)$-structures (E, φ_N) and (F, φ_P) of TN and TP respectively. Consider the spherical fibre space $p : \mathrm{BSU}(n) \to \mathrm{BSU}(n+1)$ with fibre S^{2n+1} induced from the inclusion of $\mathrm{SU}(n)$ into $\mathrm{SU}(n+1)$. Let $c_N : N \to \mathrm{BSU}(n)$ and $c_P : P \to \mathrm{BSU}(n)$ denote the classifying maps of E and F respectively. Then $p \circ c_N$ and $p \circ c_P \circ f$ are the classifying maps of $TN \oplus \theta^1_N$ and $f^*(TP) \oplus \theta^1_N$ respectively. By Theorem 2 (2), there is a homotopy $c : N \times I \to \mathrm{BSU}(n+1)$ between $p \circ c_N$ and $p \circ c_P \circ f$. Let $c^*(p) : c^*(\mathrm{BSU}(n)) \to N \times I$ be the induced fibre space. By applying the obstruction theorem ([St]), the obstructions to extending the induced sections $c^*(c_N)$ and $c^*(c_P \circ f)$ to a section defined on $N \times I$ lie in $H^i(N \times I, N \times \{0,1\}; \pi_{i-1}(S^{2n+1}))$ $(i = 0, \ldots, 2n+1)$, which vanish for all i. Hence, there exists a section $c' : N \times I \to c^*(\mathrm{BSU}(n))$ with $c' \,|\, N \times 0 = c^*(c_N)$ and $c' \,|\, N \times 1 = c^*(c_P \circ f)$. This implies that there exists an $\mathrm{SU}(n)$-bundle map of E to $f^*(F)$, which yields an $\mathrm{SU}(n)$-bundle map $B : E \to F$. Thus we obtain a bundle map $\varphi_P^{-1} \circ B \circ \varphi_N : TN \to TP$ covering f. ∎

Remark 7.1. Theorem 2 does not hold for general complex manifolds. The holomorphic fold map $f : \mathbf{CP}^1 \to \mathbf{CP}^1$ defined by $f([z]) = [z^2]$ has the property that $f^*(C_1(\mathbf{CP}^1)) = 2C_1(\mathbf{CP}^1)$. Hence $T(\mathbf{CP}^1)$ is not even stably equivalent to $f^*(T(\mathbf{CP}^1))$.

Example 7.2. (1) We consider the following Hopf manifolds (cf. [K, Example 2.9]). Let G be the infinite cyclic group generated by the automorphism g of $\mathbf{C}^n \setminus \{\mathbf{0}\}$ defined by $g(z_1, \ldots, z_n) = (\alpha_1 z_1, \ldots, \alpha_n z_n)$, where $\alpha_1, \ldots, \alpha_n$ are constants with $|\alpha_i| > 1$ $(i = 1, \ldots, n)$. Then $M(\alpha_1, \ldots, \alpha_n)$ is defined to be the quotient manifold $\mathbf{C}^n \setminus \{\mathbf{0}\}/G$, which is diffeomorphic to $S^1 \times S^{2n-1}$. Hence, its first Chern class vanishes (cf. [H]). There is a holomorphic fold map $f : M(\alpha_1, \ldots, \alpha_{n-1}, \alpha_n) \to M(\alpha_1, \ldots, \alpha_{n-1}, \alpha_n^2)$ defined by $f([z_1, \ldots, z_{n-1}, z_n]) = [z_1, \ldots, z_{n-1}, z_n^2]$, where $[*]$ refers to the element represented by $*$. The singularity submanifold of f is identified with $M(\alpha_1, \ldots, \alpha_{n-1})$, which consists of the points of the form $[z_1, \ldots, z_{n-1}, 0]$.

(2) Given integers $a_1, \ldots, a_n \geq 2$, consider the Brieskorn polynomial $p(z) = z_1^{a_1} + \cdots + z_n^{a_n}$ $(n \geq 2)$ and the hypersurface $p^{-1}(0)$. Let r be a real number greater than 1 and $\alpha_1, \ldots, \alpha_n$ be n complex numbers with $\alpha_i^{a_i} = r$ $(i = 1, \ldots, n)$. Then the group G in (1) acts on $p^{-1}(0) \setminus \{\mathbf{0}\}$. Let $B(a_1, \ldots, a_n; \alpha_1, \ldots, \alpha_n)$ denote the quotient space $(p^{-1}(0) \setminus \{\mathbf{0}\})/G$. Since G is properly discontinuous (see [K, Theorem 2.2]), it is a compact complex $n-1$ dimensional submanifold of $M(\alpha_1, \ldots, \alpha_n)$. Let $K(a_1, \ldots, a_n)$ be the Brieskorn manifolds $p^{-1}(0) \cap S_\varepsilon^{2n-1}$, where ε is a sufficiently small positive real number (see [Br] and [Mi]). We can prove that $B(a_1, \ldots, a_n; \alpha_1, \ldots, \alpha_n)$ is C^∞-diffeomorphic to $S^1 \times K(a_1, \ldots, a_n)$. We give a sketch of the proof, which is analogous to the arguments found in [K, Example 2.9].

For a real number u and $(z_1, \ldots, z_n) \neq \mathbf{0}$, define the function $\mathcal{G}(u, z_1, \ldots, z_n) = |z_1|^2 \, |\alpha_1|^{-2u} + \cdots + |z_n|^2 \, |\alpha_n|^{-2u}$. Since $\lim_{u \to \infty} \mathcal{G}(u, z_1, \ldots, z_n) = 0$, $\lim_{u \to -\infty} \mathcal{G}(u, z_1, \ldots, z_n) = \infty$ and $\mathcal{G}(u, z_1, \ldots, z_n)$ is strictly decreasing with respect to u, the equation $\mathcal{G}(u, z_1, \ldots, z_n) = \varepsilon^2$ induces the unique implicit function $u(z) = u(z_1, \ldots, z_n)$. Consider the two C^∞-maps,

$$\Phi : \mathbf{R} \times K(a_1, \ldots, a_n) \longrightarrow p^{-1}(0) \setminus \{\mathbf{0}\},$$

$$\Phi_1 : p^{-1}(0) \setminus \{\mathbf{0}\} \longrightarrow \mathbf{R} \times K(a_1, \ldots, a_n)$$

defined by $\Phi(u, \zeta_1, \ldots, \zeta_n) = (\alpha_1^u \zeta_1, \ldots, \alpha_n^u \zeta_n)$ and $\Phi_1(z_1, \ldots, z_n) = (u(z), \alpha_1^{-u(z)} z_1, \ldots, \alpha_n^{-u(z)} z_n)$ respectively. Since $\mathcal{G}(u, \alpha_1^u \zeta_1, \ldots, \alpha_n^u \zeta_n) = |\zeta_1|^2 + \cdots + |\zeta_n|^2 = \varepsilon^2$, they satisfy that $\Phi_1 \circ \Phi(u, \zeta_1, \ldots, \zeta_n) = (u, \zeta_1, \ldots, \zeta_n)$ and $\Phi \circ \Phi_1(z_1, \ldots, z_n) = (z_1, \ldots, z_n)$. Furthermore, we have the following commutative diagram:

$$\begin{array}{ccc} p^{-1}(0) \setminus \{\mathbf{0}\} & \xrightarrow{\Phi_1} & \mathbf{R} \times K(a_1, \ldots, a_n) \\ \Big\downarrow g^m & & \Big\downarrow \widetilde{m} \\ p^{-1}(0) \setminus \{\mathbf{0}\} & \xrightarrow{\Phi_1} & \mathbf{R} \times K(a_1, \ldots, a_n), \end{array}$$

where $g^m(z_1, \ldots, z_n) = (\alpha_1^m z_1, \ldots, \alpha_n^m z_n)$ and $\widetilde{m}(u, \zeta) = (u + m, \zeta)$. This is what we want.

Note that the first Chern class of $B(a_1, \ldots, a_n; \alpha_1, \ldots, \alpha_n)$ vanishes at least for $n \geq 4$ and $n = 2$, since $K(a_1, \ldots, a_n)$ is simply connected for $n \geq 4$ ([Mi, Theorem 5.2]) and $\dim K(a_1, \ldots, a_n) = 1$ for $n = 2$. Furthermore grad$(p(z))$ is equal to ${}^t(a_1 z_1^{a_1 - 1}, \ldots, a_n z_n^{a_n - 1})$, which cannot be orthogonal to all of the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}$. Hence, for any point z of $p^{-1}(0) \setminus \{\mathbf{0}\}$, there exists a number j with $1 \leq j \leq n-1$ such that $(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n)$ is a local coordinate system both for $p^{-1}(0) \setminus \{\mathbf{0}\}$ near z and for $B(a_1, \ldots, a_n; \alpha_1, \ldots, \alpha_n)$ near $[z]$.

Let β be a complex number with $\beta^2 = \alpha_n$. Then we have the fold map $f: B(a_1, \ldots, a_{n-1}, 2a_n; \alpha_1, \ldots, \alpha_{n-1}, \beta) \to B(a_1, \ldots, a_{n-1}, a_n; \alpha_1, \ldots, \alpha_{n-1}, \alpha_n)$ defined by $f([z_1, \ldots, z_{n-1}, z_n]) = ([z_1, \ldots, z_{n-1}, z_n^2])$. The singularity submanifold of f is identified with $B(a_1, \ldots, a_{n-1}; \alpha_1, \ldots, \alpha_{n-1})$, which consists of the points of the form $[z_1, \ldots, z_{n-1}, 0]$ with $z_1^{a_1} + \cdots + z_{n-1}^{a_{n-1}} = 0$.

In a forthcoming paper we will deal with a complex analogy of the results in [An2, §4]. Let F_k^m denote the space consisting of all continuous maps $(S^{k-1}, *) \to (S^{k-1}, *)$ of degree m, where S^{k-1} is the unit sphere of dimension $k-1$ and $*$ is the base point. Let F^m denote the space $\lim_{k\to\infty} F_k^m$. Let N and P be compact complex manifolds of dimension n and P be, in addition, connected. Then we will show that a holomorphic fold map $f : N \to P$ of degree m determines a homotopy class of $[P, F^m]$, which depends only on a certain equivalence class of f.

References

[A1] Y. Ando, *The homotopy type of the space consisting of regular jets and folding jets in $J^2(n,n)$*, Japan. J. Math., **24** (1998), 169–181.

[A2] ———, *Folding maps and the surgery theory on manifolds*, J. Math. Soc. Japan, **53** (2001), 357–382.

[Bo] J. M. Boardman, *Singularities of smooth mappings*, Publ. Math. I.H.E.S., **33** (1967), 21–57.

[Br] E. Brieskorn, *Beispiele zur Differentialtopologie von Singularitäten*, Invent. Math., **2** (1966), 1–14.

[E] J. M. Eliashberg, *On singularities of folding types*, Math. USSR. Izv., **4** (1970), 1119–1134.

[H] F. Hirzebruch, Topological Methods in Algebraic Geometry, Springer-Verlag, 1966.

[K] K. Kodaira, Complex Manifolds and Deformation of Complex Structures, Springer-Verlag, 1986.

[L] H. I. Levine, *Singularities of differentiable maps*, Lecture Notes in Math., Springer-Verlag, **192** (1971), 1–89.

[Ma] J. Mather, *Stability of C^∞ mappings: VI. The nice dimensions*, Lecture Notes in Math., Springer-Verlag, **192** (1971), 207–253.

[Mi] J. Milnor, Singular Points of Complex Hypersurfaces, Princeton Univ. Press, Princeton, 1968.

[Sa] O. Saeki, *Notes on the topology of folds*, J. Math. Soc. Japan, **44** (1992), 551–566.

[St] N. Steenrod, The Topology of Fibre Bundles, Princeton Univ. Press, Princeton, 1951.

[W] H. Whitney, Complex Analytic Varieties, Addison-Wesley, 1972.

Department of Mathematics
Faculty of Science
Yamaguchi University
Yamaguchi, 753-8512
Japan
andoy@po.cc.yamaguchi-u.ac.jp