T. Kubota and M. Yoshida
Nagoya Math. J.
Vol. 163 (2001), 1–11

A NOTE ON THE CONGRUENT DISTRIBUTION OF THE NUMBER OF PRIME FACTORS OF NATURAL NUMBERS

TOMIO KUBOTA AND MARIKO YOSHIDA

Abstract. Let $n = p_1 p_2 \cdots p_r$ be a product of r prime numbers which are not necessarily different. We define then an arithmetic function $\mu_m(n)$ by

$$\mu_m(n) = \rho^r \quad (\rho = e^{2\pi i/m}),$$

where m is a natural number. We further define the function $L(s, \mu_m)$ by the Dirichlet series

$$L(s, \mu_m) = \sum_{n=1}^{\infty} \frac{\mu_m(n)}{n^s} = \prod_p \left(1 - \frac{\rho}{p^s}\right)^{-1} \quad (\operatorname{Re} s > 1),$$

and will show that $L(s, \mu_m)$, $(m \geq 3)$, has an infinitely many valued analytic continuation into the half plane $\operatorname{Re} s > 1/2$.

§1. Introduction

Let $n = p_1 p_2 \cdots p_r$ be a product of r prime numbers which are not necessarily different. We define then an arithmetic function $\mu_m(n)$ by

$$\mu_m(n) = \rho^r \quad (\rho = e^{2\pi i/m}),$$

where m is a natural number. In the case of $m = 2$, $\mu_2(n)$ is related to the Möbius function $\mu(n)$ as

$$\mu(n) = \begin{cases} \mu_2(n) & \text{if } p_1, p_2, \ldots, p_r \text{ are different,} \\ 0 & \text{otherwise.} \end{cases}$$

We define the function $L(s, \mu_m)$ by the Dirichlet series

$$L(s, \mu_m) = \sum_{n=1}^{\infty} \frac{\mu_m(n)}{n^s} \quad (\operatorname{Re} s > 1). \tag{1}$$

Received January 26, 2000.
2000 Mathematics Subject Classification: Primary: 11N37, Secondary: 11M41.

On the other hand, we denote by $N_j(x)$ $(0 \leq j < m)$, the number of natural numbers n satisfying $\mu_m(n) = \rho^j$ and $n \leq x$ for a given positive real number x. Namely, N_j is the number of those natural numbers n not exceeding x which are products of r prime factors such that $r \equiv j \pmod{m}$.

The Dirichlet series $L(s, \mu_m)$ has the Euler product

$$L(s, \mu_m) = \prod_p \Big(1 - \frac{\rho}{p^s}\Big)^{-1}. \tag{2}$$

If $m = 2$, we have the equality

$$\begin{aligned} L(s, \mu_2) &= \prod_p \Big(1 + \frac{1}{p^s}\Big)^{-1} = \prod_p \Big(1 - \frac{1}{p^{2s}}\Big)^{-1} \prod_p \Big(1 - \frac{1}{p^s}\Big) \\ &= \zeta(2s)\zeta(s)^{-1} \end{aligned} \tag{3}$$

with the Riemann zeta function $\zeta(s)$. Accordingly, $L(s, \mu_2)$ is closely related to $\zeta(s)^{-1}$. If, however, $m \geq 3$, then $L(s, \mu_m)$ is of considerably different nature, although a product formula

$$\begin{aligned} \prod_{k=1}^{m-1} L(s, \mu_m^k) &= \prod_{k=1}^{m-1} \prod_p \Big(1 - \frac{\rho^k}{p^s}\Big)^{-1} = \prod_p \Big(1 - \frac{1}{p^{ms}}\Big)^{-1} \prod_p \Big(1 - \frac{1}{p^s}\Big) \\ &= \zeta(ms)\zeta(s)^{-1} \end{aligned}$$

similar to (3) holds, where

$$L(s, \mu_m^k) = \sum_{n=1}^{\infty} \frac{\mu_m(n)^k}{n^s}$$

The aim of the present paper is to investigate $L(s, \mu_m)$ to some extent, and, as a result, to obtain some informations on $N_0, N_1, \ldots, N_{m-1}$.

§2. Analytic continuation of $L(s, \mu_m)$

In this article, we will show that $L(s, \mu_m)$, $(m \geq 3)$, has an infinitely many valued analytic continuation into the half plane $\operatorname{Re} s > 1/2$.

Theorem 1. *Assume* $\operatorname{Re} s_0 > 1$ *and let* C *be a smooth path starting from* s_0*, remaining in the half plane* $\operatorname{Re} s > 1/2$*, and not passing* 1 *or any zero of* $\zeta(s)$*. Then,* $L(s, \mu_m)$ *in* (1) *can be continued analytically along* C*.*

The analytic continuation thus obtained can be expressed as

$$L(s, \mu_m) = \zeta(s)^{\rho} G(s)$$

in the half plane $\operatorname{Re} s > 1/2$*, where* $G(s)$ *is a holomorphic function in the same region.*

Proof. For a moment, suppose $\operatorname{Re} s > 1$. Then,

$$\frac{d}{ds} \log \zeta(s) = -\sum_p \frac{d}{ds} \log\Big(1 - \frac{1}{p^s}\Big) = -\sum_p \frac{\frac{1}{p^s}}{1 - \frac{1}{p^s}} \log p = -\sum_p \sum_{k=1}^{\infty} \frac{\log p}{p^{ks}},$$

and by (2)

$$\begin{aligned} \frac{d}{ds} \log L(s, \mu_m) &= -\sum_p \frac{d}{ds} \log\Big(1 - \frac{\rho}{p^s}\Big) = -\sum_p \frac{\frac{\rho}{p^s}}{1 - \frac{\rho}{p^s}} \log p \\ &= -\sum_p \sum_{k=1}^{\infty} \frac{\rho^k \log p}{p^{ks}}. \end{aligned}$$

Therefore, if we put

$$F(s) = \frac{d}{ds} \log L(s, \mu_m) - \rho \frac{d}{ds} \log \zeta(s),$$

then

$$F(s) = \sum_p \sum_{k=2}^{\infty} \frac{(\rho - \rho^k) \log p}{p^{ks}}.$$

This series is absolutely convergent in the half plane $\operatorname{Re} s > 1/2$, because $|\rho - \rho^k| \le 2$ and

$$\sum_p \sum_{k=2}^{\infty} \frac{\log p}{p^{k\sigma}} = \sum_p \frac{\frac{1}{p^{2\sigma}}}{1 - \frac{1}{p^{\sigma}}} \log p < \frac{\sqrt{2} - 1}{\sqrt{2}} \sum_p \frac{\log p}{p^{2\sigma}} < \sum_{n=1}^{\infty} \frac{\log n}{n^{2\sigma}} < \infty$$

with $\sigma = \operatorname{Re} s > 1/2$. Thus, $F(s)$ is holomorphic and one-valued in the half plane $\operatorname{Re} s > 1/2$, and the analytic continuation of $L(s, \mu_m)$ is obtained by

$$\begin{aligned} L(s, \mu_m) &= L(s_0, \mu_m) \exp\Big(\int_{s_0}^{s} \Big[\rho \frac{d}{ds} \log \zeta(s) + F(s)\Big] \, ds \Big) \\ &= \exp\Big(\int_{s_0}^{s} \rho \frac{d}{ds} \log \zeta(s) \, ds \Big) \cdot L(s_0, \mu_m) \cdot \exp\Big(\int_{s_0}^{s} F(s) \, ds \Big). \end{aligned}$$

Furthermore, since

$$\exp\Big(\int_{s_0}^{s} \rho \frac{d}{ds} \log \zeta(s)\, ds\Big) = \frac{\zeta(s)^{\rho}}{\zeta(s_0)^{\rho}}$$

holds as an equality between many valued functions, we can correspondingly put

$$G(s) = \zeta(s_0)^{-\rho} L(s_0, \mu_m) \cdot \exp\Big(\int_{s_0}^{s} F(s)\, ds\Big),$$

where $\zeta(s_0)^{\rho}$ is uniquely determined by the Euler product of $\zeta(s)$ as

$$\zeta(s_0)^{\rho} = \exp\Big(-\rho \sum_{p} \log\Big(1 - \frac{1}{p^{s_0}}\Big)\Big).$$

In these formulas, s is an arbitrary complex number with $\operatorname{Re} s > 1/2$, the integral is taken along the path C, and s is the end of C. Hence, we obtain the assertion of the theorem. □

Corollary. *If $m \geq 3$, then $L(s, \mu_m)$ is many valued, and has a logarithmic singularity like $(s-1)^{-\rho}$ at 1, and has logarithmic singularities also at zeros of $\zeta(s)$ in the half plane* $\operatorname{Re} s > 1/2$.

Remark 1. The only pole of $\frac{d}{ds} \log \zeta(s)$ is 1, and the residue is -1. At a zero of $\zeta(s)$ of order g, $\frac{d}{ds} \log \zeta(s)$ has a pole of order 1, and the residue is g. Accordingly, $\rho \frac{d}{ds} \log \zeta(s)$ has a pole of order 1 at 1 and at zeros of $\zeta(s)$, and the residues are $-\rho$ and $g\rho$, respectively. If C in the Theorem turns around 1 once in the positive direction, then the analytic continuation is multiplied by $e^{-2\pi i \rho}$. If C turns around a zero of $\zeta(s)$ of order g once in the positive direction, then the analytic continuation is multiplied by $e^{2\pi i g \rho}$.

If $m = 2$, then $-\rho$ and $g\rho$ in the Theorem are rational integers so that $L(s, \mu_2)$ is one-valued.

Theorem 2. *Assume $m \geq 3$, then an asymptotic formula of the form*

$$\sum_{n \leq x} \mu_m(n) = o(x^{\alpha}) \tag{4}$$

can not hold, whenever $\alpha < 1$.

Proof. Put

$$S_m(\alpha) = \sum_{n \le x} \mu_m(n)$$

and suppose that (4) is true for an $\alpha < 1$. Then,

$$\sum_{n=1}^{\infty} \frac{\mu_m(n)}{n^s} = \sum_{n=1}^{\infty} \frac{S_m(n) - S_m(n-1)}{n^s} = \sum_{n=1}^{\infty} S_m(n)\Big(\frac{1}{n^s} - \frac{1}{(n+1)^s}\Big) \tag{5}$$

and there exists a positive constant c_1 such that

$$\big|S_m(n)\big| < c_1 n^{\alpha}$$

for all $n \ge 1$. Therefore, we have

$$S_m(n)\Big(\frac{1}{n^s} - \frac{1}{(n+1)^s}\Big) = S_m(n)\,\frac{1}{s}\int_n^{n+1} x^{-s-1}\,dx$$

and

$$\Big|S_m(n)\Big(\frac{1}{n^s} - \frac{1}{(n+1)^s}\Big)\Big| \le c_1 n^{\alpha} \cdot \frac{1}{s}\, n^{-s-1}$$

so that the series in (1) converges at $s = \alpha + \varepsilon$ for every $\varepsilon > 0$. It follows from this and from the general theory of Dirichlet series that $L(s, \mu_m)$ is holomorphis and one-valued in the half plane $\operatorname{Re} s > \alpha$, which is a contradiction, because $s = 1$ is a logarithmic singularity of $L(s, \mu_m)$ as shown in the Corollary. ∎

§3. Asymptotic properties of $N_j(x)$, $(j = 0, 1, \ldots, m-1)$

A version of the prime number theorem is

$$\sum_{n \le x} \mu(n) = o(x) \tag{6}$$

and Riemann's hypothesis is equivalent to the fact that

$$\sum_{n \le x} \mu(x) = o(x^{\alpha}) \tag{7}$$

holds for every $\alpha > 1/2$.

If $m = 2$, the following theorem on the asymptotic behavior of N_0, N_1 is easily deduced from (6) and (7). For the sake of completeness, we state the fact as a theorem with proof.

THEOREM 3. *Let $N_0(x)$ and $N_1(x)$ be as in §1 with $m = 2$. Then, both $N_0(x)$ and $N_1(x)$ are $\frac{1}{2}x + o(x)$. If (7) is true, then both $N_0(x)$ and $N_1(x)$ are $\frac{1}{2}x + o(x^\alpha)$.*

Proof. It is enough to treat the second assertion. The definitions of $N_0(x)$, $N_1(x)$ and $\mu_2(x)$ imply

$$\sum_{n\le x} \mu_2(x) = N_0(x) - N_1(x).$$

On the other hand, since

$$\mu_2(n) = \sum_{k^2 \mid n} \mu\Big(\frac{n}{k^2}\Big) \tag{8}$$

the left hand side of (7) is equal to

$$\sum_{n\le x} \sum_{k^2\mid n} \mu\Big(\frac{n}{k^2}\Big) = \sum_{k=1}^{\infty} \sum_{n'\le x/k^2} \mu(n') \quad (n = k^2 n').$$

So, putting

$$S(x) = \sum_{n\le x} \mu(x), \quad S_2(x) = \sum_{n\le x} \mu_2(x),$$

we have

$$S_2(x) = \sum_{k=1}^{\infty} S\Big(\frac{x}{k^2}\Big). \tag{9}$$

Under the assumption (7), there is a constant c_2 such that

$$\big|S(x)\big| < c_2 x^\alpha$$

for all $x > 0$. Hence,

$$\big|x^{-\alpha} S_2(x)\big| \le \sum_{k=1}^{\infty} x^{-\alpha} \Big|S\Big(\frac{x}{k^2}\Big)\Big| \tag{10}$$

and

$$x^{-\alpha}\Big|S\Big(\frac{x}{k^2}\Big)\Big| \le c_2 x^{-\alpha}\Big(\frac{x}{k^2}\Big)^\alpha = c_2 \frac{1}{k^{2\alpha}}.$$

This means that the series in (10) has an absolutely convergent majorant, and, again by the assumption (7), each term $x^{-\alpha}\left|S\left(x/k^2\right)\right|$ tends to 0 as $x \to \infty$. Thus, we have

$$\lim_{x\to\infty} x^{\alpha} S_2(x) = 0$$

or

$$N_0(x) - N_1(x) = o(x^{\alpha}) \tag{11}$$

as desired.

Now, it is clear that

$$N_0(x) + N_1(x) \sim x \tag{12}$$

Therefore the Theorem follows from (11) and (12). □

Remark 2. Applying Möbius' inversion formula to (9), it is also shown similarly to the above proof that $S_2(x) = o(x^{\alpha})$ implies $S(x) = o(x^{\alpha})$.

We can ask whether similar asymptotic properties as appeared in Theorem 3 exist or not for $N_j(x)$ defined in §1, too. It seems to be fairly hard to answer this kind of question. But, at least, the following theorem is valid:

THEOREM 4. *If $m \geq 3$, and if asymptotic formulas of the form*

$$N_j(x) = \nu_j x + o(x^{\alpha}) \quad (\alpha < 1)$$

exist for all $j = 0, 1, \ldots, m-1$, then $\nu_0, \nu_1, \ldots, \nu_{m-1}$ can not be all equal.

Proof. Assume $\nu_0 = \nu_1 = \cdots = \nu_{m-1}$. Then, since

$$S_m(x) = \sum_{j=0}^{m-1} \rho^j N_j(x),$$

it turns out that

$$S_m(x) = o(x^{\alpha})$$

which contradicts Theorem 2. □

§4. A computational experiment

Theorem 4 denies the possibility to get an asymptotic formula $N_j(x) = \frac{1}{m}x + o(x^\alpha)$ $(j = 0, 1, \ldots, m-1)$ for $\alpha < 1$. But, for $\alpha = 1$, the possibility still remains. While, in this direction, the authors still do not possess any theoretical results, they made an experiment by Mathematica in order to examine the behavior of $\frac{1}{x}N_j(x)$ as $x \to \infty$ in the case of $m = 3$. The computation up to $x = 3 \times 10^8$ made it plausible that $\frac{1}{x}N_j(x)$, $(j = 0, 1, 2)$, tend to a common limit $1/3$.

From Theorem 3, we see that, roughly speaking, the distribution of the number of prime factors of natural numbers is uniform modulo 2. Theorem 4 shows, however, the distribution is not uniform modulo m, $(m \geq 3)$, if the uniformity is defined rather in a strong sense that $\alpha < 1$. Nevertheless, the above computational data allude that the distribution in question is uniform in the weaker sense with $\alpha = 1$.

Computational investigation supplies some more facts. Fig. 1 shows the behavior of

$$\frac{1}{\sqrt[3]{x}} \sum_{n \leq x} \mu_3(n). \tag{13}$$

up to 3×10^8 plotted at every 10^4 of n. While

$$\frac{1}{x} \sum_{n \leq x} \mu_3(n) \tag{14}$$

probably tends to 0 as $x \to \infty$, the product of (14) and $x^{3/2}$ draws the curve in Fig. 1. So, Fig. 1 shows the behavior of (14) magnified by the factor $x^{3/2}$. It is remarkable that the curve in Fig. 1 is fairly smooth.

Fig. 2 shows the behavior of

$$\frac{1}{\sqrt{x}} \sum_{n \leq x} \mu_2(n) \tag{15}$$

up to 3×10^8 plotted at every 10^4 of n. The result is far more disorderly than Fig. 1.

Tables 1 and 2 show actual values of (13) and (15) restricting n to multiples of 10^7.

Figure 1:

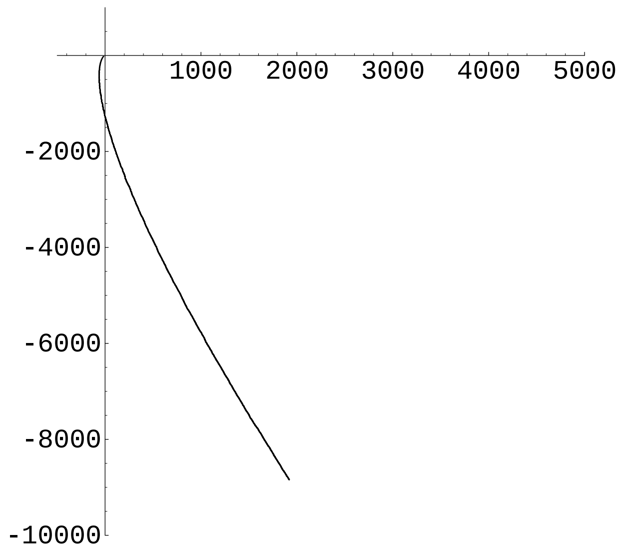

Figure 2:

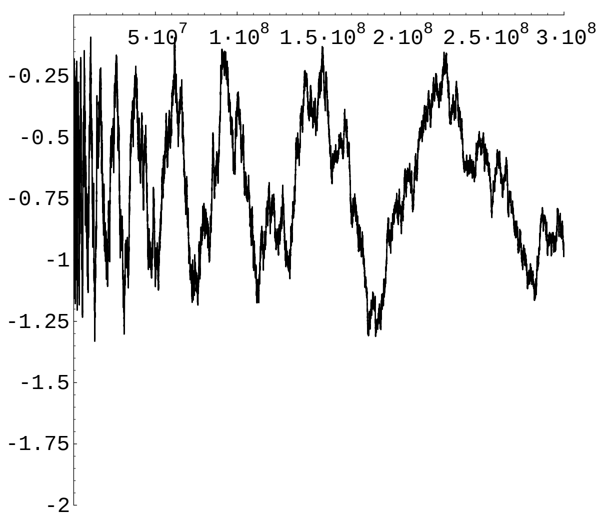

Table 1:

$n(\times 10^7)$	$\frac{1}{\sqrt[3]{x}}\sum_{n\le x}\mu_3(n)$	$n(\times 10^7)$	$\frac{1}{\sqrt[3]{x}}\sum_{n\le x}\mu_3(n)$
1	$-5.6210-1235.5215i$	16	$1103.9154-6144.6615i$
2	$85.6279-1846.3794i$	17	$1165.7678-6362.9139i$
3	$172.1854-2334.9107i$	18	$1230.5480-6576.5930i$
4	$254.8574-2752.9714i$	19	$1291.5799-6785.4313i$
5	$334.6198-3134.9381i$	20	$1349.3514-6990.6826i$
6	$411.6768-3481.0532i$	21	$1409.9050-7187.6113i$
7	$489.1593-3809.0565i$	22	$1466.2162-7384.5319i$
8	$559.2476-4115.3894i$	23	$1525.3580-7579.0326i$
9	$632.8016-4401.6940i$	24	$1588.7635-7768.3007i$
10	$703.6815-4680.1121i$	25	$1644.7499-7952.1305i$
11	$777.5754-4945.3833i$	26	$1700.4838-8135.4568i$
12	$839.3834-5201.3469i$	27	$1756.7184-8316.9325i$
13	$910.2625-5448.3369i$	28	$1812.3899-8494.9559i$
14	$973.7787-5683.9184i$	29	$1865.6500-8667.6974i$
15	$1039.1155-5914.1483i$	30	$1920.8040-8841.0852i$

Table 2:

$n(\times 10^7)$	$\frac{1}{\sqrt{x}}\sum_{n\le x}\mu_2(n)$	$n(\times 10^7)$	$\frac{1}{\sqrt{x}}\sum_{n\le x}\mu_2(n)$
1	-0.266264	16	-0.569684
2	-1.00847	17	-0.820346
3	-1.02789	18	-1.26785
4	-0.5047	19	-1.12899
5	-1.07593	20	-0.786727
6	-0.389364	21	-0.669226
7	-0.788612	22	-0.331841
8	-0.797382	23	-0.401299
9	-0.229371	24	-0.61929
10	-0.3884	25	-0.504067
11	-0.968909	26	-0.597103
12	-0.784339	27	-0.8979
13	-0.978972	28	-1.031
14	-0.476329	29	-0.95987
15	-0.264871	30	-0.961173

Added in proof. In the meantime, the author proved $\sum_{n \le x} \mu_3(n) = o(x)$, and found a more precise asymptotic formula

$$\sum_{n \le x} \mu_3(n) \sim \Gamma(\rho)^{-1} \eta_0 \cdot x (\log x)^{\rho - 1}$$

plausible.

References

[1] H. M. Edwards, Riemann's zeta function, Academic Press, 1974.
[2] S. Lang, Algebraic number theory, Addison Wesley, 1970.

Tomio Kubota
Department of Mathematics
Meijo University
Shiogamaguchi 1-501
Tempaku-ku
Nagoya, 468-8502
Japan

Mariko Yoshida
Department of Mathematics
Meijo University
Shiogamaguchi 1-501
Tempaku-ku
Nagoya, 468-8502
Japan