

PROPER HOLOMORPHIC SELF-MAPS OF QUASI-CIRCULAR DOMAINS IN \mathbf{C}^2

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Abstract. In this paper, we prove that every proper holomorphic self-map of a smoothly bounded pseudoconvex circular or Hartogs domain of finite type in \mathbf{C}^2 is biholomorphic.

§1. Introduction

In this paper, we study proper holomorphic maps between domains of finite type. A domain Ω is said to be quasi-circular if there exist integers p, q ($p + q \geq 1$) such that whenever $(z, w) \in \Omega$, $(e^{ip\theta}z, e^{iq\theta}w) \in \Omega$ for $\theta \in [0, 2\pi]$. We observe that when $p = q = 1$, Ω is circular; when $p = 0$ or $q = 0$, Ω is Hartogs. The following is the main result that we shall prove in this paper.

THEOREM 1. *Let Ω be a smoothly bounded pseudoconvex quasi-circular domain of finite type in \mathbf{C}^2 . Then every proper holomorphic self-map of Ω is a biholomorphism.*

As a consequence for classical domains, we have

COROLLARY. *Let Ω be a smoothly bounded pseudoconvex circular or Hartogs domain of finite type in \mathbf{C}^2 . Then every proper holomorphic self-map of Ω is a biholomorphism.*

Since the first result of Alexander [1] on the unit ball, it has been an open question whether every proper holomorphic self-map of a smoothly bounded domain in \mathbf{C}^n is a biholomorphism. This problem still remains open in general. However, some positive results have been proved in the case of strictly pseudoconvex domains by Pinchuk [13], and in the case of pseudoconvex domains with real-analytic boundary by Bedford [2], Bedford-Bell [3]. Also, for domains with various symmetries, see [4, 9, 10, 12]. A recent paper of Berteloot [6] solves the problem for complete Reinhardt domains with

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C^2 smooth boundary by studying the Lie algebra of holomorphic tangent vector fields of the boundary. In a recent paper [9], we studied the structure of the branch locus of proper holomorphic maps between domains of finite type. As a consequence, it was proved that every proper holomorphic self-map of a smoothly bounded pseudoconvex complete circular domain in \mathbf{C}^2 is a biholomorphism. In this paper we continue to use the main result of [9] on the branch locus to study the case of quasi-circular domains of finite type. Using the method of this paper, we also generalize the result of [9] by dropping the assumption of completeness. In fact, we shall prove a more general result. The paper is organized as follows. In Section 1, some basic facts concerning self-maps are collected. In the case of Hartogs, Theorem 1 is proved in Section 2 using a result concerning fixed points of an analytic function. The remaining case is proved in Section 3 by reducing to one variable and two-variable complex dynamics situation.

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§2. Basic facts

To proceed with the proof of the theorem, we need some basic facts. Let $F : \Omega \rightarrow D$ be a proper holomorphic map between smoothly bounded pseudoconvex domains of finite type. It is well known that F extends smoothly up to the boundary of Ω . The branch locus of F is defined to be $V_F = \{z \in \Omega : \det F' = 0\}$. It is well-known that the automorphism group action $\text{Aut}(\Omega) \times \Omega \rightarrow \Omega$, $(F, z) \mapsto F(z)$ extends smoothly to $\overline{\Omega}$. Thus we can assume that $\text{Aut}(\Omega)$ acts smoothly on $\overline{\Omega}$ and in particular on $\partial\Omega$. We say that a subgroup G of $\text{Aut}(\Omega)$ acts transversally at a boundary point p of Ω if the image of the tangent mapping $(\Psi_p)^* : T_e G \rightarrow T_p(\partial\Omega)$ associated to the mapping $\Psi_p : G \rightarrow \partial\Omega$, $F \mapsto F(p)$ is not contained in the holomorphic tangent space $H_p(\partial\Omega)$. We will denote by \mathbf{T} the Lie group of the unit circle. If \mathbf{T} is a subgroup of $\text{Aut}(\Omega)$ and acts transversally near a point $p \in \partial\Omega$, we will simply say that Ω admits a transversal \mathbf{T} -action at p .

We recall the main result of [9] concerning the branch locus of proper holomorphic maps.

THEOREM 2. *Let $F : \Omega \rightarrow D$ be a proper holomorphic map between bounded pseudoconvex domains of finite type in \mathbf{C}^2 . Suppose that Ω admits*

a \mathbf{T} -action which is transversal at p . Then for any irreducible component V of the branch locus V_F with $p \in \partial\bar{V} = \bar{V} \cap \bar{\Omega}$, $\partial\bar{V}$ is a finite union of \mathbf{T} -orbits.

Although this result was not explicitly stated in [9], the proof there would apply for a \mathbf{T} -action which is locally transversal. It is this form that we shall apply in this paper.

We recall certain general facts about boundary behavior of proper holomorphic mappings. Let $F : D_1 \rightarrow D_2$ be a proper holomorphic mapping between two smoothly bounded pseudoconvex domains in \mathbf{C}^2 . We suppose that F is smooth up to the boundary. Let r_j be the defining function of D_j . Following [2], we consider the Levi-determinant $\Lambda_{\partial D_j}$ of D_j defined as [2].

One has then

$$\Lambda_{\partial D_2}(F(p))|J_f(p)|^2 = \Lambda_{\partial D_1}(p)$$

for any $p \in \partial D_1$.

For any boundary point $p \in \partial D_j$ we consider also the order of vanishing of $\Lambda_{\partial D_j}$ at p denoted by $\tau_{\partial D_j}(p)$, which is defined as follows: we choose smooth real coordinates $x = (x_1, x_2, x_3)$ on ∂D_j such that p corresponds to $x = 0$, and the formal power series $\Lambda_{\partial D_j}(x) = \sum_{j=0}^{\infty} \sum_{|\alpha|=j} a_{\alpha} x^{\alpha}$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. We set $\tau_{\partial D_j}(p) = \min\{|\alpha| : a_{\alpha} \neq 0\}$ (of course, this definition does not depend on the choice of coordinates). The following properties of τ are well known (see [2, 3]):

(1) $\tau_{\partial D_j}(p)$ is an upper-semicontinuous function on ∂D_j .

(2) $\tau_{\partial D_2}(F(p)) \leq \tau_{\partial D_1}(p)$ and the equality holds if and only if \bar{V}_F does not contain p , i.e. F is a diffeomorphism on the boundary near p . The following lemma proves to be very useful.

LEMMA 1. Let $F : \Omega \rightarrow \Omega$ be a proper holomorphic map between smoothly bounded pseudoconvex domains of finite type in \mathbf{C}^n . Then if $V_F \neq \emptyset$, there exists an irreducible component L_n of V_{F^n} such that $L_i \neq L_j, i \neq j$, and

$$L_{n+1} \subset F^{-1}(L_n)$$

for $n = 1, 2, 3, \dots$, where F^n denotes the n -th iteration of F .

Proof. We claim that there exists a sequence (L_n) of complex varieties such that $L_n \subset V_{F^n}, L_{n+1} \subset F^{-1}(L_n)$.

We will construct the family (L_n) by induction. For every n we have $V_{F^{n+1}} = V_{F^n} \cup F^{-1}(V_{F^n})$. Fix any irreducible component L_1 in V_F . Then

$F^{-1}(L_1)$ is contained in V_{F^2} and contains an irreducible component L_2 . Assume that the components L_1, \dots, L_n are defined. Then $F^{-1}(L_n)$ is contained in $F^{-1}(V_{F^n}) \subset V_{F^{n+1}}$. So there exists an irreducible component L_{n+1} such that $L_{n+1} \subset F^{-1}(L_n)$. Note that since every restriction $F : L_{n+1} \rightarrow L_n$ is proper and $F(L_{n+1}) \subset L_n$, we have $F(L_{n+1}) = L_n$. We note that the varieties (L_n) are distinct. Indeed, suppose by contradiction that m is the first integer such that there exists p with $L_m = L_{m+p}$. If $m \geq 2$, we have $F(L_m) = F(L_{m+p})$ and so $L_{m-1} = L_{m+p-1}$. This contradicts the definition of m . So $m = 1$. Let p be the integer such that $L_1 = L_{1+p}$. Since we have $F^p(L_{1+p}) = L_1$, it follows that $F^p(L_1) = L_1$. Since $L_1 \subset V_{F^p}$, letting $Q \in \overline{L_1} \cap \Omega$, we see

$$\dots < \tau_{\partial\Omega}(F^{kp}(Q)) < \tau_{\partial\Omega}(F^{(k-1)p}(Q)) < \dots < \tau_{\partial\Omega}(F^p(Q)) < \tau_{\partial\Omega}(Q) < \infty,$$

which is obviously a contradiction.

We need a simple result from one complex variable dynamics on fixed points, which is key to the study of the case of Hartogs domains.

LEMMA 2. *Let f be an analytic function in a neighborhood U of z_0 with $f(z_0) = z_0$. Suppose that $f^n(V)$ is uniformly bounded for a neighborhood $V \subset \subset U$. Then there exists no sequence z_n in V converging to z_0 such that $f(z_{n+1}) = z_n$ for all n .*

Proof. We consider the Taylor series of f :

$$f(z) = z_0 + \lambda(z - z_0) + \dots$$

where $\lambda = f'(z_0)$, the multiplier of f at z_0 .

If $\lambda = 0$ -superattracting, by [8], there is a conformal map $\zeta = \phi(z)$ of a neighborhood of z_0 onto a neighborhood of 0 which conjugates f to ζ^p for a positive integer $p > 1$, that is

$$g(\zeta) = \phi \circ f \circ \phi^{-1}(\zeta) = \zeta^p.$$

Letting $\zeta_n = \phi(z_n)$, it follows that $\zeta_n \neq \zeta_m$, $n \neq m$, and $\zeta_n \rightarrow 0$. We see $g(\zeta_{n+1}) = \zeta_n$, therefore $\zeta_{n+1}^p = \zeta_n$. This implies $\zeta_{n+1} = \zeta_n^{1/p^n}$, a contradiction to $\zeta_n \rightarrow 0$.

If $0 < |\lambda| < 1$ -attracting, and not superattracting, by [8], there is a conformal map $\zeta = \phi(z)$ of a neighborhood of z_0 onto a neighborhood of 0 which conjugates f to $\lambda\zeta$, that is

$$g(\zeta) = \phi \circ f \circ \phi^{-1}(\zeta) = \lambda\zeta.$$

Letting $\zeta_n = \phi(z_n)$, it follows that $\zeta_n \neq \zeta_m$, $n \neq m$, and $\zeta_n \rightarrow 0$. As before, $g(\zeta_{n+1}) = \zeta_n$, which implies $\lambda\zeta_{n+1} = \zeta_n$, and $\zeta_{n+1} = (1/\lambda)^n \zeta_1$, which converges to ∞ , a contradiction.

The case that $|\lambda| > 1$ does not happen by the assumption that $f^n(V)$ is uniformly bounded.

The remaining case is when $|\lambda| = 1$. By the same assumption, and by [8] there is a conformal map $\zeta = \phi(z)$ of a neighborhood of z_0 onto a neighborhood of 0 which conjugates f to $\lambda\zeta$, that is

$$g(\zeta) = \phi \circ f \circ \phi^{-1}(\zeta) = \lambda\zeta.$$

As before, this also leads to a contradiction.

The following fact is well-known for biholomorphic maps, and its proof for proper maps is included. This lemma is very important for incomplete Hartogs and quasi-circular domains. Denote by $A(r, R) = \{r < |z| < R\}$ an annulus in the complex plane.

LEMMA 3. *If $f(z) : A(r_1, R_1) \rightarrow A(r_2, R_2)$ is a proper holomorphic map with multiplicity m , then we have*

$$\frac{R_2}{r_2} = \left(\frac{R_1}{r_1}\right)^m.$$

Proof. It is well-known that f extends continuously to the boundary. Consider $\phi(z) = \ln|f(z)|$. Then $\phi(z)$ is harmonic, continuous up to the boundary. A simple topological argument shows that

$$\lim_{|z| \rightarrow r_1} \phi(z) = a$$

where $a = \ln r_2$ or $\ln R_2$, and

$$\lim_{|z| \rightarrow R} \phi(z) = \{\ln r_2, \ln R_2\} \setminus \{a\}.$$

Suppose f maps $|z| = r_1$ to $|z| = r_2$. Then we have $\phi(z) = \ln r_2$ when $|z| = r_1$; $\phi(z) = \ln R_2$ when $|z| = R_1$. Therefore ϕ solves the Dirichlet problem. However $c(\ln |z| - \ln r_1) + \ln r_2$ also solves the same Dirichlet problem where

$$c = \frac{\ln \frac{R_2}{r_2}}{\ln \frac{R_1}{r_1}}.$$

By the uniqueness, it follows that $c(\ln |z| - \ln r_1) + \ln r_2 = \ln |f(z)|$, which gives $|f(z)| = |z|^c$. Since f is a proper holomorphic map with multiplicity m , we have $c = m$, which proves the lemma.

Suppose that f maps $|z| = r_1$ to $|z| = R_2$. We only need to consider $1/f$. The same proof then applies.

LEMMA 4. *Let $F : \Omega \rightarrow D$ be a proper holomorphic map between smoothly bounded pseudoconvex domains of finite type in \mathbf{C}^n . Suppose that E is a totally real manifold of dimension n in the boundary of Ω on which τ is constant. Then we have $\overline{V}_F \cap E = \emptyset$.*

Proof. If not so, we let $p \in \overline{V}_F \cap E$. It follows that $\tau(F(p)) < \tau(p)$, and choosing q in E not in \overline{V}_F near p (we can do so, since $\partial \overline{V}_F$ is smooth at most points [12]), we have $\tau(F(p)) < \tau(p) = \tau(q) = \tau(F(q))$, which contradicts the fact that τ is upper semicontinuous.

§3. The case of Hartogs

In this section, we shall prove Theorem 1 in case of Hartogs in two cases. When the Hartogs domain is complete, we shall make use of the base domain; when the domain is not complete, we shall exploit the incompleteness.

Case 1. Complete Hartogs domains

A domain Ω is said to be a complete Hartogs domain if whenever $(z, w) \in \Omega$, $(z, \lambda w) \in \Omega$ for $|\lambda| < 1$. The base of a complete Hartogs domain Ω is defined as

$$E = \Omega \cap \{w = 0\}.$$

LEMMA 5. *Let Ω be a smoothly bounded pseudoconvex complete Hartogs domain of finite type in \mathbf{C}^2 . Then there exists a neighborhood U of $\partial\Omega \cap \overline{E}$ so that $\partial\Omega \cap U$ is strictly pseudoconvex at every point except possibly at points of $\partial\Omega \cap \overline{E}$.*

Proof. Let $\rho(z, w)$ be a defining function of Ω . Let $(z_0, 0) \in \partial E$. We see that $\nabla\rho(z_0, 0) \neq 0$, and the complex z plane is transversal to $\partial\Omega$ at $(z_0, 0)$. Then we may assume $\rho_z(z_0, 0) \neq 0$. Define

$$r(z, w) = \int_0^{2\pi} \rho(z, e^{i\theta}w) d\theta.$$

Then r is a defining function of Ω , but $r(z, w) = r(z, |w|)$. By the linear change of coordinates, $(z, w) \rightarrow (z - z_0, w)$, we have, $r_z(0, 0) \neq 0$,

$$r(z, w) = \Re z + \phi(\Im z, |w|).$$

Since Ω is of finite type, we have

$$\phi(\Im z, |w|) = c|w|^{2k} + o(|\Im z| + |\Im zw| + |w|^{2k}).$$

It is easy to see that $\partial\Omega$ is strictly pseudoconvex when $w \neq 0$ and small.

LEMMA 6. *Let $F : \Omega \rightarrow D$ be a proper holomorphic map between pseudoconvex domains of finite type in \mathbf{C}^2 . If Ω is complete Hartogs, then there exist $z_1, z_2, \dots, z_n \in E$ such that the branch locus of F satisfies*

$$V_F \subset \bigcup_{i=1}^n \{z = z_i\} \cup \{w = 0\}.$$

Proof. First we remark that the \mathbf{T} -action on Ω is transversal whenever $(z, w) \notin \partial\Omega \cap \overline{E}$. Let V be an irreducible component of V_F , and let $p \in \partial\overline{V}$, $p = (z_0, w_0)$. If $p \notin \partial\Omega \cap \overline{E}$, By Theorem 2, $V = \{z = z_0\}$. If $p \in \partial\Omega \cap \overline{E}$, it follows $V = \{w = 0\} \cap \Omega$. Now it suffices to prove that V_F has finitely many irreducible components. If not so, we have

$$V_F \subset \bigcup_{i=1}^{\infty} \{z = z_i\} \cup \{w = 0\}.$$

Then z_j must converges to the boundary of E . By Lemma 5, there exists an neighborhood U of $\partial\Omega \cap \overline{E}$ such that $\partial\Omega$ is strictly pseudoconvex except possibly at $\partial\Omega \cap \overline{E}$; therefore $\partial\Omega \cap \{z = z_i\} \cap U$ must be strictly pseudoconvex points for i large, which contradicts the fact that branch locus never hits strictly pseudoconvex points.

Now we are ready to prove Theorem 1. Assume $F = (f, g)$. By Lemma 1, we have $L_n \subset V_{F^n}$. By Lemma 6, we have $L_n = \{z = z_n\} \cap \Omega$, for some $z_n \in E$. We claim first the sequence $\{z_n\}$ must stay away from the boundary of E since by Lemma 2, $\partial\Omega \cap U \setminus \partial\bar{E}$ is strictly pseudoconvex and the branch locus does not hit strictly pseudoconvex points. Without loss of generality, we may assume that $\{z_n\}$ converges to $z_0 \in E$. Since $L_{n+1} \subset F^{-1}(L_n)$, $F(L_{n+1}) = L_n$, which implies $f(z_{n+1}, w) = z_n$. It follows that $\frac{\partial f}{\partial w}(z, w)$, being analytic in z in E has infinitely many zeros $\{z_n\}$ accumulating at an interior point z_0 in E , therefore is identically zero. This implies that $f(z, w)$ is independent of w . Furthermore, we have $F = (f(z), g(z, w))$, $f(z_0) = z_0$ and $g(z_0, w) : L_0 \rightarrow L_0$ is a proper self map, where of course $L_0 = \{z = z_0\} \cap \Omega$. It is easy to see that f maps E into E , and f fixes z_0 such that $f(z_{n+1}) = z_n$ with $z_n \rightarrow z_0$. This is impossible by Lemma 2 and completes the proof of Theorem 1 in the case of complete Hartogs domains.

Case 2. Incomplete Hartogs domains

LEMMA 7. *Let $F : \Omega \rightarrow D$ be a proper holomorphic map between pseudoconvex domains of finite type in \mathbf{C}^2 . If Ω is incomplete Hartogs, then there exist $z_1, z_2, \dots, z_n, \dots$ such that the branch locus of F satisfies*

$$V_F \subset \bigcup_{i=1}^{\infty} \{z = z_i\} \cap \Omega.$$

Proof. In fact, let V be an irreducible component of V_F . If there exists a point $p = (z_0, w_0)$ in $\bar{V} \cap \partial\Omega$ so that the \mathbf{T} -action is transversal at p , then by Theorem 2, $V = \{z = z_0\} \cap \Omega$. Now we assume that the \mathbf{T} -action is transversal at no points of $\bar{V} \cap \partial\Omega$. Choose a smooth arc Γ in $\bar{V} \cap \partial\Omega$ so that $\tau_{\partial\Omega}(p)$ is constant for $p \in \Gamma$. This is possible since Ω is of finite type. Considering $E = \Gamma \times \mathbf{T}$, this is a totally real manifold of dimension 2 for which $\tau_{\partial\Omega}$ is constant. By Lemma 4, we have $\bar{V} \cap E = \emptyset$, which is obviously a contradiction.

Now we are ready to prove Theorem 1 when Ω is incomplete. We first claim that $w \neq 0$ on $\bar{\Omega}$. Indeed, this follows from pseudoconvexity and the continuity principle.

Assume $V_F \neq \emptyset$. By Lemma 1, there exists $L_n \subset V_{F^n}$ such that $F(L_{n+1}) = L_n$, and by Lemma 7, there exists z_n such that

$$L_n = \{z = z_n\} \cap \Omega.$$

By the claim, there exist $R_n, r_n, R_n > r_n > 0$, such that

$$L_n = \{(z_n, w) : r_n < |w| < R_n\},$$

i.e., L_n is an annulus in the w plane. Since Ω is bounded, we may assume that $R_n \rightarrow R, r_n \rightarrow r$. Since $w \neq 0$ on $\overline{\Omega}$, we have $r > 0$. On the other hand, $g(z_{n+1}, w)$ is a proper holomorphic map from $r_{n+1} < |w| < R_{n+1}$ to $r_n < |w| < R_n$; and without loss of generality, we may assume that $g(z_n, w)$ has the same multiplicity m for any n , by Lemma 3, we have

$$\frac{R_{n+1}}{r_{n+1}} = \left(\frac{R_n}{r_n}\right)^m,$$

which implies that

$$\frac{R_{n+1}}{r_{n+1}} = \left(\frac{R_1}{r_1}\right)^{nm}.$$

Letting $n \rightarrow \infty$, we see the left hand side goes to R/r , while the right hand side goes to infinity, arriving at a contradiction, and the proof of the theorem is complete.

§4. Basins of attraction of quasi-homogeneous maps

If $F : \mathbf{C}^2 \rightarrow \mathbf{C}^2$ is a holomorphic map such that $F(0) = 0$ and the eigenvalues of $F'(0)$ are smaller than one in modulus, we may associate with the basin of attraction at the origin, Ω_F , which is defined by

$$\Omega_F = \{z \in \mathbf{C}^2 \mid \lim_{k \rightarrow \infty} F^k(z) = 0\}.$$

Following [7], if $(p, q) = 1$, we say that a holomorphic map $P : \mathbf{C}^2 \rightarrow \mathbf{C}^2$ is quasi-homogeneous map of type (p, q) if the components (P_1, P_2) satisfy $P_1(t^p z, t^q w) = t^{np} P_1(z, w)$ and $P_2(t^p z, t^q w) = t^{nq} P_2(z, w)$. It is easy to see that such a map must be a polynomial map.

Now let P be a quasi-homogeneous polynomial with $P^{-1}(0) = 0$. Let Ω_P be the basin of attraction of P at the origin. One can see that Ω_P is a quasi-circular domain of type (p, q) , bounded and pseudoconvex. Moreover, P induces a non-injective proper holomorphic self-map of Ω_P . One also has that if $z \in \Omega_P$, then $\lim_{n \rightarrow \infty} P^n(z) = 0$; if $z \notin \overline{\Omega_P}$, then $\lim_{n \rightarrow \infty} \|P^n(z)\| = \infty$.

As proved in [7], we notice that the dynamics of quasi-homogeneous polynomial maps can be related to that of homogeneous ones. Let $\Phi(z, w) =$

(z^q, w^p) . There exists a unique homogeneous polynomial $Q(z, w)$ of degree n such that

$$\Phi \circ P = Q \circ \Phi.$$

Also Φ is a proper map from Ω_P to Ω_Q , where Ω_Q is the basin of attraction of Q . It also follows that

$$\Phi \circ P^n = Q^n \circ \Phi.$$

§5. Non-Hartogs case

In this section, we assume that Ω is quasi-circular of type (p, q) with $p, q \geq 1$. We also assume, without loss of generality, that $(p, q) = 1$. First we use Theorem 2 to study the structure of the branching locus of proper holomorphic maps from quasi-circular domains.

LEMMA 8. *Let $F : \Omega \rightarrow D$ be a proper holomorphic map between pseudoconvex domains of finite type in \mathbf{C}^2 . If Ω is quasi-circular of type (p, q) , then there exist p_1, p_2, \dots, p_n such that the branch locus of F satisfies*

$$V_F = \bigcup_{i=1}^n \{(\lambda^{p_i} z_i, \lambda^{q_i} w_i) : \lambda \in \mathbf{C}\} \cap \Omega,$$

where $p_i = (z_i, w_i) \in \partial\Omega$.

Proof. First the transversality of T -action can be verified as in [9, Lemma 4.1]. It is easy to see by T -action that any irreducible component of V_F is given by $\{(\lambda^p z, \lambda^q w) : \lambda \in \mathbf{C}\} \cap \Omega$ for $(z, w) \in \partial\Omega$, which contains the origin since $p, q \geq 1$. We conclude that V_F cannot have infinitely many components by the compactness of V_F near the origin. This proves the lemma.

A quasi-circular domain of type (p, q) is said to be complete provided that whenever $(z, w) \in \Omega$, $(\lambda^p z, \lambda^q w) \in \Omega$ for $|\lambda| \leq 1$. We will prove the theorem in two cases as done in the Hartogs case.

Case 1. Complete quasi-circular domain of type (p, q)

We shall first prove that if the self-map is branched, then it must be a (p, q) quasi-homogeneous polynomial map.

LEMMA 9. *Let $F : \Omega \rightarrow \Omega$ be a proper holomorphic self-map of Ω . If $V_F \neq \emptyset$, then $F = (f(z, w), g(z, w))$ is given by*

$$f(z, w) = \sum_{p\alpha+q\beta=pk} A_{\alpha\beta} z^\alpha w^\beta$$

$$g(z, w) = \sum_{p\alpha+q\beta=qk} B_{\alpha\beta} z^\alpha w^\beta,$$

for some positive integer k .

Proof. By Lemma 1, there exists $L_n \subset V_{F^n}$ such that $F(L_{n+1}) = L_n$, and F is proper from $L_{n+1} \rightarrow L_n$ where L_n are distinct. By Lemma 8, there exists $(z_n, w_n) \in \partial\Omega$ such that $L_n = \{(\lambda^p z_n, \lambda^q w_n) : \lambda \in \mathbf{C}\} \cap \Omega$. It is easy to see $F(0) = 0$. Since $\{L_n\}$ are different we may assume that $z_n \neq 0$ for $n = 1, 2, 3, \dots$. Therefore

$$L_n = \{(z, w) : w^p = a_n z^q\} \cap \Omega,$$

where $a_n = w_n^p / z_n^q$. Since $F : L_{n+1} \rightarrow L_n$ is proper, it follows that if $|\lambda| = 1$, then

$$|f(\lambda^p z_{n+1}, \lambda^q w_{n+1})| = |z_n|$$

$$|g(\lambda^p z_{n+1}, \lambda^q w_{n+1})| = |w_n|.$$

Therefore, we have the function $\phi_n(\lambda) = f(\lambda^p z_{n+1}, \lambda^q w_{n+1})$ maps $|\lambda| < 1$ properly to $|\lambda| < |z_n|$, and the function $\psi_n(\lambda) = g(\lambda^p z_{n+1}, \lambda^q w_{n+1})$ maps $|\lambda| < 1$ properly to $|\lambda| < |w_n|$. It follows that $\phi_n(\lambda)$ is a finite Blaschke product. We claim actually that $\phi_n(\lambda) = c_n \lambda^N$ where c_n is a constant and N is independent of n .

In order to prove this, we first prove that $F|_{L_{n+1}}^{-1}(0) = 0$ for sufficiently large n . In fact if not, there are infinitely many n for which there exist $z_n, 0 \neq z_n \in L_{n+1}$ such that $F(z_n) = 0$. Since L_n are only in common at 0, we see z_n are different, and therefore $F^{-1}(0)$ has infinitely many points, which contradicts the properness of F .

Now we only have to prove $\phi_n^{-1}(0) = 0$ as well since ϕ_n is a finite Blaschke product. Indeed, if there is $c \neq 0, |c| < 1$ such that $\phi_n(c) = 0$. Then since $F : L_{n+1} \rightarrow L_n, F(c^p z_{n+1}, c^q w_{n+1}) \in L_n$, there exists λ_0 such that

$$F(c^p z_{n+1}, c^q w_{n+1}) = (\lambda_0^p z_n, \lambda_0^q w_n).$$

Since $F^{-1}(0) = 0$ then $\lambda_0 \neq 0$, and therefore $\phi_n(c) = f(c^p z_{n+1}, c^q w_{n+1}) = \lambda_0^p z_n \neq 0$, which is a contradiction (notice $z_n \neq 0$). Therefore $\phi_n(\lambda) = c\lambda^k$ for some k . where k is at most of the multiplicity of F . We can also assume that all ϕ_n have the same multiplicity, say N , which is at most the multiplicity of F . Therefore $\phi_n(\lambda) = c_n \lambda^N$.

Consider the Taylor series of $f(z, w)$ in a small neighborhood of $(0, 0)$

$$f(z, w) = \sum_{\alpha, \beta} A_{\alpha\beta} z^\alpha w^\beta.$$

Rewrite $f(z, w)$ in terms of weight (p, q) ,

$$f(z, w) = \sum_{m=0}^{\infty} f_m(z, w)$$

where

$$f_m(z, w) = \sum_{p\alpha+q\beta=m} A_{\alpha\beta} z^\alpha w^\beta.$$

We have

$$\phi_n(\lambda) = \sum_{m=0}^{\infty} \lambda^m f_m(z_{n+1}, w_{n+1})$$

From $\phi_n(\lambda) = c_n \lambda^N$, for all n , it follows that $f_j(z_{n+1}, w_{n+1})$ when $j \neq N$ and for all n . However, a simple computation shows, invoking $a_{n+1} = w_{n+1}^p / z_{n+1}^q$

$$\begin{aligned} f_j(z_{n+1}, w_{n+1}) &= \sum_{p\alpha+q\beta=j} A_{\alpha\beta} z_{n+1}^\alpha w_{n+1}^\beta \\ &= z_{n+1}^{j/p} \sum_{p\alpha+q\beta=j} A_{\alpha\beta} a_{n+1}^{\beta/p}. \end{aligned}$$

Since a_n are different, we conclude that $a_n^{1/p}$ are zeros of the polynomial:

$$\sum_{p\alpha+q\beta=j} A_{\alpha\beta} t^\beta = 0,$$

where $j \neq N$. A polynomial cannot have infinitely many zero unless it is zero. Therefore, we conclude that

$$A_{\alpha\beta} = 0$$

whenever $p\alpha + q\beta \neq N$. Therefore we have that f is a quasi-homogeneous polynomial of weight (p, q) of degree N , i.e.,

$$f(z, w) = \sum_{p\alpha+q\beta=N} A_{\alpha\beta} z^\alpha w^\beta.$$

Similarly, using $\psi_n(\lambda)$ we can prove that $g(z, w)$ is a quasi-homogeneous polynomial of weight (p, q) . In fact, we notice

$$g^p(\lambda^p z_{n+1}, \lambda^q w_{n+1}) = a_n f^q(\lambda^p z_{n+1}, \lambda^q w_{n+1}),$$

which implies that $\psi_n(\lambda) = c_n \lambda^l$ for some l and that the degree of g is Nq/p . Since $(p, q) = 1$, we have $N = kp$, and the degree of g is qk . We set

$$\begin{aligned} f(z, w) &= \sum_{p\alpha+q\beta=kp} A_{\alpha\beta} z^\alpha w^\beta \\ g(z, w) &= \sum_{p\alpha+q\beta=kq} B_{\alpha\beta} z^\alpha w^\beta. \end{aligned}$$

That is to say F is a quasi-homogeneous polynomial map of weight (p, q) of degree k , so the lemma is proved.

To complete the proof of the theorem, we have to use some facts from two variables complex dynamics. Let Ω_F be the basin of attraction of F . Then there exists a unique homogeneous polynomial map Q given in the last section such that

$$\Phi \circ F = Q \circ \Phi,$$

where $\Phi = (z^q, w^p)$ as before. We first claim that $\Omega_F = \Omega$. Indeed, assume that F^n converges to G on Ω . For every $\lambda \in \mathbf{C}$, $|\lambda| < 1$, we have $F^n(\lambda^p z, \lambda^q w) \rightarrow G(\lambda^p z, \lambda^q w)$ as $n \rightarrow \infty$. However, F^n is a quasi-homogeneous polynomial map of weight (p, q) of degree k^n , therefore

$$F^n(\lambda^p z, \lambda^q w) = \lambda^{k^n} F^n(z, w)$$

which converges to zero. Hence $G = 0$ and $\Omega \subset \Omega_F$. But F is proper and $F(\partial\Omega) = \partial\Omega$. Hence Ω_F is contained in Ω .

Now we notice that $\Phi(\Omega) = \Omega_Q$. Therefore the boundary of Ω_Q is smooth except possibly when $z = 0$ or $w = 0$. Therefore Ω_Q contains strictly pseudoconvex points.

We denote by $\pi : \mathbf{C}^2 \setminus \{0\} \rightarrow \mathbf{C}P$ the canonical projection. Since Q is nondegenerate, it takes lines to lines in \mathbf{C}^2 and naturally induces a rational

mapping $\varphi : \mathbf{CP} \rightarrow \mathbf{CP}$ on the projective space. We claim that its Julia set J_φ does not coincide with \mathbf{CP} . For the proof we apply an argument of [6]. Suppose by contradiction it does. It is known (see [8, pp. 56–58]) that in this case for every point $a \in J_\varphi$ there exists a neighborhood U and a positive integer n such that $\cup_{k=1}^n \varphi^k(U)$ covers \mathbf{CP} . Take a such that $\pi^{-1}(a)$ contains a strictly pseudoconvex point p in $\partial\Omega_Q$. Then there exists a neighborhood W of p in \mathbf{C}^2 such that $\cup_{k=1}^n Q^k(W)$ covers $\partial\Omega_Q$. Since Q takes any strictly pseudoconvex point to a strictly pseudoconvex one, we get that Ω is strictly pseudoconvex and by [13] V_Q is empty: a contradiction. Thus, J_ϕ is different from \mathbf{CP} . But by the classical results J_ϕ is a closed subset of \mathbf{CP} with empty interior. Therefore $\partial\Omega_Q \cap \pi^{-1}(J_\phi)$ is a nonempty open subset of $\partial\Omega_Q$ which in view of [11], Proposition 7.1, is foliated by Riemann surfaces; this is impossible since Ω_Q is a proper image of the finite type domain Ω_Q . This completes the proof of the theorem.

Case 2. Incomplete quasi-circular domain of type (p, q) .

Now we assume that Ω is not complete. We first observe that $(0, 0) \notin \overline{\Omega}$. Indeed, this follows from using continuity principle and pseudoconvexity.

Given a point $p = (z, w)$ in $\partial\Omega$, we consider the complex curve

$$L_p = \{(\lambda^p z, \lambda^q w) : \lambda \in \mathbf{C}\}.$$

Now we go back to the proof of the theorem. By Lemma 1, we have $L_n \subset V_{F^n}$ and $F(L_{n+1}) = L_n$, and F is proper from L_{n+1} to L_n with multiplicity m for all n . By the fact proved above $L_n = L_{p_n} \cap \Omega$ where p_n is a boundary point. Since Ω is incomplete, there exist $r_n, R_n, R_n > r_n > 0$ such that

$$L_n = \{(\lambda^p z_n, \lambda^q w_n) : r_n < |\lambda| < R_n\}.$$

We assume $r_n \rightarrow r, R_n \rightarrow R$. Since $\overline{\Omega}$ does not contain $(0, 0)$, we have $r > 0$. Consider the map $\phi \circ F \circ \pi(\lambda) : \{r_{n+1} < |\lambda| < R_{n+1}\} \rightarrow \{r_n < |\lambda| < R_n\}$ where $\phi : \lambda \rightarrow (\lambda^p z_{n+1}, \lambda^q w_{n+1})$, and π is the projection from $\mathbf{C}^2 \rightarrow \mathbf{C}$. Then it is a proper map. By Lemma 3, we have

$$\frac{R_{n+1}}{r_{n+1}} = \left(\frac{R_n}{r_n}\right)^m,$$

which implies that

$$\frac{R_{n+1}}{r_{n+1}} = \left(\frac{R_1}{r_1}\right)^{nm}.$$

Letting $n \rightarrow \infty$, we see the left hand side goes to R/r , while the right hand side goes to infinity, arriving at a contradiction, and the proof of the theorem is complete.

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