I. Lahiri
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WEIGHTED SHARING AND UNIQUENESS OF MEROMORPHIC FUNCTIONS

INDRAJIT LAHIRI

Abstract. Introducing the idea of weighted sharing of values we prove some uniqueness theorems for meromorphic functions which improve some existing results.

§1. Introduction and Definitions

Let f and g be two nonconstant meromorphic functions defined in the open complex plane C. If for some $a \in C \cup \{\infty\}$ the a-points of f and g coincide in locations and multiplicities, we say that f and g share the value a CM (counting multiplicities). On the other hand, if the a-points of f and g coincide in locations only, we say that f and g share the value a IM (ignoring multiplicities).

Though we do not explain the standard notations of the value distribution theory because those are available in [2], we explain some notations which will be needed in the sequel.

Definition 1. If s is a nonnegative integer, we denote by $N(r, a; f \mid= s)$ the counting function of those a-points of f whose multiplicity is s, where each a-point is counted according to its multiplicity.

Definition 2. If s is a positive integer, we denote by $\overline{N}(r, a; f \mid\geq s)$ the counting function of those a-points of f whose multiplicities are greater than or equal to s, where each a-point is counted only once.

Definition 3. If s is a nonnegative integer, we denote by $N_s(r, a; f)$ the counting function of a-points of f where an a-point with multiplicity m is counted m times if $m \leq s$ and s times if $m > s$. We put $N_\infty(r, a; f) = N(r, a; f)$.

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Definition 4. Let f, g share a value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the counting function of those a-points of f whose multiplicities are different from multiplicities of the corresponding a-points of g, where each a-point is counted only once.

Clearly $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$.

Definition 5. Let f, g share a value a IM. We denote by $\overline{N}(r, a; f < g)$ ($\overline{N}(r, a; f > g)$) the counting function of those a-points of f whose multiplicities are less (greater) than the multiplicities of the corresponding a-points of g, where each a-point is counted only once.

We denote by I a set of infinite linear measure not necessarily the same in all its occurrences. Also $T(r)$ denotes the maximum of $T(r, f)$ and $T(r, g)$.

M. Ozawa [4] proved the following result.

Theorem A. ([4]) *Let f, g be entire functions of finite order such that f and g share 0, 1 CM. If $\delta(0, f) > 1/2$ then $f.g \equiv 1$ unless $f \equiv g$.*

Removing the order restriction in the above result H. Ueda [6] proved the following theorem.

Theorem B. ([6]) *If f, g share 0, 1, ∞ CM and*

$$\limsup_{r\to\infty} \frac{\overline{N}(r, 0; f) + \overline{N}(r, \infty, f)}{T(r, f)} < \frac{1}{2}$$

then either $f \equiv g$ or $f.g \equiv 1$.

In this direction H. X. Yi proved the following two results.

Theorem C. ([7]) *If f, g share 0, 1, ∞ CM and $\overline{N}(r, 0; f) + \overline{N}(r, \infty; f) < \{\lambda + o(1)\}T(r, f)$ for $r \in I$ and $0 < \lambda < 1/2$, then $f \equiv g$ or $f.g \equiv 1$.*

Theorem D. ([9]) *If f, g share 0, 1, ∞ CM and $N(r, 0; f \mid= 1) + N(r, \infty; f \mid= 1) < \{\lambda + o(1)\}T(r)$ for $r \in I$ and $0 < \lambda < 1/2$ then either $f \equiv g$ or $f.g \equiv 1$.*

Following examples show that in Theorem D the sharing of 0 can not be relaxed from CM to IM.

EXAMPLE 1. Let $f(z) = \left(\frac{1+e^z}{1-e^z}\right)^2$ and $g(z) = \frac{1+e^z}{1-e^z}$. Then f, g share 0, ∞ IM and 1 CM. Also $N(r, 0; f \mid= 1) \equiv N(r, \infty; f \mid= 1) \equiv 0$ but neither $f \equiv g$ nor $f.g \equiv 1$.

EXAMPLE 2. Let $f(z) = (e^z - 1)^2$ and $g(z) = e^z - 1$. Then f, g share 0 IM and 1, ∞ CM. Also $N(r, 0; f \mid= 1) \equiv N(r, \infty; f \mid= 1) \equiv \overline{N}(r, \infty; f) \equiv 0$ but neither $f \equiv g$ nor $f.g \equiv 1$.

Now one may ask: *Is it possible to relax the nature of sharing of* 0 *in the above results and if possible how far?*

The purpose of the paper is to discuss this problem. To this end we introduce a gradation of sharing of values which we call the weight of sharing.

DEFINITION 6. Let k be a nonnegative integer or infinity. For $a \in C \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$.

DEFINITION 7. Let k be a nonnegative integer or infinity. If for $a \in C \cup \{\infty\}$, $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_o is a zero of $f - a$ with multiplicity m $(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity m $(\leq k)$ and z_o is a zero of $f - a$ with multiplicity m $(> k)$ if and only if it is a zero of $g - a$ with multiplicity n $(> k)$ where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

§2. Lemmas

In this section we present some lemmas which are necessary in the sequel.

LEMMA 1. *If f, g share $(a, 0)$, $(b, 0)$, $(\infty, 0)$ where $b \neq \infty$ and $a \neq b, \infty$ then $T(r, f) \leq 3T(r, g) + S(r, f)$ and $T(r, g) \leq 3T(r, f) + S(r, g)$.*

Proof. The lemma follows as a direct consequence of the second fundamental theorem.

Lemma 2. *Let $c_1 f + c_2 g \equiv c_3$, where c_1, c_2, c_3 are nonzero constants. Then*

(i) $T(r,f) \le \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;f) + S(r,f)$,

(ii) $T(r,g) \le \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + S(r,g)$.

Proof. By the second fundamental theorem we get

$$\begin{aligned} T(r,f) &\le \overline{N}(r,0;f) + \overline{N}(r,c_3/c_1;f) + \overline{N}(r,\infty;f) + S(r,f) \\ &= \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(f,\infty;f) + S(r,f). \end{aligned}$$

In a similar manner we can prove (ii). This proves the lemma.

Lemma 3. *Let f, g share $(a,0)$ and $\phi = \frac{f'}{f-b} - \frac{g'}{g-b}$ where $a \ne \infty$, $b \ne a, \infty$. If $\overline{N}(r,a;f) \ne S(r,f)$ and $\phi \equiv 0$ then $f \equiv g$.*

Proof. Since $\phi \equiv 0$, we get $f - b = c(g - b)$, where c is a constant. Since f, g share $(a,0)$ and $\overline{N}(r,a;f) \ne S(r,f)$, there exists $z_0 \in C$ such that $f(z_0) = g(z_0) = a$. This shows that $c = 1$ because $a \ne b$. Therefore $f \equiv g$. This proves the lemma.

Lemma 4. *Let $a \ne \infty$, $b \ne a, \infty$ be two complex numbers. If f, g share $(a,1)$, $(\infty,0)$, (b,∞) and $f \not\equiv g$ then*

$$\begin{aligned} \overline{N}(r,a;f \mid \ge 2) &\le \overline{N}_*(r,\infty;f,g) + S(r,f), \\ \overline{N}(r,a;g \mid \ge 2) &\le \overline{N}_*(r,\infty;f,g) + S(r,f). \end{aligned}$$

Proof. Since the lemma is obvious when $\overline{N}(r,a;f) = S(r,f)$, we suppose that $\overline{N}(r,a;f) \ne S(r,f)$.

Let $\phi = \frac{f'}{f-b} - \frac{g'}{g-b}$. Since f, g share $(a,1)$ and $f \not\equiv g$, by Lemma 3 it follows that $\phi \not\equiv 0$. Since f, g share $(a,1)$, every multiple a-point of f is a multiple a-point of g and so it is a zero of ϕ. Hence

$$\begin{aligned} \overline{N}(r,a;f \mid \ge 2) &\le N(r,0;\phi) \le T(r,\phi) + O(1) \\ &= N(r,\phi) + m(r,\phi) + O(1) \\ &\le N(r,\phi) + m\Big(r, \frac{f'}{f-b}\Big) + m\Big(r, \frac{g'}{g-b}\Big) + O(1) \\ &= N(r,\phi) + S(r,f) + S(r,g), \end{aligned}$$

by Milloux theorem [2, p. 55].

So by Lemma 1 we get

$$\overline{N}(r,a;f\mid\geq 2)\leq N(r,\phi)+S(r,f). \tag{1}$$

Since f, g share $(a,1)$, it follows that $\overline{N}(r,a;f\mid\geq 2)=\overline{N}(r,a;g\mid\geq 2)$ and so

$$\overline{N}(r,a;g\mid\geq 2)\leq N(r,\phi)+S(r,f). \tag{2}$$

Clearly the possible poles of ϕ occur at the b-points and poles of f, g.

Let z_0 be a b-point of f with multiplicity m. Then $f-b=(z-z_0)^m\alpha(z)$ in some neighbourhood of z_0, where α is analytic at z_0 and $\alpha(z_0)\neq 0$. So $\frac{f'}{f-b}=\frac{\alpha'}{\alpha}+\frac{m}{z-z_0}$ in some neighbourhood of z_0.

Since f, g share (b,∞), in a similar manner we get $\frac{g'}{g-b}=\frac{\beta'}{\beta}+\frac{m}{z-z_0}$ in some neighbourhood of z_0, where β is analytic at z_0 and $\beta(z_0)\neq 0$.

Hence in some neighbourhood of z_0, $\phi=\frac{\alpha'}{\alpha}-\frac{\beta'}{\beta}$ so that z_0 is not a pole of ϕ.

Let z_1 be a pole of f with multiplicity m and a pole of g with multiplicity n. Then in some neighbourhood of z_1 we get $f-b=\gamma(z)/(z-z_1)^m$ and $g-b=\delta(z)/(z-z_1)^n$, where γ, δ are analytic at z_1 and $\gamma(z_1)\neq 0$, $\delta(z_1)\neq 0$. So

$$f'=\frac{\gamma'}{(z-z_1)^m}-\frac{m\gamma}{(z-z_1)^{m+1}}\quad\text{and}\quad g'=\frac{\delta'}{(z-z_1)^n}-\frac{n\delta}{(z-z_1)^{n+1}}$$

in some neighbourhood of z_1.

Hence $\phi=\frac{\gamma'}{\gamma}-\frac{\delta'}{\delta}-\frac{m-n}{z-z_1}$ in some neighbourhood of z_1. This shows that if $m\neq n$, z_1 is a simple pole of ϕ and if $m=n$, z_1 is not a pole of ϕ. Since all the poles of ϕ are simple, we get

$$N(r,\phi)=\overline{N}(r,\phi)\leq\overline{N}_*(r,\infty;f,g). \tag{3}$$

Now the lemma follows from (1), (2) and (3). This proves the lemma.

LEMMA 5. *Let $a\neq\infty$, $b\neq a,\infty$ be two complex numbers. If f, g share $(a,1)$, (b,∞), $(\infty,0)$ and $f\not\equiv g$ then*

$$N_2(r,a;f)\leq N(r,a;f\mid=1)+2\overline{N}_*(r,\infty;f,g)+S(r,f),$$

and

$$N_2(r,a;g)\leq N(r,a;f\mid=1)+2\overline{N}_*(r,\infty;f,g)+S(r,f).$$

Proof. By Lemma 4 we get

$$\begin{aligned} N_2(r,a;f) &= N(r,a;f \mid= 1) + 2\overline{N}(r,a;f \mid\geq 2) \\ &\leq N(r,a;f \mid= 1) + 2\overline{N}_*(r,\infty;f,g) + S(r,f) \end{aligned}$$

and

$$\begin{aligned} N_2(r,a;g) &= N(r,a;g \mid= 1) + 2\overline{N}(r,a;g \mid\geq 2) \\ &\leq N(r,a;f \mid= 1) + 2\overline{N}_*(r,\infty;f,g) + S(r,f). \end{aligned}$$

This proves the lemma.

LEMMA 6. *Let $a \neq \infty$, $b \neq a, \infty$ be two comlpex numbers. If f, g share $(a,1)$, (b,∞), $(\infty,1)$ and $f \not\equiv g$ then*

(i) $\overline{N}(r,\infty;f \mid\geq 2) \leq \overline{N}_*(r,\infty;f,g) + S(r,f)$, *and*

(ii) $\overline{N}(r,\infty;g \mid\geq 2) \leq \overline{N}_*(r,\infty;f,g) + S(r,f)$.

Proof. Let $F = a + \frac{(b-a)^2}{f-a}$ and $G = a + \frac{(b-a)^2}{g-a}$. Then F, G share $(a,1)$, (b,∞), $(\infty,1)$. So by Lemma 4 we get

$$\overline{N}(r,a;F \mid\geq 2) \leq \overline{N}_*(r,\infty;F,G) + S(r,f)$$

i.e.,

$$\begin{aligned} \overline{N}(r,\infty;f \mid\geq 2) &\leq \overline{N}_*(r,a;f,g) + S(r,f) \\ &\leq \overline{N}(r,a;f \mid\geq 2) + S(r,f). \end{aligned} \tag{4}$$

Again by Lemma 4 we get

$$\overline{N}(r,a;f \mid\geq 2) \leq \overline{N}_*(r,\infty;f,g) + S(r,f). \tag{5}$$

Now (i) follows from (4) and (5). Since by Lemma 1 $S(r,G) = S(r,g) = S(r,f)$, we can prove (ii) in a similar manner. This proves the lemma.

LEMMA 7. *Let $a \neq \infty$, $b \neq a, \infty$ be two complex numbers. If f, g share $(a,1)$, (b,∞), $(\infty,1)$ and $f \not\equiv g$ then*

(i) $N_2(r,\infty;f) \leq N(r,\infty;f \mid= 1) + 2\overline{N}_*(r,\infty;f,g) + S(r,f)$,

(ii) $N_2(r,\infty;g) \leq N(r,\infty;f \mid= 1) + 2\overline{N}_*(r,\infty;f,g) + S(r,f)$.

Proof. By Lemma 6 we get

$$\begin{aligned} N_2(r,\infty;f) &= N(r,\infty;f \mid= 1) + 2\overline{N}(r,\infty;f \mid\geq 2) \\ &\leq N(r,\infty;f \mid= 1) + 2\overline{N}_*(r,\infty;f,g) + S(r,f) \end{aligned}$$

and

$$\begin{aligned} N_2(r,\infty;g) &= N(r,\infty;g \mid= 1) + 2\overline{N}(r,\infty;f \mid\geq 2) \\ &\leq N(r,\infty;f \mid= 1) + 2\overline{N}_*(r,\infty;f,g) + S(r,f). \end{aligned}$$

This proves the lemma.

LEMMA 8. *Let $a \neq \infty$, $b \neq a, \infty$ be two complex numbers. If f, g share $(a,1)$, (b,∞), (∞,∞) and $f \not\equiv g$ then*

(i) $N_2(r,a;f) \leq N(r,a;f \mid= 1) + S(r,f)$,

(ii) $N_2(r,a;g) \leq N(r,a;f \mid= 1) + S(r,f)$,

(iii) $N_2(r,\infty;f) \leq N(r,\infty;f \mid= 1) + S(r,f)$, *and*

(iv) $N_2(r,\infty;g) \leq N(r,\infty;f \mid= 1) + S(r,f)$.

Proof. Since f, g share (∞,∞), $\overline{N}_*(r,\infty;f,g) \equiv 0$ and the lemma follows from Lemma 5 and Lemma 7. This proves the lemma.

LEMMA 9. ([3]) *Let f_1, f_2, f_3 be nonconstant meromorphic functions such that $f_1 + f_2 + f_3 \equiv 1$. If f_1, f_2, f_3 are linearly independent then for $i = 1,2,3$*

$$T(r,f_i) \leq \sum_{j=1}^{3} N_2(r,0;f_j) + \sum_{j=1}^{3} \overline{N}(r,\infty;f_j) + \sum_{j=1}^{3} S(r,f_j).$$

§3. Theorems

In this section we present the main results of the paper.

THEOREM 1. *Let f, g share $(0,1)$, $(\infty,0)$, $(1,\infty)$. If*

$$N(r,0;f \mid= 1) + 4\overline{N}(r,\infty;f) < \{\lambda + o(1)\}T(r) \tag{6}$$

for $r \in I$ and $0 < \lambda < 1/2$ then either $f \equiv g$ or $f.g \equiv 1$.

Theorem 2. *Let f, g share $(0,1)$, (∞,∞), $(1,\infty)$. If*

$$N(r,0;f\mid=1)+N(r,\infty;f\mid=1)<\{\lambda+o(1)\}T(r) \tag{7}$$

for $r\in I$ and $0<\lambda<1/2$ then either $f\equiv g$ or $f.g\equiv 1$.

Example 2 shows that in Theorem 1 and Theorem 2 sharing $(0,1)$ can not be relaxed to sharing $(0,0)$. Also the following example shows that the conditions (6) and (7) are sharp.

Example 3. Let $f(z)=e^z(1-e^z)$, $g(z)=e^{-z}(1-e^{-z})$. Then f, g share $(0,\infty)$, (∞,∞), $(1,\infty)$ and $N(r,0;f\mid=1)\sim\frac{1}{2}T(r)$, $N(r,\infty;f\mid=1)\equiv\overline{N}(r,\infty;f)\equiv 0$. Also neither $f\equiv g$ nor $f.g\equiv 1$.

Proof of Theorem 1. We suppose that $f\not\equiv g$. Without loss of generality, we suppose that there exists a set I of infinite linear measure such that $T(r,g)\le T(r,f)$ for $r\in I$, because otherwise we have only to interchange f and g in our discussion, noting by Lemma 1 that $S(r,f)=S(r,g)$. Let

$$h=\frac{f-1}{g-1}. \tag{8}$$

Since f, g share $(1,\infty)$, $(\infty,0)$ it follows that

$$N_2(r,0;h)\le 2\overline{N}(r,0;h)\le 2\overline{N}(r,\infty;f<g)$$

and

$$N_2(r,\infty;h)\le 2\overline{N}(r,\infty;h)\le 2\overline{N}(r,\infty;f>g).$$

Let $f_1=f$, $f_2=-gh$ and $f_3=h$. Then by (8) it follows that

$$f_1+f_2+f_3\equiv 1. \tag{9}$$

If possible, we suppose that f_1, f_2, f_3 are linearly independendnt. It is clear that a zero of h is not a zero of f_2 so that $N_2(r,0;f_2)\le N_2(r,0;g)$. Then by Lemma 9, Lemma 5 and Lemma 1 we get

$$\begin{aligned}T(r,f)&\le\sum_{j=1}^{3}N_2(r,0;f_j)+\sum_{j=1}^{3}\overline{N}(r,\infty;f_j)+S(r,f)\\&\le N_2(r,0;f)+N_2(r,0;g)+N_2(r,0;h)+\overline{N}(r,\infty;f)\\&\qquad+\overline{N}(r,\infty;gh)+\overline{N}(r,\infty;h)+S(r,f)\end{aligned}$$

$$\begin{aligned}
&\le 2N(r,0;f\mid=1)+4\overline{N}_*(r,\infty;f,g)+2\overline{N}(r,0;h)+\overline{N}(r,\infty;f)\\
&\qquad+\overline{N}(r,\infty;h(g-1))+\overline{N}(r,\infty;h)+S(r,f)\\
&\le 2N(r,0;f\mid=1)+4\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;f)\\
&\qquad+\{\overline{N}(r,0;h)+\overline{N}(r,\infty;h)\}+S(r,f)\\
&\le 2N(r,0;f\mid=1)+7\overline{N}(r,\infty;f)\\
&\qquad+\{\overline{N}(r,\infty;f<g)+\overline{N}(r,\infty;f>g)\}+S(r,f)\\
&\le 2N(r,0;f\mid=1)+8\overline{N}(r,\infty;f)+S(r,f)\\
&<\{2\lambda+o(1)\}T(r,f)\quad\text{for } r\in I,
\end{aligned}$$

which is a contradiction.

Therefore f_1, f_2, f_3 are linearly dependent and so there exist constants c_1, c_2, c_3, not all zero, such that

$$c_1f_1+c_2f_2+c_3f_3\equiv 0. \tag{10}$$

If $c_1=0$, we get from (10) $h(c_3-c_2g)\equiv 0$, which is a contradiction because f, g are nonconstant. So $c_1\ne 0$ and eliminating f_1 from (9) and (10) we get

$$cf_2+df_3\equiv 1, \tag{11}$$

where $c=1-(c_2/c_1)$ and $d=1-(c_3/c_1)$ and clearly $|c|+|d|\ne 0$.

Now we consider the following cases.

Case I. Let $c.d\ne 0$. Then from (11) and (8) we get

$$\begin{aligned}
&-cgh+dh\equiv 1, \qquad\qquad (12)\\
&\text{i.e., } -c\Big(1+\frac{f-1}{h}\Big)h+dh\equiv 1,\\
&\text{i.e., } (d-c)h-cf\equiv 1+c.
\end{aligned}$$

Since f is nonconstant, it follows that $c\ne d$. Let $c\ne -1$. Then by Lemma 2 and Lemma 5 we get

$$\begin{aligned}
T(r,f)&\le \overline{N}(r,0;f)+\overline{N}(r,0;h)+\overline{N}(r,\infty;f)+S(r,f)\\
&\le \overline{N}(r,0;f)+\overline{N}(r,\infty;f<g)+\overline{N}(r,\infty;f)+S(r,f)\\
&\le N(r,0;f\mid=1)+2\overline{N}_*(r,\infty;f,g)+2\overline{N}(r,\infty;f)+S(r,f)\\
&\le N(r,0;f\mid=1)+4\overline{N}(r,\infty;f)+S(r,f)\\
&<\{\lambda+o(1)\}T(r,f)\quad\text{for } r\in I,
\end{aligned}$$

which is a contradiction.

Let $c = -1$. Then $d \neq -1$ and from (12) we get

$$\begin{aligned}(d+1)h + f &\equiv 0,\\ \text{i.e., } (d+1)\frac{f-1}{g-1} + f &\equiv 0,\\ \text{i.e., } \frac{d+1}{f} - g &\equiv d.\end{aligned}$$

So by Lemma 2, Lemma 5 and the first fundamental theorem we get

$$T\Big(r, \frac{1}{f}\Big) \leq \overline{N}\Big(r, 0; \frac{1}{f}\Big) + \overline{N}(r, 0; g) + \overline{N}\Big(r, \infty; \frac{1}{f}\Big) + S(r, f)$$

i.e.,

$$\begin{aligned}T(r, f) &\leq 2\overline{N}(r, 0; f) + \overline{N}(r, \infty; f) + S(r, f)\\ &\leq 2N(r, 0; f \mid= 1) + 4\overline{N}_*(r, \infty; f, g) + \overline{N}(r, \infty; f) + S(r, f)\\ &\leq 2N(r, 0; f \mid= 1) + 5\overline{N}(r, \infty; f) + S(r, f)\\ &< \{2\lambda + o(1)\}T(r, f) \quad \text{for } r \in I,\end{aligned}$$

which is a contradiction. Therefore the case $c.d \neq 0$ does not arise.

Case II. Let $c.d = 0$.

Let $c = 0$. Then $d \neq 0$ and so from (11) we get $df - g \equiv d - 1$. Since $f \not\equiv g$, $d \neq 1$ and so by Lemma 2 and Lemma 5 we get

$$\begin{aligned}T(r, f) &\leq \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}(r, \infty; f) + S(r, f)\\ &\leq 2N(r, 0; f \mid= 1) + 4\overline{N}_*(r, \infty; f, g) + \overline{N}(r, \infty; f) + S(r, f)\\ &\leq 2N(r, 0; f \mid= 1) + 5\overline{N}(r, \infty; f) + S(r, f)\\ &< \{2\lambda + o(1)\}T(r, f) \quad \text{for } r \in I,\end{aligned}$$

which is a contradiction.

Therefore $c \neq 0$ and so $d = 0$. From (11) we get

$$-cf + \frac{1}{g} \equiv 1 - c. \tag{13}$$

If $c \neq 1$, by Lemma 2 and Lemma 5 we get

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,0;f) + \overline{N}(r,0;1/g) + \overline{N}(r,\infty;f) + S(r,f) \\ &\leq N(r,0;f \mid= 1) + 2\overline{N}_*(r,\infty;f,g) + 2\overline{N}(r,\infty;f) + S(r,f) \\ &\leq N(r,0;f \mid= 1) + 4\overline{N}(r,\infty;f) + S(r,f) \\ &< \{\lambda + o(1)\}T(r,f) \quad \text{for } r \in I, \end{aligned}$$

which is a contradiction.

So $c = 1$ and from (13) we get $f.g \equiv 1$. This proves the theorem.

Proof of Theorem 2. Using Lemma 8 the theorem can be proved in a similar manner noting that $\overline{N}(r,0;h) \equiv \overline{N}(r,\infty;h) \equiv 0$ and $N_2(r,0;h) \leq 2\overline{N}(r,0;h)$, $\overline{N}(r,\infty;f) \leq N_2(r,\infty;f)$.

§4. Consequences

In this section we discuss some consequences of Theorem 1 and Theorem 2.

Definition 8. For $S \subset C \cup \{\infty\}$ we denote by $E_f(S)$ the set $E_f(S) = \bigcup_{a \in S}\{z : f(z) - a = 0\}$, where an a-point of multiplicity m is counted m times.

Definition 9. For $S \subset C \cup \{\infty\}$ we define $E_f(S,k)$ as $E_f(S,k) = \bigcup_{a \in S} E_k(a;f)$, where k is a nonnegative integer or infinity.

Clearly $E_f(S) = E_f(S,\infty)$.

Gross and Osgood [1] proved the following theorem.

Theorem E. ([1]) *Let $S_1 = \{-1,1\}$, $S_2 = \{0\}$. If f and g are entire functions of finite order such that $E_f(S_j) = E_g(S_j)$ for $j = 1,2$ then $f \equiv \pm g$ or $f.g \equiv \pm 1$*

Extending this result Tohge [5] and Yi [8] proved the following two theorems.

Theorem F. ([5]) *Let $S_1 = \{1,\omega,\omega^2,\dots,\omega^{n-1}\}$, $S_2 = \{0\}$, $S_3 = \{\infty\}$ where n is an integer (≥ 2) and $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$. If $E_f(S_j) = E_g(S_j)$ for $j = 1,2,3$ then $f^n \equiv g^n$ or $f^n.g^n \equiv 1$.*

THEOREM G. ([8]) *Let* $S_1 = \{a+b, a+b\omega, \dots, a+b\omega^{n-1}\}$, $S_2 = \{a\}$, $S_3 = \{\infty\}$, *where* n *is an integer* (≥ 2), a, b $(b \neq 0)$ *are constants and* $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$. *If* $E_f(S_j) = E_g(S_j)$ *for* $j = 1,2,3$ *then* $f - a \equiv t(g-a)$ *where* $t^n = 1$ *or* $(f-a)(g-a) \equiv s$ *where* $s^n = b^{2n}$.

As an application of Theorem 2 we improve Theorem G.

THEOREM 3. *Let* S_1, S_2, S_3 *be defined as in Theorem G. If* $E_f(S_1, \infty) = E_g(S_1, \infty)$, $E_f(S_2, 1) = E_g(S_2, 1)$ *and* $E_f(S_3, \infty) = E_g(S_3, \infty)$ *then* $f - a \equiv t(g-a)$ *where* $t^n = 1$ *or* $(f-a)(g-a) \equiv s$ *where* $s^n = b^{2n}$.

Proof. Let $F = \left(\frac{f-a}{b}\right)^n$, $G = \left(\frac{g-a}{b}\right)^n$. Then F, G share $(0,1)$, $(1,\infty)$ and (∞,∞). Since $N(r,0;F \mid= 1) \equiv N(r,\infty;F \mid= 1) \equiv 0$, it follows from Theorem 2 that either $F \equiv G$ or $F.G \equiv 1$ from which the theorem follows. This proves the theorem.

Following are two simple consequences of Theorem 1 and Theorem 2.

THEOREM 4. *Let* f, g *share* $(0,0)$, $(1,\infty)$ *and* $(\infty,1)$. *If*

$$N(r,\infty;f \mid= 1) + 4\overline{N}(r,0;f) < \{\lambda + o(1)\}T(r) \quad \textit{for } r \in I,$$

where $0 < \lambda < 1/2$, *then either* $f \equiv g$ *or* $f.g \equiv 1$.

Proof. Let $F = 1/f$ and $G = 1/g$. Then F, G satisfy the conditions of Theorem 1. So either $F \equiv G$ or $F.G \equiv 1$, from which the theorem follows.

THEOREM 5. *Let* f, g *share* $(0,\infty)$, $(1,\infty)$ *and* $(\infty,1)$. *If*

$$N(r,0;f \mid= 1) + N(r,\infty;f \mid= 1) < \{\lambda + o(1)\}T(r) \quad \textit{for } r \in I,$$

where $0 < \lambda < 1/2$ *then either* $f \equiv g$ *or* $f.g \equiv 1$.

Proof. Let $F = 1/f$, $G = 1/g$. Then F, G satisfy the conditions of Theorem 2. So either $F \equiv G$ or $F.G \equiv 1$, from which the theorem follows.

Remark 1. If f has at least one zero or pole then the possibility $f.g \equiv 1$ does not arise in Theorems 1, 2, 4, 5.

DEFINITION 10. ([6]) Let $\{a_n\}$, $\{b_n\}$ and $\{p_n\}$ be three disjoint sequences with no finite limit point. If it is possible to construct a meromorphic function f in the plain C whose zeros, 1-points and poles are exactly $\{a_n\}$, $\{b_n\}$ and $\{p_n\}$ respectively, where their multiplicities are taken into consideration, then the given triad $(\{a_n\}, \{b_n\}, \{p_n\})$ is called a zero-one-pole set. Further if there exists only one meromorphic function f whose zero-one-pole set is just the given triad then the triad is called unique.

H. Ueda [6] proved the following result.

THEOREM H. ([6]) *If* $n(r, 0; f) + n(r, \infty; f) \not\equiv 0$ *and*

$$\limsup_{r\to\infty} \frac{\overline{N}(r, 0; f) + \overline{N}(r, \infty; f)}{T(r, f)} < \frac{1}{2}$$

then the zero-one-pole set of f *is unique.*

As an application of Theorem 2 and Remark 1 we can improve Theorem H.

THEOREM 6. *If* $n(r, 0; f) + n(r, \infty; f) \not\equiv 0$ *and*

$$N(r, 0; f \mid= 1) + N(r, \infty; f \mid= 1) < \{\lambda + o(1)\}T(r, f) \quad \text{for } r \in I,$$

where $0 < \lambda < 1/2$ *then the zero-one-pole set of* f *is unique.*

COROLLARY 1. *If* $n(r, 0; f) + n(r, \infty; f) \not\equiv 0$ *and* f *has at most a finite number of simple zeros and poles then zero-one-pole set of* f *is unique.*

Proof. If f is transcendental, the corollary follows from Theorem 6. Let f be rational and g have the same zero-one-pole set of f. Then g is also rational and $f = cg$, where c is a constant. Since f is rational, there exists a point $z_0 \in C$ such that $f(z_0) = 1$ and so $g(z_0) = 1$. This shows that $c = 1$ and hence $f \equiv g$. This proves the corollary.

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Department of Mathematics
University of Kalyani
West Bengal, 741235
India
`indrajit@cal2.vsnl.net.in`