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POSITIVE SOLUTIONS OF NONRESONANT SINGULAR BOUNDARY VALUE PROBLEM OF SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. This paper investigates the existence of positive solutions of nonresonant singular boundary value problem of second order differential equations. A necessary and sufficient condition for the existence of $C[0,1]$ positive solutions as well as $C^1[0,1]$ positive solutions is given by means of the method of lower and upper solutions with the fixed point theorems.

§1. Introduction

The theory of singular boundary value problems has become an important area of investigation in recent years (see [1-7] and the references therein). Consider the singular boundary value problems of second order ordinary differential equation

$$(1.1)\qquad \begin{cases} -x'' + \rho p(t)x = f(t,x), & t \in (0,1), \\ ax(0) - bx'(0) = 0, & cx(1) + dx'(1) = 0, \end{cases}$$

where $\rho > 0$ is such that

$$(1.2)\qquad \begin{cases} -x'' + \rho p(t)x = 0, & t \in (0,1), \\ ax(0) - bx'(0) = 0, & cx(1) + dx'(1) = 0 \end{cases}$$

has only the trivial solution, and where $p(t) \in C(0,1),\ p(t) \geq 0,\ t \in (0,1)$, $a \geq 0,\ b \geq 0,\ c \geq 0,\ d \geq 0,\ a+b > 0,\ c+d > 0,\ \delta = ac + ad + bc > 0$. For convenience, we list the following hypothesis.

(H_1)

$$(1.3)\qquad \int_0^1 t(1-t)p(t)\,dt < \infty; \quad \text{also}$$

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$$\lim_{t\to 0^+} t^2 p(t) = 0 \;\text{ if }\; \int_0^1 (1-t)p(t)dt = \infty; \quad \text{and} \tag{1.4}$$

$$\lim_{t\to 1^-} (1-t)^2 p(t) = 0 \;\text{ if }\; \int_0^1 tp(t)dt = \infty; \tag{1.5}$$

(H_2)

$$\int_0^1 tp(t)dt < \infty; \quad \text{also} \tag{1.6}$$

$$\lim_{t\to 0^+} t^2 p(t) = 0 \;\text{ if }\; \int_0^1 p(t)dt = \infty; \tag{1.7}$$

(H_3)

$$\int_0^1 (1-t)p(t)dt < \infty; \quad \text{also} \tag{1.8}$$

$$\lim_{t\to 1^-} (1-t)^2 p(t) = 0 \;\text{ if }\; \int_0^1 p(t)dt = \infty; \tag{1.9}$$

(H_4)

$$\int_0^1 p(t)dt < \infty; \tag{1.10}$$

(H_5) $f(t,x) \in C((0,1)\times(0,+\infty),[0,+\infty))$, $f(t,1) \not\equiv 0$ for $t \in (0,1)$, and there exist constants $\lambda,\ \mu,\ N,\ M(-\infty < \lambda < 0 < \mu < 1,\ 0 < N \le 1 \le M)$, such that, for $t \in (0,1)$ and $x \in (0,+\infty)$,

$$\ell^\mu f(t,x) \le f(t,\ell x) \le \ell^\lambda f(t,x) \quad \text{if }\; 0 \le \ell \le N\;; \tag{1.11}$$

$$\ell^\lambda f(t,x) \le f(t,\ell x) \le \ell^\mu f(t,x) \quad \text{if }\; \ell \ge M. \tag{1.12}$$

Typical functions that satisfy the above sublinear hypothesis are those taking the form

$$f(t,x) = \sum_{k=1}^n p_k(t) x^{\lambda_k}\;;$$

here $p_k(t) \in C(0,1)$, $p_k(t) > 0$ on $(0,1)$, $\lambda_k < 1$, $k = 1,\ 2,\dots,n$.

By singularity we mean that the functions $p,\ f$ in (1.1) are allowed to be unbounded at the end points $t=0$ and $t=1$. A function $x(t)\in C[0,1]\cap C^2(0,1)$ is called a $C[0,1]$ (positive) solution of (1.1) if it satisfies (1.1) ($x(t)>0$ for $t\in(0,1)$). A $C[0,1]$ (positive) solution of (1.1) is called a $C^1[0,1]$ (positive) solution if $x'(0^+)$ and $x'(1^-)$ both exist ($x(t)>0$ for $t\in(0,1)$).

In the special cases i): $b=d=0,\ p(t)=0,\ f(t,x)=p_1(t)x^{-\lambda_1},\ \lambda_1>0$ and ii): $b=d=0,\ p(t)=0,\ f(t,x)=p_1(t)x^{\lambda_1},\ 0<\lambda_1<1$, where $p_1(t)\in C(0,1),\ p_1(t)>0$ on $(0,1)$, the existence and uniqueness of positive solutions of (1.1) have been studied completely by Taliaferro in [3] with the shooting method and by Zhang in [4] with the method of lower and upper solutions, respectively. A sufficient condition for the existence of $C[0,1]$ solutions of the singular problem (1.1) in the case $b=d=0$ was given by D. O'Regan in [5] with a continuous theorem. In the special cases iii): $p(t)=0,\ f(t,x)=p_1(t)x^{-\lambda_1},\ \lambda_1>0$ and iv): $p(t)=0,\ f(t,x)=p_1(t)x^{\lambda_1},\ 0<\lambda_1<1$, where $p_1(t)\in C(0,1),\ p_1(t)>0$ on $(0,1)$, the existence of positive solutions of (1.1) has been studied by Wei in [6] and [7] with the method of lower and upper solutions.

Now, in this paper, we shall give a necessary and sufficient condition for the existence of $C[0,1]$ positive solutions as well as $C^1[0,1]$ positive solutions of the singular problem (1.1) by using the method of lower and upper solutions with the fixed point theorems, which is different from that of [3-5].

§2. Several lemmas

Lemma 1. *Suppose* (H$_1$) *holds.*

(i) *Then*

$$\begin{cases} -x''+\rho p(t)x=0, & t\in(0,1),\\ x(0)=0,\quad x'(0)=1 \end{cases} \tag{2.1}$$

has a unique positive increasing solution $e_1(t)\in C[0,1]\cap C^1[0,1)$.

(ii) *Then*

$$\begin{cases} -x''+\rho p(t)x=0, & t\in(0,1),\\ x(1)=0,\quad x'(1)=-1 \end{cases} \tag{2.2}$$

has a unique positive decreasing solution $e_2(t)\in C[0,1]\cap C^1(0,1]$.

In addition, if (H$_2$) *holds, then* $e_1(t)\in C^1[0,1]$*; if* (H$_3$) *holds, then* $e_2(t)\in C^1[0,1]$*; therefore, if* (H$_4$) *holds, then* $e_1(t),\ e_2(t)\in C^1[0,1]$.

Proof. Similar to that of Theorem 2.1 in [5], we can obtain that there exists a unique $w_1 \in C[0,1]$ with

$$w_1(t) = 1 + \frac{\rho}{t}\int_0^t \int_0^s \tau p(\tau) w_1(\tau) d\tau ds \tag{2.3}$$

and $e_1(t) = tw_1(t) \in C[0,1] \cap C^1[0,1)$ is a solution of (2.1). In the following, we shall prove that $e_1(t)$ is a positive increasing function. In fact, if $e_1(t)$ is not increasing, then from $e_1(0) = 0,\ e_1'(0) = 1,$ there exist positive numbers $0 < t^* < \eta < 1$ such that $e_1'(t^*) < 0$ and $e_1(t) > 0$ for $t \in (0, \eta)$. Therefore,

$$\int_0^{t^*} \left(-e_1''(t) + \rho p(t) t w_1(t)\right) dt \geq -e_1'(t^*) + 1 > 0,$$

which contradicts

$$-e_1''(t) + \rho p(t) t w_1(t) = 0, \quad t \in (0,1).$$

Hence, $e_1(t)$ is an increasing function. From $e_1(t) > 0$ for $t \in (0, \eta)$, we have $e_1(t) > 0$ for $t \in [0,1]$. Consequently, $w_1(t) \geq 0$ for $t \in [0,1]$ and $w_1(1) = e_1(1) > 0$.

Similarly, we can obtain that there exists a nonnegative function $w_2 \in C[0,1]$ with

$$w_2(t) = 1 + \frac{\rho}{1-t}\int_t^1 \int_s^1 (1-\tau) p(\tau) w_2(\tau) d\tau ds \tag{2.4}$$

and $e_2(t) = (1-t)w_2(t) \in C[0,1] \cap C^1(0,1]$ is a positive decreasing solution of (2.2).

Obviously, if (H_2) holds, then $e_1(t) \in C^1[0,1]$; if (H_3) holds, then $e_2(t) \in C^1[0,1]$; therefore, if (H_4) holds, then $e_1(t),\ e_2(t) \in C^1[0,1]$. The proof is complete.

Remark 1. If $p(t) = 0$, then $e_1(t) = t$, $e_2(t) = 1 - t$, $w_1(t) = w_2(t) = 1$.

By Lemma 1, we can obtain Lemma 2.

LEMMA 2. (i) *Suppose that* (H_3) *holds. Then*

$$u(t) = (ae_2(0) - be_2'(0))e_1(t) + be_2(t) \in C[0,1] \cap C^1[0,1) \tag{2.5}$$

is a positive increasing solution of the following problem

$$\begin{cases} -x'' + \rho p(t)x = 0, & t \in (0,1), \\ ax(0) - bx'(0) = 0. \end{cases} \tag{2.6}$$

(ii) *Suppose that* (H_2)*holds. Then*

$$v(t) = de_1(t) + (ce_1(1) + de_1'(1))e_2(t) \in C[0,1] \cap C^1(0,1] \tag{2.7}$$

is a positive decreasing solution of the following problem

$$\begin{cases} -x'' + \rho p(t)x = 0, & t \in (0,1), \\ cx(1) + dx'(1) = 0. \end{cases} \tag{2.8}$$

In addition, if (H_4) *holds, then* $u(t),\ v(t) \in C^1[0,1]$ *and the Wronskian*

$$\omega = \omega(t) = \begin{vmatrix} v(t) & v'(t) \\ u(t) & u'(t) \end{vmatrix} = \mathit{constant} > 0, \tag{2.9}$$

where $e_1(t)$ *and* $e_2(t)$ *are given by Lemma* 1.

LEMMA 3. *Suppose that* (H_4) *holds. Let* $x(t)$ *be a* $C^1[0,1]$ *positive solution of* (1.1)*. Then there are constants* I_1 *and* I_2*,* $0 < I_1 < I_2$*, such that*

$$I_1 u(t)v(t) \le x(t) \le I_2 u(t)v(t), \quad t \in [0,1], \tag{2.10}$$

where $u(t)$ *and* $v(t)$ *are given by Lemma* 2.

Proof. Assume that $x(t)$ is a $C^1[0,1]$ positive solution of (1.1). Then $x'(0) \ge 0$ and $x'(1) \le 0,\ x(t) > 0$ for $t \in (0,1)$. By integration of (1.1), we have

$$\int_0^1 f(t,x(t))dt \le -x'(1) + x'(0) + \rho \max_{t\in[0,1]} |x(t)| \int_0^1 p(t)dt < \infty. \tag{2.11}$$

Let $t_0 \in (0,1)$ and let a_1 be a constant sufficiently small satisfying $x(t_0) - a_1u(t_0) \ge 0$, and let $y(t) = x(t) - a_1u(t),\ t \in [0,t_0]$. Then

$$\begin{cases} -y''(t) + \rho p(t)y(t) = f(t,x(t)) \ge 0, & t \in (0,t_0], \\ ay(0) - by'(0) = 0, \quad y(t_0) = x(t_0) - a_1u(t_0) \ge 0. \end{cases}$$

By the maximum principle, we have $y(t) \geq 0$ for $t \in [0, t_0]$. Therefore,

$$x(t) \geq a_1 u(t), \quad t \in [0, t_0]. \tag{2.12}$$

On the other hand, let a_2 be a constant sufficiently large such that

$$a_2 u(t_0) - x(t_0) = r_0,$$

$$r_0 \geq (2u(t_0)/\omega^*) \int_0^{t_0} y_2(0) f(s, x(s)) ds,$$

$$r_0 \geq (2u(t_0)/\omega^*) \int_0^{t_0} y_2(s) f(s, x(s)) ds.$$

Here, $y_2(t)$ is a unique decreasing positive solution of the problem

$$\begin{cases} -y''(t) + \rho p(t) y(t) = 0, & t \in (0, t_0], \\ y(t_0) = 0, \quad y'(t_0) = -1; \end{cases}$$

and

$$\omega^* = \begin{vmatrix} y_2(t) & y_2'(t) \\ u(t) & u'(t) \end{vmatrix} = \text{constant} > 0.$$

Let $y(t) = a_2 u(t) - x(t)$. Then

$$\begin{cases} -y''(t) + \rho p(t) y(t) = -f(t, x(t)), & t \in (0, t_0], \\ ay(0) - by'(0) = 0, \quad y(t_0) = a_2 u(t_0) - x(t_0) = r_0 > 0. \end{cases} \tag{2.13}$$

By (H_4), (2.11) and Theorem 2.2 in [5], (2.13) has a unique solution $y(t)$ satisfying

$$\begin{aligned} y(t) &= \frac{u(t)}{u(t_0)} r_0 - \frac{1}{\omega^*} \int_0^t y_2(t) u(s) f(s, x(s)) ds \\ &\quad - \frac{1}{\omega^*} \int_t^{t_0} y_2(s) u(t) f(s, x(s)) ds \\ &\geq u(t) \left[\frac{r_0}{2u(t_0)} - \frac{1}{\omega^*} \int_0^{t_0} y_2(0) f(s, x(s)) ds \right] \\ &\quad + u(t) \left[\frac{r_0}{2u(t_0)} - \frac{1}{\omega^*} \int_0^{t_0} y_2(s) f(s, x(s)) ds \right] \geq 0, \quad t \in [0, t_0]. \end{aligned}$$

Hence,

$$x(t) \leq a_2 u(t), \quad t \in [0, t_0]. \tag{2.14}$$

Similarly, we can verify that there exist two numbers b_1 and b_2 satisfying

$$b_1 v(t) \leq x(t) \leq b_2 v(t), \quad t \in [t_0, 1]. \tag{2.15}$$

For $t \in [0, t_0]$, from (2.12) and (2.14), we have

$$x(t) \geq \frac{a_1}{v(0)} v(0) u(t) \geq \frac{a_1}{v(0)} u(t) v(t), \tag{2.16}$$

$$x(t) \leq \frac{a_2}{v(t_0)} v(t_0) u(t) \leq \frac{a_2}{v(t_0)} u(t) v(t). \tag{2.17}$$

For $t \in [t_0, 1]$, from (2.15), we have

$$x(t) \geq \frac{b_1}{u(1)} u(1) v(t) \geq \frac{b_1}{u(1)} u(t) v(t), \tag{2.18}$$

$$x(t) \leq \frac{b_2}{u(t_0)} u(t_0) v(t) \leq \frac{b_2}{u(t_0)} u(t) v(t). \tag{2.19}$$

Let

$$I_1 = \min\left\{ \frac{a_1}{v(0)}, \quad \frac{b_1}{u(1)} \right\}, \quad I_2 = \max\left\{ \frac{a_2}{v(t_0)}, \quad \frac{b_2}{u(t_0)} \right\}.$$

Then, (2.16)–(2.19) imply that (2.10) holds. The proof of Lemma 3 is complete.

§3. Main results

A function $\alpha(t)$ is called a lower solution of (1.1) if $\alpha(t) \in C[0,1] \cap C^2(0,1)$, and satisfies

$$\begin{cases} -\alpha''(t) + \rho p(t) \alpha(t) \leq f(t, \alpha(t)), \quad t \in (0,1), \\ a\alpha(0) - b\alpha'(0) \leq 0, \qquad c\alpha(1) + d\alpha'(1) \leq 0. \end{cases}$$

Similarly, a function $\beta(t)$ is called an upper solution of (1.1) if $\beta(t) \in C[0,1] \cap C^2(0,1)$, and satisfies

$$\begin{cases} -\beta''(t) + \rho p(t) \beta(t) \geq f(t, \beta(t)), \quad t \in (0,1), \\ a\beta(0) - b\beta'(0) \geq 0, \qquad c\beta(1) + d\beta'(1) \geq 0. \end{cases}$$

Now, we state the main results of this paper which are the following two theorems.

THEOREM 3.1. *Suppose that* (H_4) *and* (H_5) *hold. Then a necessary and sufficient condition for problem* (1.1) *to have* $C^1[0,1]$ *positive solutions is that the following inequality holds*:

$$0<\int_0^1 f(t,e(t))dt<\infty, \tag{3.1}$$

where $e(t)=u(t)v(t)$, $u(t)$, $v(t)$ *are given by* (2.5), (2.7), *respectively.*

THEOREM 3.2. *Suppose* (H_5) *holds.*

I) *If* $b=d=0$, *and* (H_1) *holds, then a necessary and sufficient condition for problem* (1.1) *to have* $C[0,1]$ *positive solutions is that the following integral conditions hold*:

$$0<\int_0^1 t(1-t)f(t,1)dt<\infty, \qquad \textit{also} \tag{3.2}$$

$$\lim_{t\to 0^+} t\int_t^1 (1-s)f(s,1)ds=0 \quad \textit{if} \quad \int_0^1 (1-s)f(s,1)ds=\infty, \tag{3.3}$$

and

$$\lim_{t\to 1^-} (1-t)\int_0^t sf(s,1)ds=0 \quad \textit{if} \quad \int_0^1 sf(s,1)ds=\infty. \tag{3.4}$$

II) *If* $b=0$, $d>0$, *and* (H_2) *holds, then a necessary and sufficient condition for problem* (1.1) *to have* $C^1(0,1]$ *positive solutions is that the following integral conditions hold*:

$$0<\int_0^1 tf(t,1)dt<\infty, \qquad \textit{also} \tag{3.5}$$

$$\lim_{t\to 0^+} t\int_t^1 f(s,1)ds=0 \quad \textit{if} \quad \int_0^1 f(s,1)ds=\infty. \tag{3.6}$$

III) *If* $b>0$, $d=0$, *and* (H_3) *holds, then a necessary and sufficient condition for problem* (1.1) *to have* $C^1[0,1)$ *positive solutions is that the following integral conditions hold*:

$$0<\int_0^1 (1-t)f(t,1)dt<\infty, \qquad \textit{also} \tag{3.7}$$

$$\lim_{t\to 1^-} (1-t)\int_0^t f(s,1)ds=0 \quad \textit{if} \quad \int_0^1 f(s,1)ds=\infty. \tag{3.8}$$

Remark 2. When $b = d = 0,\ p(t) = 0,\ f(t,x) = p_1(t)x^{-\lambda_1},\ \lambda_1 > 0$, we obtain the main results of paper [3]. When $b = d = 0,\ p(t) = 0,\ f(t,x) = p_1(t)x^{\lambda_1},\ 0 < \lambda_1 < 1$, we get the Theorems 1 and 2 in paper [4]. When $p(t) = 0,\ f(t,x) = p_1(t)x^{-\lambda_1},\ \lambda_1 > 0$, we obtain the main results of paper [6]. When $p(t) = 0,\ f(t,x) = p_1(t)x^{\lambda_1},\ 0 < \lambda_1 < 1$, we get the Theorems 1 and 2 in paper [7].

The proof of Theorem 3.1.

1. Necessity. Suppose that $x(t)$ is a $C^1[0,1]$ positive solution of (1.1). Then both $x'(0) \geq 0$ and $x'(1) \leq 0$ exist. By Lemma 3, there are constants I_1 and $I_2,\ 0 < I_1 < I_2$ such that

$$I_1 e(t) \leq x(t) \leq I_2 e(t), \quad t \in [0,1]. \tag{3.9}$$

Let c_0 be a constant satisfying $c_0 I_2 \leq N,\ 1/c_0 \geq M$. Then (1.11), (1.12) and (3.9) lead to

$$\begin{aligned} f(t,x(t)) &\geq (1/c_0)^{\lambda} f\left(t,\ \frac{c_0 x(t)}{e(t)} e(t)\right) \\ &\geq (c_0)^{\mu-\lambda} \left(\frac{x(t)}{e(t)}\right)^{\mu} f(t,e(t)) \\ &\geq (c_0)^{\mu-\lambda} I_1^{\mu} f(t,e(t)), \quad t \in (0,1). \end{aligned}$$

Consequently,

$$\begin{aligned} &0 < \int_0^1 f(t,e(t))dt \leq (c_0)^{\lambda-\mu} I_1^{-\mu} \int_0^1 f(t,x(t))dt \\ &\leq (c_0)^{\lambda-\mu} I_1^{-\mu} \left(x'(0) - x'(1) + I_2 \rho v(0) u(1) \textstyle\int_0^1 p(t)dt\right) < \infty. \end{aligned}$$

Thus (3.1) holds.

2. Sufficiency. Suppose that (3.1) holds. Let

$$h(t) = \frac{v(t)}{\omega} \int_0^t u(s) f(s,e(s))ds + \frac{u(t)}{\omega} \int_t^1 v(s) f(s,e(s))ds, \quad t \in [0,1].$$

Then $h(t) \in C^1[0,1] \cap C^2(0,1)$ and (3.9) holds if $x(t)$ is replaced by $h(t)$, and

$$I_1 = \frac{1}{u(1)v(0)\omega} \int_0^1 e(s) f(s,e(s))ds, \quad I_2 = \frac{1}{\omega} \int_0^1 f(s,e(s))ds.$$

Suppose that constant c_1 satisfies $c_1 I_1 \geq M,\ 1/c_1 \leq N$. Let $\alpha(t) = k_1 h(t),\ \beta(t) = k_2 h(t),\ \ t \in [0,1]$; here

$$k_1 = \min\left\{1,\ \ \left(I_2^{\lambda} c_1^{\lambda-\mu}\right)^{1/(1-\mu)}\right\}$$

and

$$k_2 = \max\left\{1,\ \ \left(I_2^{\mu} c_1^{\mu-\lambda}\right)^{1/(1-\mu)}\right\}.$$

For $t \in (0,1)$,

$$f(t,\alpha(t)) \geq \left(\frac{k_1}{c_1}\right)^{\mu} f\left(t, \frac{c_1 h(t)}{e(t)} e(t)\right) \geq k_1^{\mu} c_1^{\lambda-\mu} I_2^{\lambda} f(t,\ e(t)),$$

$$f(t,\beta(t)) \leq \left(\frac{1}{c_1}\right)^{\lambda} f\left(t, \frac{k_2 c_1 h(t)}{e(t)} e(t)\right) \leq k_2^{\mu} c_1^{\mu-\lambda} I_2^{\mu} f(t,\ e(t)),$$

$$-\alpha''(t) + \rho p(t)\alpha(t) = k_1 f(t,\ e(t)) \leq k_1^{\mu} c_1^{\lambda-\mu} I_2^{\lambda} f(t,\ e(t)) \leq f(t,\ \alpha(t)),$$

$$-\beta''(t) + \rho p(t)\beta(t) = k_2 f(t,\ e(t)) \geq k_2^{\mu} c_1^{\mu-\lambda} I_2^{\mu} f(t,\ e(t)) \geq f(t,\ \beta(t)).$$

So, $\alpha(t),\ \beta(t) \in C^1[0,1] \cap C^2(0,1)$ are, respectively, lower and upper solutions of (1.1) satisfying $0 < \alpha(t) \leq \beta(t)$ for $t \in (0,1)$, and $a\alpha(0) - b\alpha'(0) = 0,\ c\alpha(1) + d\alpha'(1) = 0,\ a\beta(0) - b\beta'(0) = 0,\ c\beta(1) + d\beta'(1) = 0$. Additionally, when $t \in (0,1)$ and $\alpha(t) \leq x \leq \beta(t)$, we have

$$\begin{aligned} 0 &\leq f(t,x) \leq \left(\frac{k_1}{c_1}\right)^{\lambda} f\left(t, \frac{c_1 x}{k_1 e(t)} e(t)\right) \\ &\leq \left(\frac{k_1}{c_1}\right)^{\lambda} \left(\frac{c_1 x}{k_1 e(t)}\right)^{\mu} f(t,\ e(t)) \\ &\leq \left(\frac{k_1}{c_1}\right)^{\lambda-\mu} (k_2 I_2)^{\mu} f(t,\ e(t)) = F(t). \end{aligned} \tag{3.10}$$

From (3.1), we have $\int_0^1 F(t)dt < \infty$. In the following, we shall show that problem (1.1) admits a positive solution $x(t) \in C^1[0,1] \cap C^2(0,1)$ such that $\alpha(t) \leq x(t) \leq \beta(t)$ for $t \in [0,1]$.

First of all, we define an auxiliary function

$$g(t,x) = \begin{cases} f(t,\alpha(t)), & \text{if} \quad x < \alpha(t), \\ f(t,x), & \text{if} \quad \alpha(t) \leq x \leq \beta(t), \\ f(t,\beta(t)), & \text{if} \quad x > \beta(t). \end{cases} \tag{3.11}$$

Consider the singular problem

$$(3.12)\qquad \begin{cases} -x'' + \rho p(t)x = g(t,x), & t \in (0,1), \\ ax(0) - bx'(0) = 0, & cx(1) + dx'(1) = 0, \end{cases}$$

and the corresponding integral equation

$$(3.13)\qquad x(t) = Ax(t) = \int_0^1 G(t,s)g(s,x(s))ds,$$

where

$$(3.14)\qquad G(t,s) = \begin{cases} \dfrac{v(t)u(s)}{\omega}, & s < t, \\ \dfrac{v(s)u(t)}{\omega}, & t \le s, \end{cases}$$

ω is given by (2.9). Obviously, if $x \in C[0,1] \cap C^1[0,1]$ is a solution of (3.13), then x is a $C^1[0,1]$ solution of (3.12).

By virtue of (3.1), (3.10) and (3.11), it is easy to verify that $A: X \to X = C[0,1]$ is completely continuous and $A(X)$ is a bounded set. Using the Schauder fixed point theorem, we assert that A has at least one fixed point $x^* \in X \cap C^1[0,1]$.

We claim that

$$(3.15)\qquad \alpha(t) \le x^*(t) \le \beta(t), \quad t \in [0,1]$$

and hence $x^*(t) \in C^1[0,1]$ is a positive solution of (1.1). Indeed, suppose by contradiction that there is $t^* \in [0,1]$ such that $x^*(t^*) > \beta(t^*)$. Then the relationships between $x(t)$ and $\beta(t)$ must be one of the following four cases:

Case 1: $x^*(t) > \beta(t),\ t \in [0,1]$;

Case 2: there exists $0 < s \le 1$ such that $x^*(s) = \beta(s),\ \ x^*(t) > \beta(t),\ t \in [0,s)$, and $t^* \in [0,s)$;

Case 3: there exists $0 \le r < 1$ such that $x^*(r) = \beta(r),\ \ x^*(t) > \beta(t),\ t \in (r,1]$, and $t^* \in (r,1]$;

Case 4: there exist $0 \le r < s \le 1$ such that $x^*(r) = \beta(r),\ \ x^*(s) = \beta(s),\ \ x^*(t) > \beta(t),\ t \in (r,s)$, and $t^* \in (r,s)$.

For the Case 1: for $t \in [0,1]$, we have that $g(t,x^*(t)) = f(t,\beta(t))$ and therefore

$$-x^{*''}(t) + \rho p(t)x^*(t) = f(t,\beta(t)), \quad t \in (0,1).$$

On the other hand, as β is an upper solution of (1.1), we also have

$$-\beta''(t) + \rho p(t)\beta(t) \geq f(t, \beta(t)), \quad t \in (0,1).$$

Then, setting

$$z(t) = \beta(t) - x^*(t), \quad t \in [0,1],$$

we obtain $-z''(t)+\rho p(t)z(t) \geq 0,\ t \in [0,1]$, and $az(0)-bz'(0) = 0, \quad cz(1)+dz'(1) = 0$. By the maximum principle, we can conclude that $z(t) \geq 0,\ t \in [0,1]$, that is $\beta(t) \geq x^*(t),\ t \in [0,1]$, a contradiction to the assumption $\beta(t^*) < x^*(t^*)$. The proof for the cases 2, 3 and 4 is analogous to that of the case 1. Similarly, we can show that $\alpha(t) \leq x^*(t), \quad t \in [0,1]$. Therefore, (3.15) holds, and $x^*(t)$ is a $C^1[0,1]$ positive solution of (1.1). The proof of Theorem 3.1 is complete.

The proof of Theorem 3.2. The proof for the case I): $b = d = 0$.

1. Necessity. Let $x(t) \in C[0,1]$ be a positive solution of (1.1). Then $x(0) = x(1) = 0$ and there is a $t_0 \in (0,1)$ such that $x'(t_0) = 0$. Let $c_0 > 0$ be a constant such that $c_0 x(t) \leq N$ for $t \in [0,1]$ and $1/c_0 \geq M$. From (1.11) and (1.12), we have

$$f(t, x(t)) \geq (1/c_0)^\lambda f(t, c_0 x(t)) \geq c_0^{\mu-\lambda} x^\mu(t) f(t,1) \quad \text{for} \ \ t \in (0,1).$$

According to (1.1), we have

$$c_0^{\mu-\lambda} f(t,1) \leq -x^{-\mu}(t)x''(t) + \rho p(t) x^{1-\mu}(t), \quad t \in (0,1). \tag{3.16}$$

For $t \in (0, t_0)$, by integration of (3.16), we obtain

$$\begin{aligned} c_0^{\mu-\lambda} \int_t^{t_0} f(s,1)ds &\leq -x'(s)x^{-\mu}(s)|_t^{t_0} + \int_t^{t_0} (-\mu x^{-\mu-1}(s))(x'(s))^2 ds \\ &\quad +\rho \int_t^{t_0} p(s)x^{1-\mu}(s)ds \\ &\leq x^{-\mu}(t)x'(t) + \rho \int_t^{t_0} p(s)x^{1-\mu}(s)ds, \quad t \in (0,t_0). \end{aligned} \tag{3.17}$$

Integrating (3.17), we have

$$\begin{aligned} & c_0^{\mu-\lambda} \int_0^{t_0}\int_t^{t_0} f(s,1)dsdt \leq \frac{x^{1-\mu}(t_0)}{1-\mu} + \rho K \int_0^{t_0}\int_t^{t_0} p(s)dsdt \\ &= \frac{x^{1-\mu}(t_0)}{1-\mu} + \rho K \int_0^{t_0} sp(s)ds < \infty, \end{aligned}$$

where $K = \max_{t\in[0,1]} x^{1-\mu}(t)$, so,

$$0 < \int_0^{t_0} sf(s,1)ds < \infty. \tag{3.18}$$

For $t \in (t_0, 1)$, by integration of (3.16), we obtain

$$c_0^{\mu-\lambda} \int_{t_0}^{t} f(s,1)ds \le -x^{-\mu}(t)x'(t) + \rho K \int_{t_0}^{t} p(s)ds, \quad t \in (t_0, 1). \tag{3.19}$$

By integration (3.19), we have

$$c_0^{\mu-\lambda} \int_{t_0}^{1} \int_{t_0}^{t} f(s,1)dsdt \le \frac{x^{1-\mu}(t_0)}{1-\mu} + \rho K \int_{t_0}^{1} (1-s)p(s)ds < \infty,$$

i.e.,

$$0 < \int_{t_0}^{1} (1-s)f(s,1)ds < \infty. \tag{3.20}$$

Then, (3.18) and (3.20) imply that (3.2) holds.

For $t \in (0, t_0)$, by integration of (3.17), we have

$$\begin{aligned} c_0^{\mu-\lambda} \int_0^t \int_s^{t_0} f(\tau,1)d\tau ds &\le \frac{x^{1-\mu}(t)}{1-\mu} + \rho K \int_0^t \int_s^{t_0} p(\tau)d\tau ds \\ &= \frac{x^{1-\mu}(t)}{1-\mu} + \rho K \int_0^t ds \left(\int_s^t + \int_t^{t_0} \right) p(\tau)d\tau, \end{aligned}$$

therefore,

$$c_0^{\mu-\lambda} t \int_t^{t_0} f(\tau,1)d\tau \le \frac{x^{1-\mu}(t)}{1-\mu} + \rho K \left(\int_0^t sp(s)ds + t \int_t^{t_0} p(\tau)d\tau \right). \tag{3.21}$$

Letting $t \to 0$ in (3.21) and noting condition (H_1) and $x(0) = 0$, we have

$$\lim_{t\to 0^+} t \int_t^{t_0} f(s,1)ds = 0. \tag{3.22}$$

These imply that (3.3) holds.

For $t \in (t_0, 1)$, by integration of (3.19), we have

$$c_0^{\mu-\lambda} \int_t^1 \int_{t_0}^{s} f(\tau,1)d\tau ds \le \frac{x^{1-\mu}(t)}{1-\mu} + \rho K \int_t^1 ds \left(\int_{t_0}^{t} + \int_t^{s} \right) p(\tau)d\tau.$$

Therefore,

$$(3.23)\qquad (1-t)\int_{t_0}^{t} f(\tau,1)d\tau$$
$$\leq c_0^{\lambda-\mu}\left(\frac{x^{1-\mu}(t)}{1-\mu}+\rho K\left((1-t)\int_{t_0}^{t} p(\tau)d\tau+\int_{t}^{1}(1-\tau)p(\tau)d\tau\right)\right).$$

Letting $t\to 1$ in (3.23) and noting condition (H_1) and $x(1)=0$, we obtain

$$(3.24)\qquad \lim_{t\to 1^-}(1-t)\int_{t_0}^{t} f(s,1)ds=0.$$

These imply that (3.4) holds.

2. Sufficiency. Suppose that (3.2)–(3.4) hold. By Theorem 2.2 in [5], we know

$$(3.25)\qquad \omega_0=e_2(0)=e_1(1)=\begin{vmatrix} e_2(t) & e_2'(t)\\ e_1(t) & e_1'(t)\end{vmatrix}=\text{constant}>0.$$

Here, $e_1(t), e_2(t)$ are given by Lemma 1. Choose a constant $m\geq 2$ such that $m(\mu-\lambda)>1$, and let

$$(3.26)\qquad q(t)=\frac{1}{\omega_0}\left(e_2(t)\int_0^t e_1(s)f(s,1)ds+e_1(t)\int_t^1 e_2(s)f(s,1)ds\right),$$

$$(3.27)\qquad Q(t)=(q(t))^{1/(m(\mu-\lambda))}.$$

Then $q(t),\ Q(t)\in C[0,1]\cap C^2(0,1)$ satisfying $q(t)>0,\ Q(t)>0,\ t\in(0,1)$, and

$$-q''(t)+\rho p(t)q(t)=f(t,1),\quad -Q''(t)+\rho p(t)Q(t)\geq 0,\quad \text{for}\ \ t\in(0,1)$$

and from (3.2)–(3.4), we have $q(i)=Q(i)=0,\ \text{ for } i=0,\ 1$. By the proof of Lemma 1, we obtain

$$(3.28)\qquad 0<q(t)\leq\frac{1}{\omega_0}\int_0^1 s(1-s)w_1(1)w_2(0)f(s,1)ds<\infty,\quad t\in(0,1)$$

and such that

$$
\begin{aligned}
& e_2(t)\int_0^t e_1(s)Q^{-(\mu-\lambda)}(s)f(s,1)ds \\
& \le e_2(t)\int_0^t e_1(s)\left(\frac{e_2(s)}{\omega_0}\int_0^s e_1(\tau)f(\tau,1)d\tau\right)^{-1/m} f(s,1)ds \\
& \le (e_2(t))^{1-1/m}\omega_0^{1/m}\int_0^t e_1(s)\left(\int_0^s e_1(\tau)f(\tau,1)d\tau\right)^{-1/m} f(s,1)ds \\
& = \omega_0^{1/m}(1-1/m)^{-1}(e_2(t))^{1-1/m}\left(\int_0^t e_1(s)f(s,1)ds\right)^{1-1/m} \\
& \le \omega_0^{1/m}(1-1/m)^{-1}\left(\int_0^1 e_1(s)e_2(s)f(s,1)ds\right)^{1-1/m} < \infty.
\end{aligned}
\tag{3.29}
$$

Similarly, we have

$$
\begin{aligned}
& e_1(t)\int_t^1 e_2(s)Q^{-(\mu-\lambda)}(s)f(s,1)ds \\
& \le \omega_0^{1/m}(1-1/m)^{-1}\left(\int_0^1 e_1(s)e_2(s)f(s,1)ds\right)^{1-1/m} < \infty.
\end{aligned}
\tag{3.30}
$$

Let

$$
\begin{aligned}
h_1(t) &= \frac{e_2(t)}{\omega_0}\int_0^t e_1(s)\left(\frac{e_1(s)e_2(s)}{e_1(1)e_2(0)}\right)^{\mu} f(s,1)ds \\
&\quad +\frac{e_1(t)}{\omega_0}\int_t^1 e_2(s)\left(\frac{e_1(s)e_2(s)}{e_1(1)e_2(0)}\right)^{\mu} f(s,1)ds \\
h_2(t) &= \frac{e_2(t)}{\omega_0}\int_0^t e_1(s)Q^{-\mu}(s)f(s,Q(s))ds \\
&\quad +\frac{e_1(t)}{\omega_0}\int_t^1 e_2(s)Q^{-\mu}(s)f(s,Q(s))ds + Q(t).
\end{aligned}
$$

Let $c_1 > 0$ such that $(1/c_1)Q(t) \le N < 1,\ \ c_1 \ge M > 1$. From (1.11) and (1.12), we have

$$
\begin{aligned}
& Q^{-\mu}(t)f(t,Q(t)) \le Q^{-\mu}(t)\left(Q(t)/c_1\right)^{\lambda} f(t,c_1) \\
& \le Q^{-\mu}(t)\left(Q(t)/c_1\right)^{\lambda} c_1^{\mu} f(t,1) = c_1^{\mu-\lambda}Q^{\lambda-\mu}(t)f(t,1).
\end{aligned}
\tag{3.31}
$$

Thus, (3.28)–(3.31) imply that

$$0 \le h_1(t) < \infty, \quad 0 \le h_2(t) < \infty, \quad \text{for} \quad t \in [0,1].$$

One can check that $h_i \in C[0,1] \cap C^2(0,1)$, $h_i(0) = h_i(1) = 0$, $i = 1, 2$, and

$$L_1 \frac{e_1(t)e_2(t)}{e_1(1)e_2(0)} \le h_1(t) \le L_1, \quad Q(t) \le h_2(t) \le L_2, \quad t \in [0,1],$$

$$-h_1''(t) + \rho p(t) h_1(t) = \left(\frac{e_1(t)e_2(t)}{e_1(1)e_2(0)} \right)^{\mu} f(t,1), \quad t \in (0,1), \tag{3.32}$$

$$-h_2''(t) + \rho p(t) h_2(t) \ge Q^{-\mu}(t) f(t, Q(t)), \quad t \in (0,1). \tag{3.33}$$

Here,

$$L_1 = \omega_0 \int_0^1 \left(\frac{e_1(s)e_2(s)}{e_1(1)e_2(0)} \right)^{1+\mu} f(s,1)ds,$$

$$L_2 = \frac{1}{\omega_0} \int_0^1 e_1(s)e_2(s)Q^{-\mu}(s) f(s, Q(s))ds + Q_0, \quad Q_0 = \max_{t\in[0,1]} Q(t).$$

Let $\alpha(t) = k_1 h_1(t)$, $\beta(t) = k_2 h_2(t)$, $t \in [0,1]$; here k_1, k_2 are constants satisfying $0 < k_1 \le 1 \le k_2$ and will be determined later. Suppose c_2, c_3 are constants such that $c_2 L_1 \le N$, $1/c_2 \ge M$, $c_3 \ge M$, $1/c_3 \le N$. From (1.11), (1.12), we have

$$\begin{aligned} f(t,\alpha(t)) &\ge (1/c_2)^{\lambda} f(t, c_2\alpha(t)) \ge (c_2)^{\mu-\lambda} \alpha^{\mu}(t) f(t,1) \\ &\ge (c_2)^{\mu-\lambda} (k_1 L_1)^{\mu} \left(\frac{e_1(t)e_2(t)}{e_1(1)e_2(0)} \right)^{\mu} f(t,1), \quad t \in (0,1), \end{aligned} \tag{3.34}$$

$$\begin{aligned} f(t,\beta(t)) &\le (c_3)^{\mu-\lambda} \left(\frac{\beta(t)}{Q(t)} \right)^{\mu} f(t, Q(t)) \\ &\le (c_3)^{\mu-\lambda} (k_2 L_2)^{\mu} Q^{-\mu}(t) f(t, Q(t)), \quad t \in (0,1). \end{aligned} \tag{3.35}$$

By virtue of (1.11), (1.12), we can find a k_0 such that $f(t, Q(t)) \ge k_0 Q^{\mu}(t) f(t,1)$, and hence, from the definitions of $h_1(t)$, $h_2(t)$, we have $h_1(t) \le k_0^{-1} h_2(t)$ for $t \in [0,1]$. Now we choose

$$k_1 = \min\left\{1, \ \left(L_1^{\mu} c_2^{\mu-\lambda}\right)^{1/(1-\mu)}\right\}$$

and

$$k_2 = \max\left\{1, \quad k_0^{-1}, \quad \left(L_2^{\mu} c_3^{\mu-\lambda}\right)^{1/(1-\mu)}\right\}.$$

Then $\alpha(t),\ \beta(t) \in C[0,1] \cap C^2(0,1),\ \ 0 < \alpha(t) \le \beta(t)$ for $t \in (0,1), \alpha(i) = \beta(i) = 0,\ \ i = 0,1$. From (3.32)–(3.35), we obtain that for such choice of k_1 and k_2, $\alpha(t)$ and $\beta(t)$ are lower and upper solutions of (1.1), respectively.

In the following, we shall prove problem (1.1) has at least one $C[0,1]$ positive solution $x(t)$ such that

$$\alpha(t) \le x(t) \le \beta(t), \qquad t \in [0,1]. \tag{3.36}$$

First of all, we define an auxiliary function $g(t,x)$ given by (3.11). Let $\{a_n\},\ \ \{b_n\}$ be sequences satisfying $0 < \cdots < a_{n+1} < a_n < \cdots < a_1 < 1/2 < b_1 < \cdots < b_n < b_{n+1} < \cdots < 1$, $a_n \to 0$ and $b_n \to 1$ as $n \to \infty$, and let $\{r_1^{(n)}\},\ \ \{r_2^{(n)}\}$ be sequences satisfying

$$\alpha(a_n) \le r_1^{(n)} \le \beta(a_n), \quad \alpha(b_n) \le r_2^{(n)} \le \beta(b_n), \quad n = 1,\ 2, \ldots .$$

For each n, consider the nonsingular problem

$$\begin{cases} -x'' + \rho p(t)x = g(t,x), & t \in [a_n, b_n], \\ x(a_n) = r_1^{(n)}, \quad x(b_n) = r_2^{(n)}, \end{cases} \tag{3.37}$$

and the corresponding integral equation

$$x(t) = A_n x(t) = \frac{x_{2n}(t)}{x_{2n}(a_n)} r_1^{(n)} + \frac{x_{1n}(t)}{x_{1n}(b_n)} r_2^{(n)} + \int_{a_n}^{b_n} G_n(t,s) g(s, x(s)) ds, \tag{3.38}$$

where

$$G_n(t,s) = \begin{cases} \dfrac{x_{2n}(t) x_{1n}(s)}{\omega_n}, & s < t, \\ \dfrac{x_{2n}(s) x_{1n}(t)}{\omega_n}, & t \le s, \end{cases} \tag{3.39}$$

$$\omega_n = \begin{vmatrix} x_{2n}(t) & x_{2n}'(t) \\ x_{1n}(t) & x_{1n}'(t) \end{vmatrix} = x_{2n}(a_n) = x_{1n}(b_n) = \text{constant} > 0,$$

and $x_{1n}(t) \in C^2[a_n, b_n]$ is a unique increasing positive solution of the problem

$$\begin{cases} -x''(t) + \rho p(t)x(t) = 0, & t \in [a_n, b_n], \\ x(a_n) = 0, \quad x'(a_n) = 1, \end{cases} \tag{3.40}$$

and $x_{2n}(t) \in C^2[a_n, b_n]$ is a unique decreasing positive solution of the problem

$$(3.41) \qquad \begin{cases} -x''(t) + \rho p(t)x(t) = 0, & t \in [a_n, b_n], \\ x(b_n) = 0, \quad x'(b_n) = -1. \end{cases}$$

It is easy to verify that $A_n: X_n \to X_n = C[a_n, b_n]$ is completely continuous and $A_n(X_n)$ is a bounded set. Moreover, if $x \in C^2[a_n, b_n]$ is a solution of (3.38), then x is a solution of (3.37). Using the Schauder fixed point theorem, we assert that A_n has at least one fixed point $x_n \in C^2[a_n, b_n]$.

Similarly to the proof of Theorem 3.1, we can prove that $\alpha(t) \leq x_n(t) \leq \beta(t)$, $t \in [a_n, b_n]$ and hence $x_n(t) \in C^2[a_n, b_n]$ satisfies

$$(3.42) \qquad -x_n''(t) + \rho p(t)x_n(t) = f(t,\ x_n(t)), \quad t \in [a_n, b_n].$$

Since $[a_1, b_1] \subset [a_n, b_n]$, $n = 1, 2, \ldots,$ there is, for each n, $t_n \in [a_1, b_1]$ such that $|x_n'(t_n)| = |(x_n(b_1) - x_n(a_1))/(b_1 - a_1)| \leq (2/(b_1 - a_1))(\beta(b_1) + \beta(a_1))$. This allows us to assume (substituting by subsequences if necessary) $t_n \to t_0 \in [a_n, b_n]$, $x_n(t_n) \to x_0 \in [\alpha(t_0), \beta(t_0)]$, $x_n'(t_n) \to x_0' \in R$, as $n \to \infty$.

From [8, Theorem 3.2, p.14], there is a solution $x(t)$ of the equation

$$-x'' + \rho p(t)x = f(t, x),$$

with the maximum existence interval (ω^-, ω^+) such that $x(t_0) = x_0$, $x'(t_0) = x_0'$ and there is a subsequence of $\{x_n(t)\}-$ we denote it again by $\{x_n(t)\}-$ such that $\{x_n(t)\}$ converges uniformly to $x(t)$ on any compact subintervals of (ω^-, ω^+). Because $[a_n, b_n] \subset [a_{n+1}, b_{n+1}]$, $\bigcup_{n=1}^{\infty}[a_n, b_n] = (0, 1)$, and $\alpha(t) \leq x_n(t) \leq \beta(t)$, $t \in [a_n, b_n]$, one can easily see that $\alpha(t) \leq x(t) \leq \beta(t)$ for $t \in (\omega^-, \omega^+)$. This leads additionally to the fact that $(\omega^-, \omega^+) = (0, 1)$, from the Extension Theorem. Also, $x(t)$ satisfies $x(0) = 0$, $x(1) = 0$, because $\alpha(t)$ and $\beta(t)$ do. Thus $x(t)$ is a $C[0, 1]$ positive solution of problem (1.1).

This completes the proof of Theorem 3.2 for the case I): $b = d = 0$.

The proof for the case II): $b = 0,\ d > 0$

1. Necessity. Let $x(t) \in C[0, 1] \cap C^1(0, 1]$ be a positive solution of (1.1). Then $x(0) = 0$. By the proof of Lemma 3, we see that $x(t)$ satisfies (2.15). And (2.15) implies $x(1) > 0$, $x'(1) \leq 0$. Then there is a $t_0 \in (0, 1]$

such that $x'(t_0) = 0$. Hence, there are two cases, 1): $0 < t_0 < 1$ and 2): $t_0 = 1$.

For the case 1): $0 < t_0 < 1$, let $c_0 > 0$ be a constant such that $c_0 x(t) \leq N$ for $t \in [0,1]$ and $1/c_0 \geq M$. Then (3.16)–(3.22) hold. By integration of (3.16), we obtain

$$c_0^{\mu-\lambda} \int_{t_0}^{1} f(s,1)ds \leq -x^{-\mu}(1)x'(1) + \rho K \int_{t_0}^{1} p(s)ds < \infty, \tag{3.43}$$

where $K = \max_{t \in [0,1]} x^{1-\mu}(t)$. Then, (3.18) and (3.43) imply that (3.5) holds, and (3.22) and (3.43) imply that (3.6) holds.

For the case 2): $t_0 = 1$ is similar to that of the case 1): $0 < t_0 < 1$.

2. Sufficiency. Suppose that (3.5) and (3.6) hold. By Theorem 2.2 in [5], we know

$$\omega = \begin{vmatrix} v(t) & v'(t) \\ e_1(t) & e_1'(t) \end{vmatrix} = \text{constant} > 0. \tag{3.44}$$

Here, $e_1(t), v(t)$ are given by Lemmas 1, 2 respectively. Choose a constant $m \geq 2$ such that $m(\mu - \lambda) > 1$, and let

$$q(t) = \frac{1}{\omega}\left(v(t) \int_0^t e_1(s) f(s,1)ds + e_1(t) \int_t^1 v(s) f(s,1)ds \right), \tag{3.45}$$

$$Q(t) = (q(t))^{1/(m(\mu-\lambda))}. \tag{3.46}$$

We can check that $q,\ Q \in C[0,1] \cap C^1(0,1] \cap C^2(0,1)$, $q(0) = Q(0) = 0$, $cq(1) + dq'(1) = 0$, $cQ(1) + dQ'(1) \geq 0$. Let

$$\begin{aligned} h_1(t) &= \frac{v(t)}{\omega} \int_0^t e_1(s) \left(\frac{e_1(s)v(s)}{e_1(1)v(0)} \right)^{\mu} f(s,1)ds \\ &\quad + \frac{e_1(t)}{\omega} \int_t^1 v(s) \left(\frac{e_1(s)v(s)}{e_1(1)v(0)} \right)^{\mu} f(s,1)ds \end{aligned}$$

$$\begin{aligned} h_2(t) &= \frac{v(t)}{\omega} \int_0^t e_1(s) Q^{-\mu}(s) f(s,Q(s))ds \\ &\quad + \frac{e_1(t)}{\omega} \int_t^1 v(s) Q^{-\mu}(s) f(s,Q(s))ds + Q(t). \end{aligned}$$

Then $h_i \in C[0,1]\cap C^1(0,1]\cap C^2(0,1)$, $h_i(0)=0,\ i=1,\ 2$, $ch_1(1)+dh_1'(1)=0$, $ch_2(1)+dh_2'(1)=cQ(1)+dQ'(1)\geq 0$. Let

$$L_1=\frac{e_1(1)v(0)}{\omega}\int_0^1\left(\frac{e_1(s)v(s)}{e_1(1)v(0)}\right)^{1+\mu}f(s,1)ds,$$

$$L_2=\frac{1}{\omega}\int_0^1 e_1(s)v(s)Q^{-\mu}(s)f(s,Q(s))ds+Q_0,\quad Q_0=\max_{t\in[0,1]}Q(t).$$

By virtue of (1.11), (1.12), we can find a k_0 such that $f(t,Q(t))\geq k_0Q^\mu(t)f(t,1)$, and hence, from the definitions of $h_1(t),\ h_2(t)$, we have $h_1(t)\leq k_0^{-1}h_2(t)$ for $t\in[0,1]$. Suppose $c_2,\ c_3$ are constants such that $c_2L_1\leq N,\ 1/c_2\geq M,\ c_3\geq M,\ 1/c_3\leq N$. Now we choose

$$k_1=\min\left\{1,\quad\left(L_1^\mu c_2^{\mu-\lambda}\right)^{1/(1-\mu)}\right\}$$

and

$$k_2=\max\left\{1,\ k_0^{-1},\quad\left(L_2^\mu c_3^{\mu-\lambda}\right)^{1/(1-\mu)}\right\}.$$

Let $\alpha(t)=k_1h_1(t),\ \beta(t)=k_2h_2(t),\quad t\in[0,1]$. A similar argument to that we have checked in the sufficiency proof of case I): $b=d=0$ in Theorem 3.2 yields $\alpha(t),\ \beta(t)\in C^1(0,1]\cap C^2(0,1),\quad 0<\alpha(t)\leq\beta(t)$ for $t\in(0,1]$, $\alpha(0)=\beta(0)=0,\ \ c\alpha(1)+d\alpha'(1)=0,\quad c\beta(1)+d\beta'(1)\geq 0,\quad \alpha(t)$ and $\beta(t)$ are lower and upper solutions of (1.1), respectively.

In the following, we shall prove problem (1.1) has at least one $C[0,1]\cap C^1(0,1]$ positive solution $x(t)$ such that

$$\alpha(t)\leq x(t)\leq\beta(t),\qquad t\in[0,1]. \tag{3.47}$$

First of all, we define an auxiliary function $g(t,x)$ given by (3.11). Let $\{a_n\}$ be a sequence satisfying $0<\cdots<a_{n+1}<a_n<\cdots<a_1<1/2$, $a_n\to 0$ as $n\to\infty$, and let $\{r_1^{(n)}\}$ be a sequence satisfying

$$\alpha(a_n)\leq r_1^{(n)}\leq\beta(a_n),\quad n=1,\ 2,\ldots.$$

For each n, consider the singular problem

$$\begin{cases} -x''+\rho p(t)x=g(t,x), & t\in[a_n,1),\\ x(a_n)=r_1^{(n)},\quad cx(1)+dx'(1)=0. \end{cases} \tag{3.48}$$

Then there exist constants $K_n,\ J$ such that $0 < K_n \le \alpha(t) \le \beta(t) \le J$ for $t \in [a_n, 1]$. Take constants c_n such that $c_n \ge M,\ K_n/c_n \le N$. Then when $t \in [a_n, 1],\ \alpha(t) \le x \le \beta(t)$, we have

$$0 \le f(t,x) = f(t, \frac{c_n x}{K_n}\frac{K_n}{c_n}) \le \left(\frac{c_n J}{K_n}\right)^{\mu}\left(\frac{K_n}{c_n}\right)^{\lambda} f(t,1) = F(t). \tag{3.49}$$

Therefore,

$$0 \le \int_{a_n}^{1} F(s)ds \le \frac{1}{a_n}\int_{a_n}^{1} sF(s)ds < \infty. \tag{3.50}$$

By virtue of the proof of the sufficiency of Theorem 3.1, noting (3.49) and (3.50), we can obtain the following conclusion: For each n, the singular problem (3.48) has at least a positive solution $x_n \in C^1[a_n, 1]$ such that $\alpha(t) \le x_n(t) \le \beta(t),\quad t \in [a_n, 1]$. Hence, we have $|x_n(1)| \le \beta(1)$, $|x_n'(1)| \le |(c/d)x_n(1)| \le (c/d)\beta(1),\ n = 1,\ 2,\ \ldots,$. This allows us to assume (substituting by subsequences if necessary)$x_n(1) \to x_0 \in [\alpha(1), \beta(1)]$, $x_n'(1) \to -(c/d)x_0$, as $n \to \infty$.

From [8, Theorem 3.2, p.14], there is a solution $x(t)$ of the equation

$$-x'' + \rho p(t)x = f(t,x),$$

with the maximum existence interval $(\omega^-, 1]$ such that $x(1) = x_0,\quad x'(1) = -(c/d)x_0$ and there is a subsequence of $\{x_n(t)\}-$ we denote it again by $\{x_n(t)\}-$ such that $\{x_n(t)\}$ and $\{x_n'(t))\}$ converge uniformly to $x(t)$ and $x'(t)$ on any compact subintervals of $(\omega^-, 1]$. Because $\bigcup_{n=1}^{\infty}[a_n, 1] = (0, 1]$, and $\alpha(t) \le x_n(t) \le \beta(t),\quad t \in [a_n, 1]$, one can easily see that $\alpha(t) \le x(t) \le \beta(t)$ for $t \in (\omega^-, 1]$. This leads additionally to the fact that $(\omega^-, 1] = (0, 1]$, from the Extension Theorem. Also, $x(t)$ satisfies $x(0) = 0$, because $\alpha(t)$ and $\beta(t)$ do, and $cx(1)+dx'(1) = 0$. Thus $x(t)$ is a $C^1(0, 1]$ positive solution of problem (1.1).

This completes the proof of Theorem 3.2 for the case II): $b = 0,\ d > 0$.

The proof for the case III): $b > 0,\ d = 0$ is almost the same as for the case II). The proof of Theorem 3.2 is complete.

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