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Nagoya Math. J.
Vol. 163 (2001), 87–106

A BOUND ON CERTAIN LOCAL COHOMOLOGY MODULES AND APPLICATION TO AMPLE DIVISORS

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Abstract. We consider a positively graded noetherian domain $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ for which R_0 is essentially of finite type over a perfect field K of positive characteristic and we assume that the generic fibre of the natural morphism $\pi : Y = \mathrm{Proj}(R) \to Y_0 = \mathrm{Spec}(R_0)$ is geometrically connected, geometrically normal and of dimension > 1. Then we give bounds on the "ranks" of the n-th homogeneous part $H^2_{R_+}(R)_n$ of the second local cohomology module of R with respect to $R_+ := \bigoplus_{m>0} R_m$ for $n < 0$. If Y is in addition normal, we shall see that the R_0-modules $H^2_{R_+}(R)_n$ are torsion-free for all $n < 0$ and in this case our bounds on the ranks furnish a vanishing result. From these results we get bounds on the first cohomology of ample invertible sheaves in positive characteristic.

§1. Introduction

Let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ a positively graded noetherian domain such that R_0 is essentially of finity type over a field K of positive characteristic. Assume that the generic fibre of the natural morphism $Y := \mathrm{Proj}(R) \xrightarrow{\pi} Y_0 := \mathrm{Spec}(R_0)$ is geometrically connected, geometrically normal and of dimension > 1. For all $n \in \mathbb{Z}$ let $H^2_{R_+}(R)_n$ denote the n-th homogeneous part of the second local cohomology module $H^2_{R^+}(R)$ of R with respect to the irrelevant ideal $R_+ := \bigoplus_{m>0} R_m$ of R.

Our aim is to bound "the size" of the R_0-module $H^2_{R_+}(R)_n$ for negative values of n. More precisely, if L_0 is the quotient field of R_0, we give bounds on the numbers.

$$h^2_{R_+}(R)_n := \dim_{L_0}\bigl(L_0 \otimes_{R_0} H^2_{R_+}(R)_n\bigr)$$

in the range $n < 0$ (cf. (3.8)). In addition, we shall see that the R_0-modules $H^2_{R_+}(R)_n$ are torsion-free for all $n < 0$, whenever Y is normal. So, in this

Received April 2, 1999.
Revised June 8, 2000.
2000 Mathematics Subject Classification: 13D03, 14B15, 14F15, 14F05.
[1]Author supported by Swiss National Foundation; Grant Nr. 21-36526.92

case, we have bounded the rank of the R_0-modules $H^2_{R_+}(R)_n$ in the range $n < 0$.

We apply the previous bounding result in the following geometric context: Let X be an integral projective scheme over an affine integral scheme X_0 which is essentially of finite type over a perfect field K. Assume that the generic fibre of X over X_0 is geometrically connected, geometrically normal and of dimension > 1. Let $\mathcal{L}$ be an ample invertible sheaf over X. Under these hypotheses we bound the first Serre-cohomology modules $H^1(X, \mathcal{L}^{\otimes n})$ in the range $n < 0$, (s. (4.5)). In particular we consider the case where X is a geometrically connected and geometrically normal projective scheme over a perfect field K of positive characteristic (cf. (4.6), (4.8)). In the special situation in which K is algebraically closed we recover a result of [A], (s. (4.7)).

Finally, we apply our bounds to projective varieties which have only finitely many non-normal points, (s. (5.2)). In the special case of a surface X with only finitely many non-normal points we see that the spaces $H^1(X, \mathcal{L}^{\otimes n})$ have the same dimension for all $n < 0$ if the so called sectional genus $\sigma_{\mathcal{L}}(X)$ of X with respect to $\mathcal{L}$ is smaller than the dimension of the complete linear system $|\mathcal{L}|$, (s. (5.5)). In particular this shows the vanishing of $H^1\big(X, \mathcal{O}_X(-1)\big)$ if $X \subseteq \mathbb{P}^r$ is a nondegenerate normal projective surface with sectional genus $\sigma(X) < r$. In many cases this latter vanishing result allows to avoid the use of the vanishing theorems of Kodaira [K] or Mumford [Mu] and thus for example gives rise to a characteristic-free approach to projective surfaces of low degree: An application to sectionally rational surfaces immediately shows that the main results of [B-V] remain valid in arbitrary characteristic. Further use of this idea is made in [B$_2$], [B$_3$].

Let us recall, that in general, for an ample invertible sheaf $\mathcal{L}$ on a normal projective surface X, $H^1(X, \mathcal{L}^{\otimes -1})$ need not vanish (s. [Mu]), even if X is smooth (s. [Ra]). Observe also that in [L-R] an example of a smooth projective variety $X \subseteq \mathbb{P}^r$ of dimension 6 is constructed for which $H^1\big(X, \mathcal{O}_X(-1)\big) \neq 0$. On the other hand it seems that even the powerful vanishing results found in [D-I] and [E-V] and the new techniques developed in [S] do not give the vanishing of $H^1\big(X, \mathcal{O}_X(-1)\big)$ for smooth projective varieties of dimension ≤ 5 in arbitrary characteristic.

As for unexplained terminology we refer to [H$_2$] (concerning algebraic geometry) and to [M] (concerning commutative algebra).

We thank the referee for his valuable suggestions and hints.

§2. The Frobenius sequence

Let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ be a positively graded noetherian domain such that R_0 is essentially of finite type over a given perfect ground field K of characteristic $p > 0$. Choose some positive integer $e \in \mathbb{N}$ and let $R^{(p^e)} := \bigoplus_{n\in\mathbb{N}_0} R_{np^e} \subseteq R$ be the p^e-th *Veronesean subring* of R. Moreover, consider the e-th *Frobenius homomorphism*

$$F_e : R \longrightarrow R^{(p^e)};\ (x \longmapsto x^{p^e},\ \forall x \in R). \tag{2.1}$$

Remark 2.2. Observe that $F_e : R \to R^{(p^e)}$ is an injective (graded) homomorphism between positively graded noetherian rings. By our hypotheses R is essentially of finite type over the perfect field K, so that the homomorphism F_e is in addition finite.

For a graded R-module $M = \bigoplus_{n\in\mathbb{Z}} M_n$, let $M^{(p^e)} := \bigoplus_{n\in\mathbb{Z}} M_{np^e}$ denote the p^e-th *Veronesean transform* of M, considered as an $R^{(p^e)}$-module in the obvious way. Moreover, consider the graded R-module

$$M^{[e]} := M^{(p^e)}\restriction_{F_e}, \tag{2.3}$$

where $\cdot\restriction_{F_e}$ denotes scalar restriction to R by means of the homomorphism $F_e : R \to R^{(p^e)}$.

Remark 2.4. A) Observe that the assignment $M \mapsto M^{[e]}$ extends naturally to graded homomorphisms and thus gives rise to covariant exact functor $\cdot^{[e]} : {}^*\mathcal{C}(R) \to {}^*\mathcal{C}(R)$ from the category ${}^*\mathcal{C}(R)$ of graded R-modules to itself. Moreover, by (2.2) this functor preserves the property of being a finitely generated (graded) module.

B) For each $n \in \mathbb{N}$ let $\cdot_n : {}^*\mathcal{C}(R) \to \mathcal{C}(R_0)$ denote the covariant exact functor from the category of graded R-modules to the category $\mathcal{C}(R_0)$ of R_0-modules, which is given by taking n-th homogeneous parts. Then, for each graded R-module M and for each $n \in \mathbb{Z}$ we have $(M^{[e]})_n = (M_{np^e})\restriction_{F_{e,0}}$, where $\cdot\restriction_{F_{e,0}}$ denotes scalar restriction to R_0 by means of the restricted Frobenius homorphism in degree 0, that is $F_{e,0} := F_e\restriction_{R_0} : R_0 \to (R^{(p^e)})_0 = R_0$, $(x \mapsto x^{p^e})$.

Definition and Remark 2.5. Keep the previous hypotheses and notation. Observe that $R^{[e]}$ carries a natural ring structure inherited from $R^{(p^e)}$

and that the homomorphism of rings $F_e : R \to R^{[e]}$ is a graded homomorphism between finitely generated and graded R-modules.

A) Let $C_{[e]}$ be the cokernel of the homomorphism $F_e : R \to R^{[e]}$, a finitely generated and graded R-module. The induced graded short exact sequence

$$\text{(i)} \qquad 0 \longrightarrow R \xrightarrow{F_e} R^{[e]} \longrightarrow C_{[e]} \longrightarrow 0$$

is called the e-th *arithmetic Frobenius sequence of* R.

B) Keep the above notations and set in addition $Y := \mathrm{Proj}(R)$. Let $\mathcal{K}_{[e]} := (C_{[e]})^{\sim}$ be the coherent sheaf of $\mathcal{O}_Y$-modules induced by $C_{[e]}$ and let $\mathcal{O}_Y^{[e]}$ denote the coherent sheaf of $\mathcal{O}_Y$-modules induced by $R^{[e]}$. Observe that $\mathcal{O}_Y^{[e]}$ carries a natural structure of sheaf of $\mathcal{O}_Y$-algebras which results from the fact that $F_e : R \to R^{[e]}$ is a homomorphism of rings. Now, the induced short exact sequence of coherent sheaves of $\mathcal{O}_Y$-modules

$$\text{(ii)} \qquad 0 \longrightarrow \mathcal{O}_Y \xrightarrow{\tilde{F}_e} \mathcal{O}_Y^{[e]} \longrightarrow \mathcal{K}_{[e]} \longrightarrow 0$$

is called the e-th (*geometric*) *Frobenius sequence of* R. It incorporates the "iterated Frobenius morphism" $\tilde{F}_e : \mathcal{O}_Y \to \mathcal{O}_Y$ (cf. [H_1, §6, pg. 128]) into a short exact sequence, which shall play a crucial rôle in our arguments. For $y \in Y$, the corresponding short exact sequence of finitely generated $\mathcal{O}_{Y,y}$-modules

$$\text{(iii)} \qquad 0 \longrightarrow \mathcal{O}_{Y,y} \xrightarrow{(\tilde{F}_e)_y} (\mathcal{O}_Y^{[e]})_y \longrightarrow (\mathcal{K}_{[e]})_y \longrightarrow 0$$

ist called the e-th (*local*) *Frobenius sequence of* R *at* y.

The splitting of the Frobenius sequence is known to furnish vanishing statements of Kodaira type in positive characteristic, (s. [Me-Ram]). Clearly, in our situation we cannot expect the Frobenius sequence to be split. As a certain substitute for the splitting in question we now shall prove that the cokernel $\mathcal{K}_{[e]}$ of $\tilde{F}_e$ is torsion-free, provided that Y is normal.

LEMMA 2.6. *Let the notations and hypotheses be as in* (2.5) *and let* $y \in Y = \mathrm{Proj}(R)$ *be a regular point. Then, the* e*-th local Frobenius sequence of* R *at* y

$$0 \longrightarrow \mathcal{O}_{Y,y} \xrightarrow{(\tilde{F}_e)_y} (\mathcal{O}_Y^{[e]})_y \longrightarrow (\mathcal{K}_{[e]})_y \longrightarrow 0$$

splits. Moreover if $L = \kappa(Y)$ *is the field of rational functions of* Y, *the* $\mathcal{O}_{Y,y}$-*module* $(\mathcal{O}_Y^{[e]})_y$ *is free of rank* $r := [L : L^p]^e$ $(< \infty)$ *and the* $\mathcal{O}_{Y,y}$-*module* $(\mathcal{K}_{[e]})_y$ *is free of rank* $r - 1$.

Proof. Keep in mind that $(\mathcal{O}_Y^{[e]})_y$ carries a natural ring structure and that we may naturally identify this ring with $\mathcal{O}_{Y,y}$. Under this identification, $(\tilde{F}_e)_y : \mathcal{O}_{Y,y} \to (\mathcal{O}^{[e]})_y$ corresponds to the e-th Frobenius map $f_e : \mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}$, $(a \mapsto a^{p^e})$.

By our assumptions, $\mathcal{O}_{Y,y}$ is a regular local ring and so the flatness-criterion of Kunz [Ku] shows that the finite homomorphism $f_e : \mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}$ is flat and hence splits. Therefore the e-th local Frobenius sequence of R at y splits and $(\mathcal{K}_{[e]})_y$ becomes a free $\mathcal{O}_{Y,y}$-module of rank $r - 1$ where $r = [L : L^p]^e$. ∎

Proposition 2.7. *Let the notations and hypotheses be as in* (2.5) *and let* $y \in Y = \mathrm{Proj}(R)$ *be a normal point. Then, the stalk* $(\mathcal{K}_{[e]})_y$ *is a torsion-free* $\mathcal{O}_{Y,y}$-*module.*

Proof. By our hypothesis, $\mathcal{O}_{Y,y}$ is a noetherian normal local ring. Let $\mathfrak{p} \in \mathrm{Ass}_{\mathcal{O}_{Y,y}}((\mathcal{K}_{[e]})_y)$. We have to show that $\mathrm{height}(\mathfrak{p}) = 0$. The localized Frobenius sequence $0 \to (\mathcal{O}_{Y,y})_{\mathfrak{p}} \to ((\mathcal{O}_Y^{[e]})_y)_{\mathfrak{p}} \to ((\mathcal{K}_{[e]})_y)_{\mathfrak{p}} \to 0$ shows that $\mathrm{depth}((\mathcal{O}_Y^{[e]})_y)_{\mathfrak{p}} = 0$ or that $\mathrm{depth}(\mathcal{O}_{Y,y})_{\mathfrak{p}} \leq 1$. As $R^{[e]}$ is a torsion free R-module, $(\mathcal{O}_Y^{[e]})_y$ is a torsion free module over $\mathcal{O}_{Y,y}$. As $\mathcal{O}_{Y,y}$ is a normal noetherian ring, it satisfies the second Serre condition S_2. Altogether we thus obtain that $\mathrm{height}(\mathfrak{p}) \leq 1$. Now, let $z \in Y$ be the point which corresponds to $\mathfrak{p}$, so that $y \in \overline{\{z\}}$ and $\mathcal{O}_{Y,z} = (\mathcal{O}_{Y,y})_{\mathfrak{p}}$. As $\mathcal{O}_{Y,y}$ is normal, $\mathrm{height}(\mathfrak{p}) \leq 1$ implies that $\mathcal{O}_{Y,z}$ is a regular ring and therefore we get by (2.6) that $(\mathcal{K}_{[e]})_z \cong ((\mathcal{K}_{[e]})_y)_{\mathfrak{p}}$ is a free module over $(\mathcal{O}_{Y,y})_{\mathfrak{p}}$. As $\mathfrak{p} \in \mathrm{Ass}((\mathcal{K}_{[e]})_y)$ we thus get $\mathfrak{p} \in \mathrm{Ass}(\mathcal{O}_{Y,y})$ and hence our claim. ∎

Corollary 2.8. *Let the notations and hypotheses be as in* (2.5) *and assume that* $Y = \mathrm{Proj}(R)$ *is normal. Then* $\mathcal{K}_{[e]}$ *is a torsion-free sheaf of* $\mathcal{O}_Y$-*modules.*

§3. Structure of the second local cohomology module

Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be as in Section 2 and let $R_+ := \bigoplus_{n \in \mathbb{N}} R_n$ be the irrelevant ideal of R. For an R-module M and for $i \in \mathbb{N}_0$ let $D_{R_+}(M) :=$

$\varinjlim_t \mathrm{Hom}_R((R_+)^t, M)$ denote the R_+-transform of M and let $H^i_{R_+}(M)$ denote the i-th local cohomology module of M with respect to R_+. Keep in mind that the R_+-transform and local cohomology with respect to R_+ give rise respectively to a left exact covariant functor $D_{R_+} : {}^*\mathcal{C}(R) \to {}^*\mathcal{C}(R)$ from the category of graded R-modules to itself or a universal δ-functor $(H^i_{R_+})_{i\in\mathbb{N}_0} : {}^*\mathcal{C}(R) \to {}^*\mathcal{C}(R)$.

Remark 3.1. A) Let M be a graded R-module. Remember that there is a natural exact sequence of graded R-modules

$$0 \longrightarrow H^0_{R_+}(M) \longrightarrow M \xrightarrow{\eta_{R_+}} D_{R_+}(M) \longrightarrow H^1_{R_+}(M) \longrightarrow 0$$

and that there are natural isomorphisms of graded R-modules $\mathcal{R}^i D_{R_+}(M) \cong H^{i+1}_{R_+}(M)$ for all $i \in \mathbb{N}$, where $\mathcal{R}^i$ denotes the i-th right derivation of covariant functors in the category $\mathcal{C}(R)$ or in the category ${}^*\mathcal{C}(R)$ (s. [B-S, Chap. 13]).

B) Let $e \in \mathbb{N}$. Then, in the notations of Section 2 we have $\sqrt{F_e(R_+)R^{(p^e)}} = (R^{(p^e)})_+ = (R_+)^{(p^e)}$. Now, let M be a graded R-module and keep in mind that taking the ideal transform or local cohomology of M with respect to graded ideals commutes with graded scalar restriction (s. [B-S, 13.1.3, 13.1.6]) and with Veronesean transforms (s. [B-S, 12.4.6]). We thus get natural isomorphisms of graded R-modules

$$\begin{aligned} D_{R_+}(M^{[e]}) &= D_{R_+}\big(M^{(p^e)}\restriction_{F_e}\big) \cong D_{F_e(R_+)R^{(p^e)}}\big(M^{(p^e)}\big)\restriction_{F_e} \\ &= D_{(R_+)^{(p^e)}}\big(M^{(p^e)}\big)\restriction_{F_e} \cong D_{R_+}(M)^{(p^e)}\restriction_{F_e} = D_{R_+}(M)^{[e]}, \end{aligned}$$

hence

(i)
$$D_{R_+}\big(M^{[e]}\big) \cong D_{R_+}(M)^{[e]}$$

and, similarly

(ii)
$$H^i_{R_+}\big(M^{[e]}\big) \cong H^i_{R_+}(M)^{[e]}, \quad \text{for all } i \in \mathbb{N}_0.$$

C) Fix $e \in \mathbb{N}$. If we apply the cohomology sequence derived from the functor D_{R_+} to the e-th arithmetic Frobenius sequence (2.5) A) (i) and observe the natural isomorphism $\mathcal{R}^1 D_{R_+}(R) \cong H^2_{R_+}(R)$ of (3.1) A) and the above natural isomorphisms (3.1) B) (i) and (ii), we get the following exact sequence of graded R-modules

(iii)
$$0 \longrightarrow D_{R_+}(R) \longrightarrow D_{R_+}(R)^{[e]} \longrightarrow D_{R_+}(C_{[e]}) \xrightarrow{\delta_e} H^2_{R_+}(R) \longrightarrow H^2_{R_+}(R)^{[e]}.$$

So, for each $n \in \mathbb{Z}$, (2.4) B) gives rise to an exact sequence of R_0-modules

$$\textbf{(iv)} \qquad 0 \longrightarrow D_{R_+}(R)_n \longrightarrow \big(D_{R_+}(R)_{np^e}\big)\restriction_{F_{e,0}} \longrightarrow D_{R_+}(C_{[e]})_n \xrightarrow{\delta_{e,n}} H^2_{R_+}(R)_n \longrightarrow \big(H^2_{R_+}(R)_{np^e}\big)\restriction_{F_{e,0}}.$$

LEMMA 3.2. *Assume that* $Y := \mathrm{Proj}(R)$ *is normal and that* $\mathrm{height}(R_+) > 2$. *Let* $e \in \mathbb{N}$ *and let* L *denote the field of rational functions* $\kappa(Y)$ *on* Y. *Then, the graded* R*-module* $D_{R_+}(C_{[e]})$ *is finitely generated and torsion-free of rank* $[L : L^p]^e - 1$. *Moreover, the* R*-modules* $H^i_{R_+}(R)$ *are finitely generated for* $i = 1, 2$.

Proof. As Y is normal, the sheaf $\tilde{R} = \mathcal{O}_Y$ satisfies the second Serre condition S_2, so that $\mathrm{depth}(R_\mathfrak{q}) \geq \min\{2, \mathrm{height}(\mathfrak{q})\}$ for each prime $\mathfrak{q} \in \mathrm{Proj}(R)$. As R is essentially of finite type over a field, it is catenarian. As moreover R is a domain and as $\mathrm{height}(R_+) > 2$ it follows easily that $\mathrm{depth}(R_\mathfrak{q}) + \mathrm{height}\big((\mathfrak{q} + R_+)/\mathfrak{q}\big) \geq 3$ for all $\mathfrak{q} \in \mathrm{Proj}(R)$. So, by (the graded version of) Grothendiecks Finiteness Theorem for local cohomology (s. [B-S, 13.1.7]), the R-modules $H^i_{R_+}(R)$ are finitely generated for all $i \leq 2$. But obviously, now the four term exact sequence of (3.1) A) applied to $M = R$ tells us that $D_{R_+}(R)$ is finitely generated. So, by the observations made in (2.4) A), the R-module $D_{R_+}(R)^{[e]}$ is finitely generated. By the exact sequence (3.1) C) (iii) the R-module $D_{R_+}(C_{[e]})$ is finitely generated.

Observe that in view of the exact sequence (3.1) A), the exactness of the functor $\tilde{}$ and the fact that R_+-torsion modules induce the zero sheaf, we may write $D_{R_+}(C_{[e]})^\sim \cong (C_{[e]})^\sim$.

Now, by (2.8) the sheaf of $\mathcal{O}_Y$-modules $D_{R_+}(C_{[e]})^\sim \cong (C_{[e]})^\sim = \mathcal{K}_{[e]}$ is torsion free, so that $\mathrm{Ass}_R\big(D_{R_+}(C_{[e]})\big) \subseteq \{0\} \cup \mathrm{Var}(R_+)$. As $D_{R_+}(C_{[e]})$ has no R_+-torsion (s. [B-S, 2.2.8]) we thus see that $D_{R_+}(C_{[e]})$ is torsion-free. Finally, by (2.6) we get that $\mathrm{rank}_{R_+}\big(D_{R_+}(C^{[e]})\big) = \mathrm{rank}_{\mathcal{O}_{Y,0}}\big((\mathcal{K}_{[e]})_0\big) = [L : L^p]^e - 1$. ∎

LEMMA 3.3. *Assume that* $Y := \mathrm{Proj}(R)$ *is normal and that* $\mathrm{height}(R_+) > 2$. *Then*:

a) *There is some* $e_0 \in \mathbb{N}$ *such that the homomorphism* $\delta_{e,n} : D_{R_+}(C_{[e]})_n \to H^2_{R_+}(R)_n$ *in the sequence* (3.1) C) (iv) *is an isomorphism for all* $e \geq e_0$ *and all* $n < 0$.

b) *The R_0-modules $H^2_{R_+}(R)_n$ are torsion-free for all $n < 0$.*

Proof. By (3.2) and (3.1) A) we know that $H^2_{R_+}(R)$ and $D_{R_+}(R)$ are finitely generated R-modules. So, there is a $t \in \mathbb{N}$ such that $H^2_{R_+}(R)_m = D_{R_+}(R)_m = 0$ for all $m \leq -t$. Let $e_0 \in \mathbb{N}$ be such that $p^{e_0} \geq t$ and let $e \geq e_0$. Then the homomorphism $\delta_{e,n}$ in the exact sequence (3.1) C) (iv) is an isomorphism for each $n < 0$. This proves claim a). Claim b) is immediate from claim a) and the fact that $D_{R_+}(C_{[e]})$ is torsion-free, which was shown in (3.2). ∎

Now, let L_0 be the quotient field of the domain R_0, e.g. the field $\kappa(Y_0)$ of rational functions on the affine scheme $Y_0 := \operatorname{Spec}(R_0)$. Let M be a finitely generated and graded R-module. As $H^i_{R_+}(M)_n$ is a finitely generated R_0-module for all $n \in \mathbb{Z}$ and all $i \in \mathbb{N}_0$, it makes sense to introduce the numbers

$$(3.4) \qquad h^i_{R_+}(M)_n := \dim_{L_0}(L_0 \otimes_{R_0} H^i_{R_+}(M)_n)$$

for all such n and i.

Remark 3.5. A) Let R'_0 be a flat and noetherian R_0-algebra. Then $R'_0 \otimes_{R_0} R$ carries a natural structure of positively graded ring, by a grading which is given by $(R'_0 \otimes_{R_0} R)_n = R'_0 \otimes_{R_0} R_n$ for all $n \in \mathbb{Z}$. As R is of finite type over R_0, the ring $R'_0 \otimes_{R_0} R$ is of finite type over R'_0 and hence is noetherian. Moreover, if $M = \bigoplus_{n\in\mathbb{Z}} M_n$ is a graded R-module, the $R'_0 \otimes_{R_0} R$-module $(R'_0 \otimes_{R_0} R) \otimes_R M = R'_0 \otimes_{R_0} M$ carries a natural grading, given by $(R'_0 \otimes_{R_0} M)_n = R'_0 \otimes_{R_0} M_n$ for all $n \in \mathbb{Z}$.

B) Keep the hypotheses and notation of A). Then, the graded version of the flat base change property for local cohomology induces natural isomorphisms of R'_0-modules $(R'_0 \otimes_{R_0} H^i_{R_+}(M))_n = R'_0 \otimes_{R_0} H^i_{R_+}(M)_n \cong H^i_{(R'_0 \otimes R)_+}(R'_0 \otimes_R M)_n$ for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$ (s. [B-S, 14.2.6]). If we apply this in the case where $R'_0 = L'_0$ is an arbitrary extension field of L_0, we thus get

$$\begin{aligned} h^i_{R_+}(M)_n &= \dim_{L'_0}\big(L'_0 \otimes_{R_0} H^i_{R_+}(M)_n\big) \\ &= \dim_{L'_0}\big(H^i_{(L'_0 \otimes_{R_0} R)_+}(L'_0 \otimes_{R_0} M)_n\big) \\ &= h^i_{(L'_0 \otimes_{R_0} R)_+}(L'_0 \otimes_{R_0} M)_n \end{aligned}$$

for each finitely generated and graded R-module M and for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$.

Now, we shall attack the principal goal of this section: to give upper bounds on the numbers $h^2_{R_+}(R)_n$ for $n < 0$. We start with the following auxiliary result:

LEMMA 3.6. *Assume that* $Y := \mathrm{Proj}(R)$ *is normal and that* $\mathrm{height}(R_+) > 2$. *Let* $t \in \mathbb{N}$ *and let* $f \in R_t \backslash \{0\}$. *Then*

a) *The multiplication homomorphism*

$$f : H^2_{R_+}(R)_n \longrightarrow H^2_{R_+}(R)_{n+t}$$

is injective for all $n < -t$.

b) *If in addition* R_0 *is a perfect field, the multiplication homomorphism*

$$f : H^2_{R_+}(R)_{-t} \longrightarrow H^2_{R_+}(R)_0$$

is injective, too.

Proof. Let $n \in \mathbb{Z}$ with $n \leq -t$. For each $e \in \mathbb{N}$ we have the following commutative diagram, in which the vertical maps are the connecting homomorphism in the sequences (3.1) C) (iv).

$$\begin{array}{ccc}
D_{R_+}(C_{[e]})_n & \xrightarrow{\quad f \quad} & D_{R_+}(C_{[e]})_{n+t} \\
\Big\downarrow \delta_{e,n} & & \Big\downarrow \delta_{e,n+t} \\
H^2_{R_+}(R)_n & \xrightarrow{\quad f \quad} & H^2_{R_+}(R)_{n+t}
\end{array}$$

By (3.2) we know that $D_{R_+}(C_{[e]})$ is a torsion free R-module, so that the upper horizontal map is injective. By (3.3) a) we may choose $e \in \mathbb{N}$ such that $\delta_{e,m}$ is an isomorphism for all $m < 0$. This proves statement a).

Assume now that R_0 is a perfect field. Then, the Frobenius homomorphism $F_{e,0} : R_0 \to R_0$ is an isomorphism so that $D_{R_+}(R)_0$ and $\big(D_{R_+}(R)_0\big)\lceil_{F_{e,0}}$ are R_0-vector spaces of the same dimension (which is finite, as $D_{R_+}(R)$ is a finitely generated R-module by (3.2) and (3.1) A)). But now, the sequence (3.1) C) (iv), applied with $n = 0$, shows that the map $\delta_{e,0}$ is injective, and this proves statement b). $\square$

LEMMA 3.7. *Let* V *and* W *be non-zero vector spaces of finite dimension over an algebraically closed field* K. *Let* $r \in \mathbb{N}$ *and let* $l_1, \dots, l_r : V \to W$ *be* K*-linear maps.*

a) *If* $\sum_{i=1}^{r} \alpha_i l_i : V \to W$ *is injective for all* $(\alpha_1, \ldots, \alpha_r) \in K^r \backslash \{(\underline{0})\}$, *then*

$$\dim_K(W) \geq \dim_K(V) + r - 1\,.$$

b) *If* $\sum_{i=1}^{r} \alpha_i l_i : V \to W$ *is surjective for all* $(\alpha_1, \ldots, \alpha_r) \in K^r \backslash \{(0)\}$, *then*

$$\dim_K(V) \geq \dim_K(W) + r - 1.$$

Proof. [B_1, (3.1), (3.2)]. □

Now, we are ready to prove the main result of this section.

THEOREM 3.8. *Assume that the generic fibre of the natural morphism*

$$\pi : Y = \mathrm{Proj}(R) \longrightarrow Y_0 = \mathrm{Spec}(R_0)$$

is geometrically connected, geometrically normal and of dimension > 1. *For each* $t \in \mathbb{N}$, *let* $r_t := \mathrm{rank}_{R_0}(R_t)$. *Then:*

a) *For each* $t \in \mathbb{N}$ *with* $r_t > 0$ *and for each* $n \in \mathbb{Z}$ *with* $n \leq -t$, *we have*

$$h^2_{R_+}(R)_n \leq \max\bigl\{0, h^2_{R_+}(R)_{n+t} - r_t + 1\bigr\}.$$

b) *If* Y *is in addition normal, then* $H^2_{R_+}(R)_n$ *is a torsion-free* R_0*-module of rank* $h^2_{R_+}(R)_n$ *for all* $n < 0$. *Moreover, if* $r_1 > 1$, *then* $H^2_{R_+}(R)_n = 0$ *as soon as* $n \leq \min\left\{-1, -\dfrac{h^2_{R_+}(R)_0}{r_1 - 1}\right\}$.

Proof. a): Fix some $t \in \mathbb{N}$ with $r_t > 0$ and some $n \in \mathbb{Z}$ with $n \leq -t$. Let L'_0 be an algebraic closure of the quotient field L_0 of R_0. Then, the graded ring $R' := L'_0 \otimes_{R_0} R$ is a flat and integral extension of the graded ring $L_0 \otimes_{R_0} R =: R''$. Therefore $\mathrm{height}(R'_+) = \mathrm{height}(R''_+) = \dim(Z) + 1$ where $Z := \mathrm{Proj}(R'')$ is the generic fibre of the morphism $\pi : Y \to Y_0$. So $\mathrm{height}(R'_+) > 2$. As $Z \to \mathrm{Spec}(L_0)$ is geometrically connected and geometrically normal, we know that $Y' := \mathrm{Proj}(R') \cong \mathrm{Spec}(L'_0) \times_{\mathrm{Spec}(L_0)} Z$ is a connected and normal scheme over $Y'_0 := \mathrm{Spec}(L'_0)$. In particular Y' is integral. As R' is flat and hence torsion-free over R, this shows that R' is an integral domain.

Keep in mind that $R'_t \cong L'_0 \otimes_{L_0} (L_0 \otimes_{R_0} R_t)$ is a vector-space of dimension r_t over L'_0.

Now, choose a basis $f'_1, \dots, f'_{r_t}$ of R'_t and keep in mind that $R'_0 = L'_0$ is an algebraically closed and hence perfect field and that Y' is normal. So, by (3.6) the multiplication map $\sum_{i=1}^{r_t} \alpha_i f'_i : H^2_{R'_+}(R')_n \to H^2_{R'_+}(R')_{n+t}$ is injective whenever $(\alpha_1, \dots, \alpha_{r_t}) \in L'_0 \backslash \{(\underline{0})\}$. So, (3.7) a) gives $h^2_{R'_+}(R')_n \leq \max\{0, h^2_{R'_+}(R')_{n+t} - r_t + 1\}$. In view of the equations given in (3.5) B) this proves our claim.

b): Let Y in addition be normal. Then (3.3) b) shows that $H^2_{R_+}(R)_n$ is torsion-free over R_0 for all $n < 0$. This allows to conclude by statement a), applied repeatedly with $t = 1$. ∎

Remark 3.9. If in (3.8) b) R_0 is a perfect field, (3.6) b) allows to replace

$$\min\Big\{-1, -\frac{h^2_{R_+}(R)_0}{r_1 - 1}\Big\} \quad \text{by} \quad -\frac{h^2_{R_+}(R)_0}{r_1 - 1}.$$

§4. Ample invertible sheaves

Let X_0 be an affine integral scheme which is essentially of finite type over a perfect field K of positive characteristic p. Moreover let X be an integral and projective scheme over X_0 with surjective structure morphism $\varrho : X \to X_0$. Finally, let $\mathcal{L}$ be an ample invertible sheaf of $\mathcal{O}_X$-modules.

Notation and Remark 4.1. A) We use $\mathcal{L}^*$ to denote the tensor algebra $\bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$ so that $\mathcal{L}^*$ is a sheaf of positively graded integral $\mathcal{O}_X$-algebras. Moreover we shall consider the positively graded ring $\Gamma(\mathcal{L}^*) := \Gamma(X, \mathcal{L}^*) = \bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$ and the induced projective scheme $Y(\mathcal{L}) := \mathrm{Proj}(\Gamma(\mathcal{L}^*))$. As $\mathcal{L}^*$ is a sheaf of integral domains, $\Gamma(\mathcal{L}^*)$ is a domain and $Y(\mathcal{L})$ is an integral projective scheme over $\mathrm{Spec}(\Gamma(X, \mathcal{O}_X)) =: Y_0$. Finally, if $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, we denote by $\Gamma(\mathcal{F}, \mathcal{L}^*)$ the graded $\Gamma(\mathcal{L}^*)$-module

$$\bigoplus_{n\geq 0} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}).$$

B) By [H_2, II, Thm. 7.6] there is some $r \in \mathbb{N}$ such that $\mathcal{L}^{\otimes r}$ is very ample with respect to the to the structure morphism ϱ. But this means that the r-th Veronesean subring $\Gamma(\mathcal{L}^*)^{(r)} = \Gamma\big((\mathcal{L}^{\otimes r})^*\big) = \bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}^{\otimes rn})$ is noetherian and that for each coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ the r-th Veronesean transform $\Gamma(\mathcal{F}, \mathcal{L}^*)^{(r)} = \Gamma\big(\mathcal{F}, (\mathcal{L}^{\otimes r})^*\big) = \bigoplus_{n\geq 0} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes rn})$ is finitely generated over $\Gamma(\mathcal{L}^*)^{(r)}$ [G, Cor. (2.3.2)]. If we apply this to the coherent sheaves $\mathcal{L}^{\otimes i} \otimes \mathcal{F}$ we then get that the $\Gamma(\mathcal{L}^*)^{(r)}$-module $\Gamma(\mathcal{F}, \mathcal{L}^*) = \bigoplus_{i=0}^{r-1} \Gamma\big(\mathcal{L}^{\otimes i} \otimes \mathcal{F}, (\mathcal{L}^{\otimes r})^*\big) = \bigoplus_{i=0}^{r-1} \Gamma(\mathcal{L}^{\otimes i} \otimes \mathcal{F}, \mathcal{L}^*)^{(r)}$ is finitely generated. If we apply this with $\mathcal{F} = \mathcal{O}_X$, we see that $\Gamma(\mathcal{L}^*)$ is a finite integral extension of $\Gamma(\mathcal{L}^*)^{(r)}$. In particular we now see that $\Gamma(\mathcal{L}^*)$ is noetherian and that $\Gamma(\mathcal{F}, \mathcal{L}^*)$ is a finitely generated $\Gamma(\mathcal{L}^*)$-module for each coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$.

Remark 4.2. (cf. [G]) Keep the previous hypotheses and notation. Then, there is a natural isomorphism of schemes.

$$\alpha_{\mathcal{L}} := \big(\alpha_{\mathcal{L}}, \alpha_{\mathcal{L}}^{\#}\big) : (X, \mathcal{O}_X) \longrightarrow \big(Y(\mathcal{L}), \mathcal{O}_{Y(\mathcal{L})}\big) .$$

Now, let $k \in \mathbb{Z}$ and let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Then we have a natural isomorphism of sheaves of $\mathcal{O}_{Y(\mathcal{L})}$-modules

$$\eta^{(k)}_{\mathcal{F},\mathcal{L}} = \eta^{(k)} : \big(\Gamma(\mathcal{F}, \mathcal{L}^*)(k)\big)^{\sim} \xrightarrow{\cong} (\alpha_{\mathcal{L}})_* \big(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes k}\big) .$$

in which $\cdot (k)$ is used to denote the k-th shift functor on graded $\Gamma(\mathcal{L}^*)$-modules.

Remark 4.3. Keep the previous notation and hypothesis. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules and let $k \in \mathbb{Z}$ and $i \in \mathbb{N}$. Then, there is a natural isomorphism of $\Gamma(X, \mathcal{O}_X)$-modules

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes k}) \cong H^{i+1}_{\Gamma(\mathcal{L}^*)_+}\big(\Gamma(\mathcal{F}, \mathcal{L}^*)\big)_k ,$$

induced by the isomorphism $\eta^{(k)}$ of (4.2) and the Serre-Grothendieck correspondence (s. [B-S, 20.4.4]).

For $i \in \mathbb{N}_0$ and for an arbitrary coherent sheaf $\mathcal{G}$ of $\mathcal{O}_X$-modules, we use the notation

$$h^i(X, \mathcal{G}) := \dim_{\kappa(Y_0)} \big(\kappa(Y_0) \otimes_{\Gamma(X, \mathcal{O}_X)} H^i(X, \mathcal{G})\big) , \tag{4.4}$$

where $\kappa(Y_0)$ is the field of rational functions on the scheme $Y_0 := \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$, (s. (4.1) A)).

Now, we are ready to prove the main result of this section. We assume that X, X_0 and $\varrho : X \to X_0$ are as introduced at the beginning of this section.

THEOREM 4.5. *Assume that the generic fibre of the structural morphism* $\varrho : X \to X_0$ *is geometrically connected, geometrically normal and of dimension* > 1. *Let* $\mathcal{L}$ *be an ample invertible sheaf of* $\mathcal{O}_X$*-modules. Then*:

a) *For each* $t \in \mathbb{N}$ *with* $h^0(X, \mathcal{L}^{\otimes t}) > 0$ *and for each* $n \in \mathbb{Z}$ *with* $n \leq -t$, *we have*

$$h^1(X, \mathcal{L}^{\otimes n}) \leq \max\bigl\{0, h^1\bigl(X, \mathcal{L}^{\otimes (n+t)}\bigr) - h^0\bigl(X, \mathcal{L}^{\otimes t}\bigr) + 1\bigr\}\,.$$

b) *If* X *is in addition normal, then* $H^1(X, \mathcal{L}^{\otimes n})$ *is a torsion-free* $\Gamma(X, \mathcal{O}_X)$*-module of rank* $h^1(X, \mathcal{L}^{\otimes n})$ *for all* $n < 0$. *If* $h^0(X, \mathcal{L}) > 1$, *then* $H^1(X, \mathcal{L}^{\otimes n})$ *vanishes as soon as* $n \leq \min\Bigl\{-1, -\dfrac{h^1(X, \mathcal{O}_X)}{h^0(X, \mathcal{L}) - 1}\Bigr\}$.

Proof. By our hypothesis, $\Gamma(X, \mathcal{O}_X)$ is a finite integral extension domain of $\Gamma(X_0, \mathcal{O}_{X_0})$. So, in view of (4.2) and using the notation of (4.1) A), we get the following commutative diagram of schemes

$$\begin{array}{ccc} X & \xrightarrow{\quad\varrho\quad} & X_0 \\ {\scriptstyle \alpha_{\mathcal{L}}}\big\downarrow{\scriptstyle \cong} & & \big\uparrow{\scriptstyle \nu} \\ Y(\mathcal{L}) = \mathrm{Proj}(\Gamma(\mathcal{L}^*)) & \xrightarrow{\quad\pi\quad} & \mathrm{Spec}(\Gamma(\mathcal{L}^*)_0) = Y_0 \end{array}$$

in which the morphism ν is finite.

As $\Gamma(\mathcal{L}^*)_0 = \Gamma(X, \mathcal{O}_X)$ is an integral extension domain of $\Gamma(X_0, \mathcal{O}_{X_0})$, we have

$$\kappa(Y_0) = \kappa(X_0) \otimes_{\Gamma(X_0, \mathcal{O}_{X_0})} \Gamma(\mathcal{L}^*)_0\,,$$

where $\kappa(X_0)$ denotes the field of rational functions on X_0. So the generic fibre

$$F_\pi := \mathrm{Spec}\bigl(\kappa(Y_0)\bigr) \times_{Y_0} Y = \mathrm{Proj}\bigl(\kappa(Y_0) \otimes_{\Gamma(\mathcal{L}^*)_0} \Gamma(\mathcal{L}^*)\bigr)$$

of π may be written as

$$\begin{aligned} \mathrm{Proj}\bigl(\kappa(X_0) \otimes_{\Gamma(X_0, \mathcal{O}_{X_0})} \Gamma(\mathcal{L}^*)\bigr) &\cong \mathrm{Spec}\bigl(\kappa(X_0)\bigr) \times_{X_0} Y(\mathcal{L}) \\ &\cong \mathrm{Spec}\bigl(\kappa(X_0)\bigr) \times_{X_0} X =: F_\varrho \end{aligned}$$

and thus is X_0-isomorphic to the generic fibre F_ϱ of ϱ. This induces, that the generic fibre F_π of π is of dimension > 1. Now, let L' be an arbitrary algebraic extension field of $\kappa(Y_0)$. Then L' is an algebraic extension field of $\kappa(X_0)$ and therefore by our hypothesis on F_ϱ we see that

$$\mathrm{Spec}(L') \times_{\mathrm{Spec}(\kappa(X_0))} F_\pi \cong \mathrm{Spec}(L') \times_{\mathrm{Spec}(\kappa(X_0))} F_\varrho$$

is a normal and integral scheme. Moreover, the natural morphism

$$\mathrm{Spec}(L') \times_{\mathrm{Spec}(\kappa(Y_0))} F_\pi \longrightarrow \mathrm{Spec}(L') \times_{\mathrm{Spec}(\kappa(X_0))} F_\pi$$

is a closed immersion between two schemes of the same dimension. Hence it is an isomorphism. So $\mathrm{Spec}(L') \times_{\mathrm{Spec}(\kappa(Y_0))} F_\pi$ is a normal and integral scheme. This shows that the generic fibre of π is geometrically connected and geometrically normal.

Next observe that $\Gamma(\mathcal{L}^*)_t = H^0(X, \mathcal{L}^{\otimes t})$ is a torsion-free $\Gamma(\mathcal{L}^*)_0$-module of rank $h^0(X, \mathcal{L}^{\otimes t})$ for all $t \in \mathbb{N}$. Finally (4.3) gives us isomorphisms of $\Gamma(\mathcal{L}^*)_0$-modules $H^1(X, \mathcal{L}^{\otimes n}) \cong H^2_{\Gamma(\mathcal{L}^*)_+}\big(\Gamma(\mathcal{L}^*)\big)_n$ and hence $h^1(X, \mathcal{L}^{\otimes n}) = h^2_{\Gamma(\mathcal{L}^*)_+}(\Gamma(\mathcal{L}^*))_n$ for all $n \in \mathbb{Z}$.

Therefore we get our statements if we apply (3.8) with $R = \Gamma(\mathcal{L}^*)$. ∎

Corollary 4.6. *Let X be a geometrically connected and geometrically normal projective scheme of dimension > 1 over a perfect field K of positive characteristic p. Let $\mathcal{L}$ be an ample invertible sheaf of $\mathcal{O}_X$-modules. Then*:

a) *For each $t \in \mathbb{N}$ with $h^0(X, \mathcal{L}^{\otimes t}) > 0$ and for each $n \in \mathbb{N}$ with $n \leq -t$ we have*

$$h^1(X, \mathcal{L}^{\otimes n}) \leq \max\big\{0, h^1(X, \mathcal{L}^{\otimes (n+t)}) - h^0(X, \mathcal{L}^{\otimes t}) + 1\big\}.$$

b) *If $h^0(X, \mathcal{L}) > 1$, then $H^1(X, \mathcal{L}^{\otimes n}) = 0$ for all integers*

$$n \leq -\frac{h^1(X, \mathcal{O}_X)}{h^0(X, \mathcal{L}) - 1}.$$

Proof. Clear from (4.5) and (3.9). ∎

Corollary 4.7. (s. [A, (5.6), (5.8)]) *Let X be a normal projective variety of dimension > 1 over an algebraically closed field K of positive characteristic p. Let $\mathcal{L}$ be an ample invertible sheaf of $\mathcal{O}_X$-modules. Then*:

a) *For each $t \in \mathbb{N}$ with $h^0(X, \mathcal{L}^{\otimes t}) > 0$ and for each $n \in \mathbb{Z}$ with $n \leq -t$ we have*

$$h^1(X, \mathcal{L}^{\otimes n}) \leq \max\big\{0, h^1(X, \mathcal{L}^{\otimes (n+t)}) - h^0(X, \mathcal{L}^{\otimes t}) + 1\big\}.$$

b) *If* $h^0(X,\mathcal{L}) > 1$*, then* $H^1(X,\mathcal{L}^{\otimes n}) = 0$ *for all integers*

$$n \leq -\frac{h^1(X,\mathcal{O}_X)}{h^0(X,\mathcal{L})-1}\,.$$

PROPOSITION 4.8. *Let X be as in* (4.6) *and let D be an effective ample Cartier divisor on X. Then, the restriction homomorphism* $\Gamma(X,\mathcal{O}_X) \to \Gamma(X,\mathcal{O}_D)$ *is an isomorphism.*

Proof. Let $\mathcal{L} := \mathcal{L}(D)$ be the invertible sheaf associated to D. Then, $\mathcal{L}$ is ample and as D is effective, we have a short exact sequence of coherent sheaves of $\mathcal{O}_X$-modules $0 \to \mathcal{L}^{\otimes -1} \xrightarrow{\varphi} \mathcal{O}_X \to \mathcal{O}_D \to 0$. If we apply cohomology we thus see, that it is sufficient to prove that the induced homomorphism $H^1(X,\varphi) : H^1(X,\mathcal{L}^{\otimes -1}) \xrightarrow{\overline{\varphi}} H^1(X,\mathcal{O}_X)$ is injective.

To do so, consider the induced monomorphism

$$0 \longrightarrow \Gamma(X,\mathcal{O}_X) \xrightarrow{\Gamma(X,\varphi\otimes\mathcal{L})} \Gamma(X,\mathcal{L}).$$

Then, $f := \Gamma(X,\varphi\otimes\mathcal{L})(1) \in \Gamma(X,\mathcal{L}) = \Gamma(\mathcal{L}^*)_1$, $f \neq 0$ and $\overline{\varphi}$ corresponds to the multiplication map $f : H^2_{\Gamma(\mathcal{L}^*)_+}(\Gamma/(\mathcal{L}^*))_{-1} \to H^2_{\Gamma(\mathcal{L}^*)_+}(\Gamma(\mathcal{L}^*))_0$ under the natural isomorphisms of (4.3). Now, we may conclude by (3.6) b), applied with $R = \Gamma(\mathcal{L}^*)$, $Y = Y(\mathcal{L})$ and $t = 1$. ∎

COROLLARY 4.9. (s. [Mu, Prop. 3], [A, (5.17)]) *Let X be a normal projective variety of dimension > 1 over an algebraically closed field K of positive characteristic p. Let D be an effective ample Cartier divisor on X. Then* $\Gamma(X,\mathcal{O}_D) = K$.

§5. Applications to projective varieties

In this section, we apply the previous results to ample invertible sheaves over projective varieties of dimension > 1 over an algebraically closed field K of positive characteristic. Our main interest is focused to the case of surfaces.

For a reduced and irreducible variety X of dimension > 1 over an algebraically closed field K we introduce the invariant.

$$(5.1)\qquad e^1(X) := \sum_{p\in X,\, p \text{ closed}} \operatorname{length}_{\mathcal{O}_{X,p}}\left(H^1_{\mathfrak{m}_{X,p}}(\mathcal{O}_{X,p})\right),$$

which is finite and which counts in a “weighted way” the number of (closed) points $p \in X$ in which X has depth ≤ 1 (s. [B, (5.7)]).

Throughout this section, let K be an algebraically closed field of positive characteristic p.

Proposition 5.2. *Let X be a projective variety of dimension > 1 over K. Assume that X has only finitely many non-normal points. Let $\mathcal{L}$ be an ample invertible sheaf of $\mathcal{O}_X$-modules such that $h^0(X,\mathcal{L}) \geq 1$. Then*

a) $e^1(X) \leq h^1(X,\mathcal{L}^{\otimes n}) \leq \max\big\{e^1(X), h^1(X,\mathcal{L}^{\otimes(n+1)}) - h^0(X,\mathcal{L}) + 1\big\}$ *for all negative integers n.*

b) $e^1(X) \leq h^1(X,\mathcal{L}^{\otimes n}) \leq \max\big\{e^1(X), h^1(X,\mathcal{O}_X) + n(h^0(X,\mathcal{L}) - 1)\big\}$ *for all $n \leq 0$.*

Proof. Let $\nu : \tilde{X} \to X$ be the normalization of X and let $Z \subseteq X$ be the finite set of non-normal points of X. Then, we have an exact sequence of coherent sheaves of $\mathcal{O}_X$-modules $0 \to \mathcal{O}_X \xrightarrow{\nu^{\#}} \nu_*\mathcal{O}_{\tilde{X}} \to \mathcal{F} \to 0$ with $\mathrm{supp}(\mathcal{F}) = Z$.

Now, let $p \in X$ be an arbitrary closed point. As $\tilde{X}$ is normal, the direct image $\nu_*\mathcal{O}_{\tilde{X}}$ has the second Serre property S_2, so that $H^1_{\mathfrak{m}_{X,p}}\big((\nu_*\mathcal{O}_{\tilde{X}})_p\big) = 0$. As $\dim(Z) = 0$ we moreover have $\mathcal{F}_p \cong H^0_{\mathfrak{m}_{X,p}}(\mathcal{F}_p)$. So, passing to stalks in the above exact sequence and applying local cohomology at all closed points $p \in X$ we see that $\mathrm{length}_{\mathcal{O}_X}(\mathcal{F}) = e^1(X)$. As $\mathcal{L}$ is invertible we thus obtain $\mathrm{length}_{\mathcal{O}_X}\big(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}\big) = e^1(X)$ and hence $h^0(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = e^1(X)$ and moreover $H^1(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0$ for all $n \in \mathbb{Z}$.

Moreover, for each $n \in \mathbb{Z}$ we get an exact sequence

(∗)
$$\begin{aligned} 0 &\longrightarrow H^0(X,\mathcal{L}^{\otimes n}) \longrightarrow H^0(X,\nu_*\mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \longrightarrow H^0(X,\mathcal{F} \otimes_{\mathcal{O}_x} \mathcal{L}^{\otimes n}) \\ &\longrightarrow H^1(X,\mathcal{L}^{\otimes n}) \longrightarrow H^1(X,\nu_*\mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \longrightarrow 0\,. \end{aligned}$$

By use of the projection formula [H_2, II Ex. 5.1 (d)], we have

$$\nu_*\mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \cong \nu_*(\mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}_{\tilde{X}}} \nu^*(\mathcal{L}^{\otimes n})) \cong \nu_*\nu^*(\mathcal{L}^{\otimes n}) \cong \nu_*((\nu^*\mathcal{L})^{\otimes n})\,.$$

As ν is an affine morphism we thus get isomorphisms $H^i(X, \nu_*\mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \cong H^i(\tilde{X}, (\nu^*\mathcal{L})^{\otimes n})$ for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$, (s. [H_2, III Ex. 4.1]). As ν is finite and surjective, $\nu^*\mathcal{L}$ is an ample invertible sheaf of $\mathcal{O}_{\tilde{X}}$-modules (s. [H_2, III Ex. 5.7 (d)]).

So, first of all $H^0(X, \nu_*\mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ vanishes for $n < 0$ and equals K for $n = 0$. But now, the sequences (∗) show that $h^1(X,\mathcal{L}^{\otimes n}) = e^1(X) + h^1(\tilde{X}, (\nu^*\mathcal{L})^{\otimes n})$ for all $n \leq 0$.

Moreover, if we apply the sequence $(*)$ with $n = 1$ we see that $h^0(\tilde{X}, \nu^*\mathcal{L}) \geq h^0(X, \mathcal{L})$. Therefore we get our claims if we apply (4.7) a) to the normal projective variety $\tilde{X}$ and the ample invertible sheaf $\nu^*\mathcal{L}$ with $t = 1$. ∎

For a coherent sheaf $\mathcal{F}$ over a projective variety X let $\chi(\mathcal{F})$ denote the characteristic of $\mathcal{F}$ so that $\chi(\mathcal{F}) = \sum_{i\geq 0}(-1)^i h^i(X, \mathcal{F})$.

Definition and Remark 5.3. A) Let X be a projective surface over K and let $\mathcal{L}$ be an ample invertible sheaf of $\mathcal{O}_X$-modules. We define the *sectional genus of X with respect to $\mathcal{L}$* by

$$\sigma_{\mathcal{L}}(X) := \chi(\mathcal{L}^{\otimes -1}) - \chi(\mathcal{O}_X) + 1\,.$$

As X is a surface, $\sigma_{\mathcal{L}}(X)$ coincides indeed with the sectional genus of the pair $(X, \mathcal{L})$ as introduced by Fujita in [F, pg. 25].

B) Let $f \in \Gamma(X, \mathcal{L})\backslash\{0\}$. Then, there is an effective divisor D_f on X with $\mathcal{L} = \mathcal{L}(D_f)$ and we have an exact sequence

$$0 \longrightarrow \mathcal{L}^{\otimes -1} \xrightarrow{f\cdot} \mathcal{O}_X \longrightarrow \mathcal{O}_{D_f} \longrightarrow 0$$

which tells us that

$$\sigma_{\mathcal{L}}(X) = -\chi(\mathcal{O}_{D_f}) + 1 = h^1(D_f, \mathcal{O}_{D_f}) - h^0(D_f, \mathcal{O}_{D_f}) + 1\,.$$

If X is normal, (4.9) gives $h^0(D_{f'} \mathcal{O}_{D_f}) = h^0(X, \mathcal{O}_{D_f}) = 1$ and so we get $\sigma_{\mathcal{L}}(X) = h^1(D_f, \mathcal{O}_{D_f}) \geq 0$ if $h^0(X, \mathcal{L}) > 0$.

C) Assume now that $\mathcal{L}$ is very ample, so that it occurs as the twisting sheaf of some non-degenerate closed immersion $X \overset{i}{\hookrightarrow} \mathbb{P}^r_K$. If we choose $f \in \Gamma(X, \mathcal{L})\backslash\{0\}$ generically, then D_f is a generic hyperplane section of X in $\mathbb{P}^r_K$ and thus is reduced and irreducible by Bertini. Therefore $h^0(D_f, \mathcal{O}_{D_f}) = 1$ and it follows again that $\sigma_{\mathcal{L}}(X) = h^1(D_f, \mathcal{O}_{D_f})$, but this time without the assumption that X is normal. In particular $\sigma_{\mathcal{L}}(X)$ coincides with the arithmetic genus of the generic hyperplane section D_f and thus is nothing else than the usual sectional genus of X with respect to the embedding $i : X \hookrightarrow \mathbb{P}^r_K$.

Proposition 5.4. (s. [A, (5.19)]) *Let X be a normal projective surface over K. Let $\mathcal{L}$ be an ample invertible sheaf of $\mathcal{O}_X$-modules such that $0 \leq \sigma_{\mathcal{L}}(X) < h^0(X, \mathcal{L}) - 1$. Then*:

a) $H^1(X, \mathcal{L}^{\otimes n}) = 0$ *for all* $n < 0$.

b) $H^2(X, \mathcal{L}^{\otimes n}) = 0$ *for all* $n \geq 0$.

c) $h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{L}^{\otimes -1}) = \sigma_{\mathcal{L}}(X)$.

Proof. By our hypothesis we have $\Gamma(X, \mathcal{L}) \neq 0$. Choose $f \in \Gamma(X, \mathcal{L})\backslash\{0\}$ arbitrarily. As – in the notations of (5.3) B) – we have $H^0(X, \mathcal{L}^{\otimes -1}) = H^2(D_f, \mathcal{O}_{D_f}) = 0$, the short exact sequence of (5.3) B) together with (4.9) gives rise to an exact sequence

$$\begin{aligned} 0 \longrightarrow H^1(X, \mathcal{L}^{\otimes -1}) &\xrightarrow{f\cdot} H^1(X, \mathcal{O}_X) \longrightarrow H^1(D_f, \mathcal{O}_{D_f}) \\ \longrightarrow H^2(X, \mathcal{L}^{\otimes -1}) &\xrightarrow{f\cdot} H^2(X, \mathcal{O}_X) \longrightarrow 0\,. \end{aligned}$$

By (5.3) B) we have $h^1(D_f, \mathcal{O}_{D_f}) = \sigma_{\mathcal{L}}(X)$ so that $h^1(X, \mathcal{O}_X) - h^1(X, \mathcal{L}^{\otimes -1})$ and $h^2(X, \mathcal{L}^{\otimes -1}) - h^2(X, \mathcal{O}_X)$ are both $\leq \sigma_{\mathcal{L}}(X) < h^0(X, \mathcal{L}) - 1$. If we make run f through all of $\Gamma(X, \mathcal{L})\backslash\{0\}$ we now may conclude from (3.7) a) that $H^1(X, \mathcal{L}^{\otimes -1}) = 0$ and from (3.7) b) that $H^2(X, \mathcal{O}_X) = 0$. Now, the above sequence gives statement c). In particular we have $h^1(X, \mathcal{O}_X) \leq \sigma_{\mathcal{L}}(X) < h^0(X, \mathcal{L}) - 1$ and so, statement a) follows from (4.7) b). Finally, statement b) is a consequence of the epimorphisms $H^2(X, \mathcal{L}^{\otimes n}) \xrightarrow{f} H^2(X, \mathcal{L}^{\otimes (n+1)}) \to 0$ for all $n \in \mathbb{Z}$ and all $f \in \Gamma(X, \mathcal{O}_X)\backslash\{0\}$. $\square$

Corollary 5.5. *Let X be a projective surface over K which has only finitely many non-normal points and let $e^1(X)$ be as in* (5.1). *Assume that* $0 \leq \sigma_{\mathcal{L}}(X) < h^0(X, \mathcal{L}) - 1$. *Then*:

a) $h^1(X, \mathcal{L}^{\otimes n}) = e^1(X)$ *for all* $n < 0$.

b) $h^2(X, \mathcal{L}^{\otimes n}) = 0$ *for all* $n \geq 0$.

c) $h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{L}^{\otimes -1}) = \sigma_{\mathcal{L}}(X) + e^1(X)$.

Proof. Let $\nu : \tilde{X} \to X$ be the normalization of X and let $Z \subseteq X$ be the finite set of non-normal points of X and consider the short exact sequence $0 \to \mathcal{O}_X \xrightarrow{\nu^\#} \mathcal{O}_{\tilde{X}} \to \mathcal{F} \to 0$ in which $\mathcal{F}$ is a coherent sheaf with support Z and of length $e^1(X)$ as we have seen in the proof of (5.2). We know already from that same proof that $\nu^*\mathcal{L}$ is an ample invertible sheaf of $\mathcal{O}_{\tilde{X}}$-modules and that $h^i(X, \nu_*\mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}X} \mathcal{L}^{\otimes n}) = h^i(\tilde{X}, (\nu^*\mathcal{L})^{\otimes n})$ for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$. So, we may apply cohomology to the exact sequences

$$0 \longrightarrow \mathcal{L}^{\otimes n} \longrightarrow \nu_*\mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \longrightarrow 0$$

in order to get $h^2(\tilde{X},(\nu^*\mathcal{L})^{\otimes n}) = h^2(X,\mathcal{L}^{\otimes n})$ for all $n \in \mathbb{Z}$. Moreover we have $h^0(X,(\nu^*\mathcal{L})^{\otimes n}) = h^0(X,\mathcal{L}^{\otimes n}) = 0$ for all $n \leq 0$. Finally we know from the proof of (5.2) that $h^1(X,\mathcal{L}^{\otimes n}) = e^1(X) + h^1(\tilde{X},(\nu^*\mathcal{L})^{\otimes n})$ for all $n \leq 0$.

Thus, we get

$$\begin{aligned}\sigma_{\nu^*\mathcal{L}}(\tilde{X}) &= \chi((\nu^*\mathcal{L})^{\otimes -1}) - \chi(\mathcal{O}_{\tilde{X}}) + 1\\ &= h^2(X,\mathcal{L}^{\otimes -1}) - \big(h^1(X,\mathcal{L}^{\otimes -1}) - e^1(X)\big)\\ &\qquad - h^2(X,\mathcal{O}_X) + \big(h^1(X,\mathcal{O}_X) - e^1(X)\big) - 1 + 1\\ &= \chi(\mathcal{L}^{\otimes -1}) - \chi(\mathcal{O}_X) + 1 = \sigma_{\mathcal{L}}(X)\,.\end{aligned}$$

Finally, if we apply the sequence $(*)$ of the proof of (5.2) with $n = 1$, we see that $h^0(\tilde{X},\nu^*\mathcal{L}) \geq h^0(X,\mathcal{L})$ and hence that $\sigma_{\nu_*\mathcal{L}}(\tilde{X}) \leq h^0(\tilde{X},\nu_*\mathcal{L}) - 1$.

Now, we get our claims if we apply (5.4) to the normal surface $\tilde{X}$ and the ample invertible sheaf of $\mathcal{O}_{\tilde{X}}$-modules $\nu^*\mathcal{L}$. ∎

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