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EXISTENCE OF FUNCTIONS IN WEIGHTED SOBOLEV SPACES

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Dedicated to Professor Masayuki Itô on the occasion of his sixtieth birthday

Abstract. The aim of this paper is to determine when there exists a quasi-continuous Sobolev function $u \in W^{1,p}(\mathbf{R}^n;\mu)$ whose trace $u|_{\mathbf{R}^{n-1}}$ is the characteristic function of a bounded set $E \subset \mathbf{R}^{n-1}$, where $d\mu(x) = |x_n|^\alpha dx$ with $-1 < \alpha < p-1$.

As application we discuss the existence of harmonic measures for weighted p-Laplacians in the unit ball.

§1. Introduction

For $p > 1$ and a Borel measure μ, consider the relative (p,μ)-capacity $\mathrm{cap}_{p,\mu}(\cdot;\Omega)$ for an open set $\Omega \subset \mathbf{R}^n$. For a compact set $K \subset \Omega$, it is defined by

$$\mathrm{cap}_{p,\mu}(K;\Omega) = \inf \int_\Omega |\nabla u|^p d\mu,$$

where the infimum is taken over all functions $u \in C_c^\infty(\Omega)$ such that $u \geq 1$ on K; here $C_c^\infty(\Omega)$ denotes the space of infinitely differentiable functions with compact support in Ω. We extend the capacity $\mathrm{cap}_{p,\mu}(\cdot;\Omega)$ in the usual way (see Section 2).

For a subset E of $\mathbf{R}^n$, denote the characteristic function of E by χ_E. We use the notation $W_E^{1,p}(\mathbf{R}^n;\mu)$ to denote the class of (p,μ)-quasicontinuous Sobolev functions $u \in W^{1,p}(\mathbf{R}^n;\mu)$ such that $u|_{\mathbf{R}^{n-1}}$ is equal to χ_E (p,μ)-q.e. on $\mathbf{R}^{n-1}$; here we identify $\mathbf{R}^{n-1}$ with the hyperplane $\mathbf{R}^{n-1} \times \{0\}$. Throughout this paper, for $-1 < \alpha < p-1$ we consider the Borel measure

$$d\mu(x) = |x_n|^\alpha dx,$$

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where $x = (x_1, ..., x_{n-1}, x_n) \in \mathbf{R}^n$ and dx denotes the usual Lebesgue measure.

Motivated by the work of Nakai on the existence of Dirichlet finite harmonic measures (cf. [10, 11, 12]), we consider the problem to determine when $W_E^{1,p}(\mathbf{R}^n;\mu) \neq \emptyset$. Our aim in this paper is to show the following result, as a generalization of Herron-Koskela [4] which treats the case $\alpha = 0$.

THEOREM 1.1. *Let $E \subset \mathbf{R}^{n-1}$ be a bounded Borel set, $1 < p < \infty$ and $-1 < \alpha < p-1$. Suppose $W_E^{1,p}(\mathbf{R}^n;\mu) \neq \emptyset$.*

(a) *If $p > n + \alpha$, then $E = \emptyset$.*
(b) *If $2 + \alpha \leq p \leq n + \alpha$, then $\mathcal{H}^{n-1}(E) = 0$.*
(c) *If $\mathcal{H}^{n-1}(E) = 0$, then $\mathrm{cap}_{p,\mu}(E) = 0$.*

Here $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure.

In case $p \geq 2 + \alpha$, E has (p,μ)-capacity zero if $W_E^{1,p}(\mathbf{R}^n;\mu) \neq \emptyset$. In Section 5, we give some examples of bounded Borel sets $E \subset \mathbf{R}^{n-1}$ with $\mathcal{H}^{n-1}(E) > 0$ and $W_E^{1,p}(\mathbf{R}^n;\mu) \neq \emptyset$ for $1 < p < 2 + \alpha$.

As applications of our results, we discuss the existence of harmonic measures for the weighted p-Laplace equation

$$-\mathrm{div}\ (\omega(x)|\nabla u(x)|^{p-2}\nabla u(x)) = 0 \tag{1.1}$$

in the unit ball $\mathbf{B}$ in $\mathbf{R}^n$. In this paper, we consider

$$\omega(x) = |1 - |x||^{\alpha} \qquad (-1 < \alpha < p-1)$$

as the weight function, which is in the Muckenhoupt A_p class. Further, letting $d\nu(x) = \omega(x)dx$, we say that a function w is (p,ν)-Dirichlet finite if

$$\int_{\mathbf{B}} |\nabla w(x)|^p \ d\nu(x) < \infty. \tag{1.2}$$

A (p,ν)-Dirichlet finite function u on $\mathbf{B}$ is said to be (p,ν)-harmonic if it is a continuous weak solution of (1.1) in $\mathbf{B}$. We say that w is a (p,ν)-harmonic measure on $\mathbf{B}$ if w is (p,ν)-harmonic in $\mathbf{B}$ and the greatest (p,ν)-harmonic minorant of $\min\{w, 1-w\}$ is zero. We see that $0 \leq w \leq 1$ if w is a (p,ν)-harmonic measure. For elementary properties of (p,ν)-harmonic functions and (p,ν)-harmonic measures, see Heinonen-Kilpeläinen-Martio [3] and Nakai [10, 11, 12].

COROLLARY 1.1. *Let $1 < p < \infty$ and $-1 < \alpha < p - 1$.*

(1) *If $p \geq 2+\alpha$, then every (p,ν)-Dirichlet finite (p,ν)-harmonic measure on $\mathbf{B}$ is constant.*

(2) *For each $1 < p < 2 + \alpha$, there exists a non-constant (p,ν)-Dirichlet finite (p,ν)-harmonic measure on $\mathbf{B}$.*

§2. Preliminaries

Let $\mathbf{R}^n$ be the n-dimensional Euclidean space ($n \geq 2$). For a point $x \in \mathbf{R}^n$, we write $x = (x', x_n)$, where $x' = (x_1, x_2, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$. Let $B(x,r)$ denote the open ball centered at x with radius r, and $\mathcal{H}^s$ the s-dimensional Hausdorff measure. For $1 < p < \infty$, let $W^{1,p}(\mathbf{R}^n; \mu)$ denote the Sobolev space of all functions $u \in L^p(\mathbf{R}^n; \mu)$ whose gradient, denoted by ∇u, belongs to $L^p(\mathbf{R}^n; \mu)$, where $d\mu(x) = |x_n|^\alpha dx$ with $-1 < \alpha < p - 1$ as above.

Suppose that $\Omega \subset \mathbf{R}^n$ is open. For a compact subset K of Ω, let

$$W(K,\Omega) = \{u \in C_c^\infty(\Omega) : u \geq 1 \text{ on } K\}$$

and define

$$\mathrm{cap}_{p,\mu}(K;\Omega) = \inf_{u \in W(K,\Omega)} \int_\Omega |\nabla u|^p d\mu.$$

Further, we set

$$\mathrm{cap}_{p,\mu}(U;\Omega) = \sup_{\substack{K \subset U \\ K:\text{compact}}} \mathrm{cap}_{p,\mu}(K;\Omega)$$

for an open set $U \subset \Omega$, and then

$$\mathrm{cap}_{p,\mu}(E;\Omega) = \inf_{\substack{E \subset U \subset \Omega \\ U:\text{open}}} \mathrm{cap}_{p,\mu}(U;\Omega)$$

for an arbitrary set $E \subset \Omega$.

The number $\mathrm{cap}_{p,\mu}(E;\Omega)$ is called the (*variational*) (p,μ)-capacity of E relative to Ω. We know that $\mathrm{cap}_{p,\mu}(E;\Omega)$ is an outer Choquet capacity (see [2, 3, 7]). We say that $E \subset \mathbf{R}^n$ has (p,μ)-capacity zero, denoted by $\mathrm{cap}_{p,\mu}(E) = 0$, if $\mathrm{cap}_{p,\mu}(E \cap \Omega; \Omega) = 0$ for all open sets $\Omega \subset \mathbf{R}^n$.

We say that a property holds (p,μ)-quasieverywhere, often abbreviated to (p,μ)-q.e., if it holds except on a set of (p,μ)-capacity zero. A function u on a bounded open set Ω is said to be (p,μ)-quasicontinuous if given

$\varepsilon > 0$ there exists an open set $E \subset \Omega$ such that $\mathrm{cap}_{p,\mu}(E;\Omega) < \varepsilon$ and u is continuous as a function on $\Omega \setminus E$. Further we say that a function u is (p,μ)-quasicontinuous in $\mathbf{R}^n$ if u is (p,μ)-quasicontinuous in Ω for all bounded open set Ω.

The following lemma can be obtained by an elementary calculation and our assumption on α.

LEMMA 2.1. *Let $x = (x', 0) \in \mathbf{R}^n$ and $r > 0$. Then*

$$\mu(B(x,r)) = c_1 r^{n+\alpha},$$

where $c_1 = \mu(B(0,1)) < \infty$.

LEMMA 2.2. *Let $x = (x', 0) \in \mathbf{R}^n$ and $r > 0$. Then*

$$\mathrm{cap}_{p,\mu}(B(x,r); B(x,2r)) = c_2 r^{n+\alpha-p},$$

where $c_2 = \mathrm{cap}_{p,\mu}(B(0,1); B(0,2)) < \infty$.

Proof. For $a > 0$, let $u \in W(\overline{B}(0,a), B(0,2a))$ where $\overline{B}(x,r)$ denotes the closure of $B(x,r)$. If we set $v(y) = u((y-x)/r)$ for $r > 0$, then $v \in W(\overline{B}(x,ra), B(x,2ra))$ and it follows from the definition of the capacity that

$$\begin{aligned}\mathrm{cap}_{p,\mu}(\overline{B}(x,ra); B(x,2ra)) &\le \int_{B(x,2ra)} |\nabla v(y)|^p d\mu(y) \\ &= r^{-p+n+\alpha} \int_{B(0,2a)} |\nabla u(y)|^p d\mu(y),\end{aligned}$$

which gives

$$\mathrm{cap}_{p,\mu}(\overline{B}(x,ra); B(x,2ra)) \le c(a) r^{n+\alpha-p},$$

where $c(a) = \mathrm{cap}_{p,\mu}(\overline{B}(0,a); B(0,2a))$. Hence we obtain

$$c(a) \le c(ra) r^{-(n+\alpha-p)} \le c(a),$$

and thus

$$\mathrm{cap}_{p,\mu}(\overline{B}(x,r); B(x,2r)) = c_2 r^{n+\alpha-p},$$

where $c_2 = c(1) < \infty$, by Lemma 2.1. It is also clear from the above equality that

$$\mathrm{cap}_{p,\mu}(B(x,r); B(x,2r)) = \mathrm{cap}_{p,\mu}(\overline{B}(x,r); B(x,2r)).$$

In the same way as in [5, Theorem 22], we have the following result (see also [3, Theorem 2.32]).

COROLLARY 2.1. *If* $E \subset \mathbf{R}^{n-1}$ *and* $\mathrm{cap}_{p,\mu}(E) = 0$*, then* E *has Hausdorff dimension at most* $n - p + \alpha$*.*

LEMMA 2.3. (cf. [3, Theorems 4.4, 4.12] and [7])

(1) *For each* $u \in W^{1,p}(\mathbf{R}^n; \mu)$*, there exists a* (p, μ)*-quasicontinuous representative* u^* *which is equal to* u *a.e. on* $\mathbf{R}^n$*.*

(2) *If* u^* *and* v^* *are* (p, μ)*-quasicontinuous and* $u^* = v^*$ *a.e. on* $\mathbf{R}^n$*, then* $u^* = v^*$ (p, μ)*-q.e. on* $\mathbf{R}^n$*.*

A set A is called (p, μ)-thin at a point x if

$$\int_0^1 \left(\frac{\mathrm{cap}_{p,\mu}(A \cap B(x, r); B(x, 2r))}{\mathrm{cap}_{p,\mu}(B(x, r); B(x, 2r))} \right)^{1/(p-1)} r^{-1} dr < \infty;$$

A is (p, μ)-thick at x if A is not (p, μ)-thin at x. We say that a function u is (p, μ)-finely continuous at x if there exists a set A which is (p, μ)-thin at x such that

$$u(x) = \lim_{\substack{y \to x \\ y \notin A}} u(y).$$

LEMMA 2.4. (cf. Meyers [6] and [3, Section 12]) *Let* $u \in W^{1,p}(\mathbf{R}^n; \mu)$*. If* u *is* (p, μ)*-quasicontinuous on* $\mathbf{R}^n$*, then it is* (p, μ)*-finely continuous* (p, μ)*-q.e. on* $\mathbf{R}^n$*.*

LEMMA 2.5. (cf. [3, Section 12] and [9]) *Let* $p > n + \alpha$*.*

(1) $E \subset \mathbf{R}^{n-1}$ *has capacity zero if and only if* E *is empty.*

(2) *If* $u \in W^{1,p}(\mathbf{R}^n; \mu)$ *is* (p, μ)*-quasicontinuous on* $\mathbf{R}^n$*, then* $u|_{\mathbf{R}^{n-1}}$ *is continuous on* $\mathbf{R}^{n-1}$*.*

Proof. In this proof, we identify $x' \in \mathbf{R}^{n-1}$ with $(x', 0) \in \mathbf{R}^{n-1} \times \mathbf{R}^1$. First we show that if $x' \in \mathbf{R}^{n-1} \cap B(0, N)$, $N > 0$, then

$$\int_{B(0,N)} |x' - y|^{1-n} f(y) dy \le C N^{(p-n-\alpha)/p} \|f\|_{L^p(\mathbf{R}^n; \mu)} \tag{2.1}$$

for every nonnegative measurable function f on $\mathbf{R}^n$. In fact, we have by Hölder's inequality

$$\begin{aligned}&\int_{B(0,N)} |x'-y|^{1-n} f(y)dy\\ &\le \left(\int_{B(0,N)} |x'-y|^{(1-n)p'} |y_n|^{-\alpha p'/p} dy\right)^{1/p'} \left(\int_{B(0,N)} f(y)^p |y_n|^{\alpha} dy\right)^{1/p}\\ &\le C(N+|x'|)^{(p-n-\alpha)/p} \|f\|_{L^p(\mathbf{R}^n;\mu)},\end{aligned}$$

where $1/p+1/p'=1$.

In view of (2.1), we see easily that

$$\mathrm{cap}_{p,\mu}(\{x'\}; B(0,N)) \ge CN^{-(p-n-\alpha)},$$

so that $\mathrm{cap}_{p,\mu}(\{x'\}) > 0$, which proves (1).

As another application of (2.1), we show that the potential

$$u(x') = \int_{B(0,N)} |x'-y|^{1-n} f(y)dy \qquad (x' \in \mathbf{R}^{n-1})$$

is continuous on $\mathbf{R}^{n-1} \cap B(0,N)$ for every $f \in L^p(\mathbf{R}^n;\mu)$. To show this, we note that if $z' \in \mathbf{R}^{n-1} \cap B(0,N)$ and $0 < s < N - |z'|$, then

$$\int_{B(0,N)-B(z',s)} |x'-y|^{1-n} f(y)dy$$

is continuous at z'. Further, in view of (2.1), if $x' \in \mathbf{R}^{n-1} \cap B(z',s)$, then

$$\int_{B(z',s)} |x'-y|^{1-n} f(y)dy \le \int_{B(x',2s)} |x'-y|^{1-n} f(y)dy$$

is small enough when s is small. Hence $u(x')$ is continuous at z'. With the aid of Sobolev's integral representation of $u \in W^{1,p}(\mathbf{R}^n;\mu)$ (cf. [9]), we see that $u|_{\mathbf{R}^{n-1}}$ is continuous, so that (2) follows.

§3. The main lemma

Here we prepare the following technical lemma needed for the proof of Theorem 1.1.

LEMMA 3.1. *Let Ω be a bounded open set in $\mathbf{R}^n$. If A is a Borel set with its closure in Ω and $\mathcal{H}^n(A) > 0$, then*

$$\iint_{A\times(\Omega\setminus A)} |x-y|^{-n-1}\, dxdy = \infty.$$

Proof. Set

$$\tilde{A} = \left\{ x \in A \ : \ \lim_{r\to 0} \frac{\mathcal{H}^n(A\cap B(x,r))}{\mathcal{H}^n(B(x,r))} = 1 \right\}.$$

Then $\mathcal{H}^n(A\setminus\tilde{A}) = 0$ (cf. [2]), so that we may assume that $A = \tilde{A}$, that is, every point of A is the Lebesgue point of A. Further we may assume without loss of generality that the origin $0 \in A$. By our assumption, there exists $r_0 > 0$ such that $\mathcal{H}^n(A\setminus B(0,r_0)) > 0$ and $B(x,r)\subset\Omega$ whenever $B(x,r)\cap A \neq \emptyset$ and $0 < r < r_0$. Set $P(x) = \{tx : t \geq r_0\}$ for each $x \in \mathbf{R}^n$ and $S = \{x \in \mathbf{R}^n : |x| = 1 \text{ and } P(x)\cap A \neq \emptyset\}$. Then we see from Fubini's theorem that $\mathcal{H}^{n-1}(S) > 0$.

Take r_1 such that $0 < r_1 < r_0$. By the intermediate value theorem, for each $x \in S$ we can find $B(x^{(1)}, r_x^{(1)})$ such that $x^{(1)} \in P(x)$, $0 < r_x^{(1)} < r_1$ and

$$\frac{\mathcal{H}^n(A\cap B(x^{(1)}, r_x^{(1)}))}{\mathcal{H}^n(B(x^{(1)}, r_x^{(1)}))} = \frac{3}{4}.$$

Now we denote by $B^*(x,r)$ the projection of $B(x,r)$ to $\partial B(0,1)$. By the covering lemma, we can find a countable family $\{B_{1j}\}$, where $B_{1j} = B(z_j^{(1)}, r_j^{(1)})$, such that $|z_j^{(1)}| \geq r_0$, $r_j^{(1)} < r_1$, $\{B_{1j}^*\}$ is disjoint and

$$\bigcup_{x\in S} B^*(x^{(1)}, r_x^{(1)}) \subset \bigcup_j B^*(z_j^{(1)}, 5r_j^{(1)}).$$

Since $\mathcal{H}^{n-1}(B^*(x,r)) \leq c_3 r^{n-1}$ for $|x| \geq r_0$ with a constant $c_3 > 0$, depending only on r_0, we have

$$\begin{aligned} c_3 \sum_j (5r_j^{(1)})^{n-1} &\geq \sum_j \mathcal{H}^{n-1}(B^*(z_j^{(1)}, 5r_j^{(1)})) \\ &\geq \mathcal{H}^{n-1}\Big(\bigcup_{x\in S} B^*(x^{(1)}, r_x^{(1)})\Big) \\ &\geq \mathcal{H}^{n-1}(S). \end{aligned}$$

Hence we can find a positive integer N_1 such that

$$\sum_{j=1}^{N_1} (r_j^{(1)})^{n-1} \geq c_4, \tag{3.1}$$

where $c_4 = (2c_3 5^{n-1})^{-1}\mathcal{H}^{n-1}(S) > 0$. Since $\{B_{1j}\}_{j=1}^{N_1}$ is disjoint, we have

$$\begin{aligned}
&\iint_{A\times(\Omega\setminus A)} |x-y|^{-n-1}\, dxdy \\
&\geq \int_{(\bigcup_{j=1}^{N_1} B_{1j})\cap A} \left(\int_{\Omega\setminus A} |x-y|^{-n-1} dx \right) dy \\
&\geq \sum_{j=1}^{N_1} \int_{B_{1j}\cap A} \left(\int_{B_{1j}\setminus A} |x-y|^{-n-1} dx \right) dy \\
&\geq \sum_{j=1}^{N_1} (2r_j^{(1)})^{-n-1} \mathcal{H}^n(B_{1j}\setminus A)\mathcal{H}^n(B_{1j}\cap A) \\
&= \sum_{j=1}^{N_1} (2r_j^{(1)})^{-n-1} \frac{1}{4}\mathcal{H}^n(B_{1j})\frac{3}{4}\mathcal{H}^n(B_{1j}) \\
&> 2^{-n-4}\sigma_n^2 \sum_{j=1}^{N_1} (r_j^{(1)})^{n-1} > c_5,
\end{aligned}$$

in view of (3.1), where $c_5 = 2^{-n-5}\sigma_n^2 c_4$. The above inequalities also imply that

$$\sum_{j=1}^{N_1} \int_{(B_{1j}\cap A)\setminus G} \left(\int_{B_{1j}\setminus A} |x-y|^{-n-1} dx \right) dy \geq c_5 \tag{3.2}$$

whenever $G \subset \mathbf{R}^n$ satisfies $\mathcal{H}^n(B_{1j}\cap A \setminus G) \geq \mathcal{H}^n(B_{1j})/2$ for $1 \leq j \leq N_1$.

For $\gamma_1 = \min_{1\leq j\leq N_1} r_j^{(1)}$ and $\varepsilon_1 > 0$, take r_2 such that $0 < r_2 < \varepsilon_1\gamma_1$. Next, for each $x \in S$, find $B(x^{(2)}, r_x^{(2)})$ such that $x^{(2)} \in P(x)$, $0 < r_x^{(2)} < r_2$ and

$$\frac{\mathcal{H}^n(A\cap B(x^{(2)}, r_x^{(2)}))}{\mathcal{H}^n(B(x^{(2)}, r_x^{(2)}))} = \frac{3}{4}.$$

By the same considerations as above, we can take a family $\{B_{2j}\}_{j=1}^{N_2}$, where $B_{2j} = B(z_j^{(2)}, r_j^{(2)})$, such that $|z_j^{(2)}| \geq r_0$, $r_j^{(2)} < r_2$, $\{B_{2j}^*\}_{j=1}^{N_2}$ is disjoint and

$$\sum_{j=1}^{N_2} \int_{(B_{2j}\cap A)\setminus G} \left(\int_{B_{2j}\setminus A} |x-y|^{-n-1} dx \right) dy \geq c_5 \tag{3.3}$$

whenever $G \subset \mathbf{R}^n$ satisfies $\mathcal{H}^n(B_{2j} \cap A \setminus G) \geq \mathcal{H}^n(B_{2j})/2$ for $1 \leq j \leq N_2$. Since $\{B_{2j}^*\}_{j=1}^{N_2}$ is disjoint, we see from Fubini's theorem that

$$\mathcal{H}^n(B_{1j_1} \cap \bigcup_{j_2=1}^{N_2} B_{2j_2}) \leq (\max_{j_2} 2r_{j_2}^{(2)}) c_6 \mathcal{H}^{n-1}(B_{1j_1}^*) \leq 2c_3c_6(r_{j_1}^{(1)})^{n-1} r_2,$$

so that

$$\begin{aligned}\mathcal{H}^n(B_{1j_1} \cap A \setminus \bigcup_{j_2=1}^{N_2} B_{2j_2}) &\geq \mathcal{H}^n(B_{1j_1} \cap A) - \mathcal{H}^n(B_{1j_1} \cap \bigcup_{j_2=1}^{N_2} B_{2j_2}) \\ &\geq 3\mathcal{H}^n(B_{1j_1})/4 - 2c_3c_6(r_{j_1}^{(1)})^{n-1} r_2 \\ &\geq (3/4 - c_7\varepsilon_1)\mathcal{H}^n(B_{1j_1}),\end{aligned}$$

where c_6 and c_7 are positive constants. Taking $\varepsilon_1 = 2^{-3}/c_7$, we have by (3.2) and (3.3)

$$\begin{aligned}&\iint_{A\times(\Omega\setminus A)} |x-y|^{-n-1}\, dxdy \\ &\quad \geq \int_{\bigcup_{j_1=1}^{N_1} B_{1j_1}\cap A\setminus\bigcup_{j_2=1}^{N_2} B_{2j_2}} \left(\int_{\Omega\setminus A} |x-y|^{-n-1} dx \right) dy \\ &\qquad\quad + \int_{\bigcup_{j_2=1}^{N_2} B_{2j_2}\cap A} \left(\int_{\Omega\setminus A} |x-y|^{-n-1} dx \right) dy \geq 2c_5.\end{aligned}$$

Setting $\varepsilon_i = 2^{-i-2}/c_7$ for each nonnegative integer i, we apply the above arguments repeatedly, and obtain $\{B_{ij}\}_{j=1}^{N_i}$, where $B_{ij} = B(z_j^{(i)}, r_j^{(i)})$, such that $\{B_{ij}^*\}_{j=1}^{N_i}$ is disjoint, $|z_j^{(i)}| \geq r_0$, $r_j^{(i)} < \varepsilon_{i-1} \min\{r_j^{(i-1)} : 1 \leq j \leq N_{i-1}\}$, $\mathcal{H}^n(A \cap B_{ij}) = 3\mathcal{H}^n(B_{ij})/4$ and

$$\sum_{j=1}^{N_i} \int_{B_{ij}\cap A\setminus G} \left(\int_{B_{ij}\setminus A} |x-y|^{-n-1} dx \right) dy \geq c_5 \tag{3.4}$$

whenever $G \subset \mathbf{R}^n$ satisfies $\mathcal{H}^n(B_{ij} \cap A \setminus G) \geq \mathcal{H}^n(B_{ij})/2$ for $1 \leq j \leq N_i$. For $1 \leq i \leq k-1$ and $1 \leq j_i \leq N_i$, we have

$$\mathcal{H}^n(B_{ij_i} \cap A \setminus G_i) \geq (3/4 - c_7 \sum_{l=i}^{k-1} \varepsilon_l)\mathcal{H}^n(B_{ij_i}) \geq \mathcal{H}^n(B_{ij_i})/2,$$

where $G_i = \cup_{l=i+1}^{k} \cup_{j_l=1}^{N_l} B_{lj_l}$. Thus we have by (3.4)

$$\begin{aligned}
&\iint_{A\times(\Omega\setminus A)} |x-y|^{-n-1}\, dxdy \\
&\geq \sum_{i=1}^{k} \int_{\bigcup_{j_i=1}^{N_i} B_{ij_i}\cap A\setminus G_i} \left(\int_{\Omega\setminus A} |x-y|^{-n-1} dx \right) dy \\
&\geq \sum_{i=1}^{k} \sum_{j_i=1}^{N_i} \int_{B_{ij_i}\cap A\setminus G_i} \left(\int_{B_{ij}\setminus A} |x-y|^{-n-1} dx \right) dy \\
&\geq kc_5,
\end{aligned}$$

which proves the present lemma by the arbitrariness of k.

§4. Proof of Theorem 1.1

We are now ready to prove Theorem 1.1.

Proof of (a). In case $p > n + \alpha$, Lemma 2.5 shows that $W_E^{1,p}(\mathbf{R}^n; \mu)$ is empty for every bounded nonempty set $E \subset \mathbf{R}^{n-1}$.

Proof of (b). Assume that $2 + \alpha \leq p \leq n + \alpha$ and $W_E^{1,p}(\mathbf{R}^n; \mu) \neq \emptyset$. If $u \in W^{1,p}(\mathbf{R}^n; \mu)$ is (p, μ)-quasicontinuous on $\mathbf{R}^n$, then, in view of [9, 13]

$$\iint_{\Omega\times\Omega} \frac{|u(x',0) - u(y',0)|^p}{|x'-y'|^{n+p-(2+\alpha)}}\, dx'dy' < \infty$$

whenever Ω is a bounded open set in $\mathbf{R}^{n-1}$, so that

$$\iint_{E\times(\Omega\setminus E)} |x'-y'|^{-n-p+2+\alpha}\, dx'dy' < \infty,$$

which together with Lemma 3.1 implies that E is of measure zero. Thus (b) holds.

Proof of (c). We claim that if $\mathcal{H}^{n-1}(E) = 0$, then $F = \mathbf{R}^{n-1} \setminus E$ is (p, μ)-thick at each point of E. Fix $x \in E$ and $r > 0$. Let $K \subset F \cap B(x, r)$ be compact and $v \in W(K, B(x, 2r))$. Then we have

$$\int \left| \frac{\partial v(y', y_n)}{\partial y_n} \right| dy_n \geq 2 \qquad \text{for all } y' \in K.$$

Hence it follows from Hölder's inequality and Fubini's theorem that

$$\int |\nabla v|^p d\mu \geq c r^{-p+1+\alpha} \mathcal{H}^{n-1}(K)$$

for a positive constant c. Consequently, we have

$$\operatorname{cap}_{p,\mu}(F \cap B(x, r); B(x, 2r)) \geq c r^{-p+1+\alpha} \mathcal{H}^{n-1}(F \cap B(x, r)).$$

In view of the assumption $\mathcal{H}^{n-1}(E) = 0$ and Lemma 2.2, we see that

$$\frac{\operatorname{cap}_{p,\mu}(F \cap B(x, r); B(x, 2r))}{\operatorname{cap}_{p,\mu}(B(x, r); B(x, 2r))} \geq c'$$

for a positive constant c'. Now our claim is proved.

Fix $u \in W_E^{1,p}(\mathbf{R}^n; \mu)$ such that u is (p, μ)-quasicontinuous on $\mathbf{R}^n$ and $u|_{\mathbf{R}^{n-1}} = \chi_E$ on $\mathbf{R}^{n-1} \setminus A_1$, where A_1 has (p, μ)-capacity zero. By Lemma 2.4, u is (p, μ)-finely continuous on $\mathbf{R}^n \setminus A_2$, where A_2 has (p, μ)-capacity zero. Since F is (p, μ)-thick at each point of E as was shown above,

$$u = 0 \qquad \text{on } E \setminus A,$$

where $A = A_1 \cup A_2$. On the other hand, $u = 1$ on $E \setminus A$, so that $E \setminus A$ is empty. Thus it follows that $\operatorname{cap}_{p,\mu}(E) = 0$.

§5. Proof of Corollary 1.1

Let ν denote the Borel measure on $\mathbf{R}^n$ defined by

$$d\nu(y) = |1 - |y||^\alpha dy.$$

Proof of (1). Suppose that w is a (p, ν)-Dirichlet finite (p, ν)-harmonic measure on $\mathbf{B}$. Then $w(1 - w) \in W^{1,p}(\mathbf{B}; \nu)$ since $0 \leq w \leq 1$. By using [3, Theorem 3.17], there exists a solution $u \in W^{1,p}(\mathbf{B}; \nu)$ of (1.1) in $\mathbf{B}$ with $u - w(1-w) \in W_0^{1,p}(\mathbf{B}; \nu)$. Since $0 \leq w \leq 1$, it follows from [3, Lemma 7.37] that u is a (p, ν)-harmonic minorant of both w and $1-w$. By the definition of harmonic measures, we see that $u \leq 0$ in $\mathbf{B}$. Hence Theorem 1.1 implies that either $\operatorname{cap}_{p,\nu}(\{\zeta \in \partial\mathbf{B} : w(\zeta) = 0\}) = 0$ or $\operatorname{cap}_{p,\nu}(\{\zeta \in \partial\mathbf{B} : w(\zeta) = 1\}) = 0$, so that w is constant by [3, Lemma 7.37].

Proof of (2). Let E be a domain in $\partial\mathbf{B}$ with (nonempty) smooth boundary and take $u \in W^{1,p}_E(\mathbf{R}^n;\nu)$ as in Proposition 6.1 given in the next section. By [3, Theorem 3.17] we find $w \in W^{1,p}(\mathbf{B};\nu)$ which is the solution of (1.1) in $\mathbf{B}$ with $w - u \in W^{1,p}_0(\mathbf{B};\nu)$. Clearly, u and hence w is not constant. Set $\hat{w} = \min\{w, 1-w\}$ and let v be any (p,ν)-harmonic minorant of $\hat{w}$. Since the boundary values of $w(1-w)$ are zero (p,ν)-q.e, $\hat{w}$ has boundary values zero (p,ν)-q.e. Hence [3, Lemma 7.37] implies that $v \le 0$. It follows that w is a (p,ν)-harmonic measure on $\mathbf{B}$, which proves (2).

§6. Further results

First we note the following result.

LEMMA 6.1. *If* $0 < \delta < 1$ *and* Ω *is a bounded domain with* C^1 *boundary, then*

$$\iint_{\Omega\times(\mathbf{R}^n\setminus\Omega)} |x-y|^{-n-\delta}\, dxdy < \infty.$$

In fact, it suffices to note that

$$\int_{\mathbf{R}^n\setminus\Omega} |x-y|^{-n-\delta} dy \le \int_{\mathbf{R}^n\setminus B(x,d(x))} |x-y|^{-n-\delta} dy \le c d(x)^{-\delta},$$

for $x \in \Omega$,where c is a positive constant and $d(x) = \operatorname{dist}(x,\partial\Omega)$ denotes the distance of x from the boundary $\partial\Omega$.

PROPOSITION 6.1. *If* $1 < p < 2+\alpha$ *and* Ω *is a bounded domain on* $\mathbf{R}^{n-1}$ *with* C^1 *boundary, then* $W^{1,p}_\Omega(\mathbf{R}^n;\mu) \ne \emptyset$.

Proof. In view of Lemma 6.1, we see that χ_Ω satisfies

$$\iint_{\mathbf{R}^{n-1}\times\mathbf{R}^{n-1}} \frac{|\chi_\Omega(x') - \chi_\Omega(y')|^p}{|x'-y'|^{n+p-(2+\alpha)}}\, dx'dy' < \infty.$$

Hence χ_Ω belongs to the Lipschitz space $\Lambda^{p,p}_\beta(\mathbf{R}^{n-1})$ with $\beta = 1-(\alpha+1)/p$, so that the Poisson integral $u(x) = P_{x_n} * \chi_\Omega(x')$ satisfies

$$\int_{\mathbf{D}} |\nabla u(x)|^p x_n^\alpha dx < \infty,$$

where $\mathbf{D} = \{x = (x', x_n) : x_n > 0\}$; see Stein [13] or the second author [9]. Then we can extend u to a function in $W^{1,p}_\Omega(\mathbf{R}^n;\mu)$, so that $W^{1,p}_\Omega(\mathbf{R}^n;\mu)$ is not empty.

For $K \subset \mathbf{R}^{n-1}$, let $\Lambda_\beta^{p,p}(K)$ be the space of all functions $u \in \Lambda_\beta^{p,p}(\mathbf{R}^{n-1})$ such that $u = 0$ outside K. Finally we discuss whether $\Lambda_\beta^{p,p}(K)$ with $\beta = 1 - (\alpha+1)/p$ is trivial, or not. For this purpose we consider a capacity inequality introduced by Carleson [1, Theorem 2 in Section 4] (see also [8]).

PROPOSITION 6.2. *Let K be a compact set in $\mathbf{R}^{n-1}$. Let G_1 and G_2 be bounded open sets in $\mathbf{R}^n$ such that $K \subset G_1 \subset \overline{G_1} \subset G_2$. Set $\omega_1 = G_1 \cap \mathbf{R}^{n-1}$. If $\beta = 1 - (1+\alpha)/p$ and*

$$\operatorname{cap}_{p,\mu}(\omega_1 \setminus K; G_2) < \operatorname{cap}_{p,\mu}(\omega_1; G_2), \tag{6.1}$$

then $\Lambda_\beta^{p,p}(K)$ has non-zero element.

Proof. We can find a (p,μ)-quasicontinuous function $u_1 \in W^{1,p}(\mathbf{R}^n;\mu)$ such that

(i) $u_1 = 0$ (p,μ)-q.e. outside G_2 ;

(ii) $u_1 = 1$ (p,μ)-q.e. $\omega_1 \setminus K$;

(iii) $\int_{G_2} |\nabla u_1|^p d\mu = \operatorname{cap}_{p,\mu}(\omega_1 \setminus K; G_2)$;

(iv) $\int_{G_2} |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \varphi d\mu = 0$ for all $\varphi \in C_c^\infty(G_2)$ such that $\varphi = 0$ on $\omega_1 \setminus K$.

Similarly, we find a (p,μ)-quasicontinuous function $u_2 \in W^{1,p}(\mathbf{R}^n;\mu)$ such that

(v) $u_2 = 0$ (p,μ)-q.e. outside G_2 ;

(vi) $u_2 = 1$ (p,μ)-q.e. ω_1 ;

(vii) $\int_{G_2} |\nabla u_2|^p d\mu = \operatorname{cap}_{p,\mu}(\omega_1; G_2)$;

(viii) $\int_{G_2} |\nabla u_2|^{p-2} \nabla u_2 \cdot \nabla \varphi d\mu = 0$ for all $\varphi \in C_c^\infty(G_2)$ such that $\varphi = 0$ on ω_1.

Then we see from (6.1) that $u_1 \neq u_2$. Consequently, for $\varphi \in C_c^\infty(G_1)$, $[\varphi(u_2 - u_1)]|_{\mathbf{R}^{n-1}}$ is shown to be a non-zero element of $\Lambda_\beta^{p,p}(K)$.

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