

## ANALYTIC DISCS IN SYMPLECTIC SPACES

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**Abstract.** We develop some symplectic techniques to control the behavior under symplectic transformation of analytic discs  $A$  of  $X = \mathbb{C}^n$  tangent to a real generic submanifold  $R$  and contained in a wedge with edge  $R$ .

We show that if  $A^*$  is a lift of  $A$  to  $T^*X$  and if  $\chi$  is a symplectic transformation between neighborhoods of  $p_o$  and  $q_o$ , then  $A$  is orthogonal to  $p_o$  if and only if  $\tilde{A} := \pi\chi A^*$  is orthogonal to  $q_o$ . Also we give the (real) canonical form of the couples of hypersurfaces of  $\mathbb{R}^{2n} \simeq \mathbb{C}^n$  whose conormal bundles have clean intersection. This generalizes [10] to general dimension of intersection.

Combining this result with the quantized action on sheaves of the “tuboidal” symplectic transformation, we show the following: If  $R, S$  are submanifolds of  $X$  with  $R \subset S$  and  $p_o \in T_S^*X|_R$  but  $ip_o \notin T_R^*X$ , then the conditions  $\text{cod}_{T^c S}(T^c R) = \text{cod}_{TS}(TR)$  (resp.  $\text{cod}_{T^c S}(T^c R) = 0$ ) can be characterized as opposite inclusions for the couple of closed half-spaces with conormal bundles  $\chi(T_R^*X)$  and  $\chi(T_S^*X)$  at  $\chi(p_o)$ .

In §3 we give some partial applications of the above result to the analytic hypoellipticity of  $CR$  hyperfunctions on higher codimensional manifolds by the aid of discs (cf. [2], [3] as for the case of hypersurfaces).

### §1. Real symplectic manifolds

Let  $X$  be a real manifold and  $T^*X$  the cotangent bundle to  $X$ ,  $(x, \xi)$  symplectic coordinates,  $\alpha = \xi dx$  the canonical one form,  $\sigma$  the two form,  $H$  the Hamiltonian isomorphism,  $\nu$  the Euler vector field,  $\chi : T^*X \rightarrow T^*X$  a real symplectic transformation.

Let  $D$  be a  $C^1$  manifold,  $D^*$  a  $C^1$  section of  $T^*X$  over  $D$ . Suppose

$$\begin{array}{ccc} D^* & \xrightarrow{\chi} & \chi(D^*) \\ \downarrow \pi & & \downarrow \pi \\ D & & \tilde{D}, \end{array}$$

and let  $p_o = (x_o, \xi_o)$ ,  $q_o = \chi(p_o) = (\tilde{x}_o, \eta_o)$ .

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PROPOSITION 1.  $\xi_o$  is orthogonal to  $T_{x_o}D$  if and only if  $\eta_o$  is orthogonal to  $T_{\tilde{x}_o}\tilde{D}$

*Proof.* We have

$$\begin{aligned} \langle \xi_o, T_{x_o}D \rangle &= \langle \xi_o, \pi' T_{p_o}D^* \rangle = \langle \pi^* \xi_o, T_{p_o}D^* \rangle \\ &= \sigma(H\pi^* \xi_o, T_{p_o}D^*) = \sigma(-\nu(p_o), T_{p_o}D^*) \\ &= \sigma(-\chi' \nu(p_o), \chi' T_{p_o}D^*) = \sigma(-\nu(q_o), T_{q_o}(\chi D^*)) \\ &= \sigma(H\pi^*(\eta_o), T_{q_o}(\chi D^*)) = \langle \pi^* \eta_o, T_{q_o}(\chi D^*) \rangle \\ &= \langle \eta_o, \pi' T_{q_o}\chi(D^*) \rangle = \langle \eta_o, T_{\tilde{x}_o}\tilde{D} \rangle. \end{aligned}$$

□

A Lagrangian submanifolds  $\Lambda$  of  $T^*X$  is a  $C^1$  submanifold whose tangent plane  $\lambda(p) = T_p\Lambda$  verifies  $\lambda(p)^\perp = \lambda(p)$ ,  $\forall p$  (with  $\perp$  denoting the  $\sigma$ -orthogonal). The intersection  $\Lambda_1 \cap \Lambda_2$  is said to be clean when it is a manifold and when  $T(\Lambda_1 \cap \Lambda_2) = T\Lambda_1 \cap T\Lambda_2$ . All manifolds will be conic i.e. invariant under  $\mathbb{R}^+$ .

Fix  $p_o = (x_o, \xi_o) \in T^*X$ :

PROPOSITION 2. Let  $M_1, M_2$  hypersurfaces,  $p_o$  a point of  $T_{M_1}^*X \cap T_{M_2}^*X$  and set

$$R = \pi(T_{M_1}^*X \cap T_{M_2}^*X).$$

Then  $T_{M_1}^*X \cap T_{M_2}^*X$  is clean if and only if  $R$  is a manifold and there exist real coordinates  $t = (t_1, t', t'')$  such that

$$\begin{cases} M_1 = \{t_1 = 0\}, \\ R = \{t_1 = t' = 0\}, \\ M_2 = \{t_1 = Q(t') + O(t')o(t', t'')\}, \quad Q \text{ non degenerate.} \end{cases}$$

*Proof.* Since  $\pi|_{T_{M_1}^*X \cap T_{M_2}^*X}$  has fiber-dimension  $\equiv 1$ , then clearly  $T_{M_1}^*X \cap T_{M_2}^*X$  is a manifold if and only if  $R$  is so. Take then real coordinates  $t = (t, t', t'')$  in  $\mathbb{R}^N \simeq X$  such that

$$M_1 = \{t_1 = 0\}, \quad R = \{t_1 = 0, t' = 0\}, \quad M_2 = \{t_1 = g(t', t'')\},$$

and  $p_o = (0; dt_1)$ ,  $g(0, 0) = 0$ ,  $dg(0, 0) = 0$ . We have

$$\begin{aligned} T_{p_o}T_{M_1}^*X &= \{(u; t dt_1) ; u \in TM_1, t \in \mathbb{R}\}, \\ T_{p_o}T_{M_2}^*X &= \{(u; t dt_1 + \text{Hess}(g)u ; u \in TM_2, t \in \mathbb{R}\}. \end{aligned}$$

Since  $g|_R \equiv 0$  and  $dg|_R \equiv 0$ , then  $\text{Hess}(g)u = 0$  if  $u'' = 0$ ; therefore  $g = Q(t') + O(t')o(t', t'')$ . Next cleanness is equivalent to the implication: “ $\text{Hess}(g)u' = 0$  implies  $u' = 0$ ” which is in turn equivalent to non-degeneracy of  $Q$ .  $\square$

*Remark 3.* When  $\text{cod}_{T_{M_1}^*X}(T_{M_1}^*X \cap T_{M_2}^*X) = 1$ , then  $Q$  is necessarily definite (positive or negative). Hence  $R = M_1 \cap M_2$  and  $M_1, M_2$  intersect to the order 2 along  $R$ . Let  $M_1^+, M_2^+$  denote the (closed) half-spaces with boundary  $M_1, M_2$  (and inward conormal  $p$ ). By the above remarks we must then have either  $M_2^+ \subset M_1^+$  or  $M_1^+ \subset M_2^+$ .

## §2. Complex symplectic manifolds

Let  $X$  be a complex manifold of dimension  $n$ ,  $T^*X$  the cotangent bundle to  $X$  with symplectic coordinates  $(z, \zeta)$ ,  $\sigma (= d\zeta \wedge dz)$  the canonical 2-form on  $T^*X$ ,  $R$  a real submanifold of  $X$ ,  $T_R^*X$  the conormal bundle to  $R$  in  $X$ ,  $p_o = (z_o, \zeta_o)$  a point of  $T_R^*X$  with  $ip_o \notin T_R^*X$ . In this situation we can identify, by a choice of coordinates,  $T_R^*X_{z_o}$  to a totally real plane  $\mathbb{R}_{y'}^l \subset \mathbb{C}^n \simeq T_{z_o}^*X$ .

For a vector  $\zeta \in \mathbb{C}^n$  we shall denote by  $|\zeta|$  the Euclidean norm  $|\zeta| = (\sum_i |\zeta_i|^2)^{1/2}$ . If  $|\Im \zeta| < |\Re \zeta|$  we also define  $\|\zeta\| = (\sum_i \zeta_i^2)^{1/2}$  (for the determination of the square root which is positive over  $\mathbb{R}^+$ ). If  $B$  is a neighborhood of  $z_o$ , and  $\Gamma_z$  for  $z \in R \cap B$  is a continuous distribution of cones in  $T_R^*X_z$  such that  $\Gamma_{z_o}$  is conic neighborhood of  $\zeta_o$  in  $T_R^*X_{z_o}$ , we consider the neighborhood  $\Sigma = \{(z, \Gamma_z) ; z \in R \cap B\}$  of  $p_o$  and denote by  $\Sigma_\varepsilon$  its  $\varepsilon$ -truncation.

We have an identification

$$(1) \quad \begin{array}{ccc} \Sigma_\varepsilon & \longrightarrow & W \\ (z'; \zeta) & \longmapsto & z' + \frac{|\zeta| \zeta}{\|\zeta\|}. \end{array}$$

Here  $W$  is a wedge of  $X$  with edge  $R$ ; for an identification  $X \simeq \mathbb{C}^n$  (in coordinates)

$$W \supset ((R \cap B) + \Gamma) \cap B$$

with  $\Gamma$  a cone of  $\mathbb{R}^l \subset X$ . In fact we see that if  $\zeta$  and  $\zeta_1$  belong to  $\Gamma_z$  with  $\zeta \neq \zeta_1$ , then  $\zeta/\|\zeta\| \neq \zeta_1/\|\zeta_1\|$  because  $\Gamma_z \cap i\Gamma_z = \emptyset$ . On the other hand the normals issued from different points of the  $C^2$  manifold  $R$  cannot have nontrivial intersection in a neighborhood of  $R$ ; and this is still true if one replaces normal directions  $\zeta/|\zeta|$  by  $\zeta/\|\zeta\|$ .

In the identification (1) we shall call  $z'$  the  $R$ -components of  $z$  and  $|\zeta|$  the distance to  $R$ . Thus  $X \setminus R$  is foliated by the surfaces of fixed distance:

$$(2) \quad \tilde{R}_t = \left\{ z = z' + t \frac{\zeta}{\|\zeta\|} ; (z'; \zeta) \in T_R^* X \times_X B \right\}, \quad t > 0 \text{ small.}$$

We consider the symplectic transformation  $\chi = \chi_t$  of  $T^* X$  into itself:

$$\chi : (z; \zeta) \mapsto \left( z + t \frac{\zeta}{\|\zeta\|}; \zeta \right).$$

Let  $s_R^\pm(p)$  denote the number of respectively positive and negative eigenvalues for the Levi form  $L_R(p)$  and also set

$$\gamma_R(z) = \dim(T_R^* X_z \cap iT_R^* X_z)$$

and

$$d_R(p) = \text{cod}(R) + s_R^-(p) - \gamma_R.$$

Consider now a new manifold  $S \supset R$ , suppose  $p \in R \times_S T_S^* X$  and note that

$$(3) \quad d_S \leq d_R \leq d_S + \text{cod}_S R.$$

Also notice that  $T_S^* X \cap T_R^* X$  is clean. Let  $\tilde{R} = \tilde{R}_t$ ,  $\tilde{S} = \tilde{S}_t$  ( $t \ll 1$ ) be the subspaces defined by (2). Denote by  $\tilde{S}^+$ ,  $\tilde{R}^+$  the closed half spaces with boundary  $\tilde{S}$ ,  $\tilde{R}$  and inward conormal  $q_o$ .

**THEOREM 4.** *Let  $R \subset S \subset X$ , and let  $p_o \in R \times_S T_S^* X$ ,  $ip_o \notin T_R^* X$ .*

(i) *Assume*

$$(4) \quad \gamma_R = \gamma_S.$$

*Then  $\tilde{R}$ ,  $\tilde{S}$  intersect at the order 2 along  $\pi(T_{\tilde{S}}^* X \cap T_{\tilde{R}}^* X)$  with  $\tilde{S}^+ \subset \tilde{R}^+$ .*

(ii) *Assume*

$$(5) \quad \gamma_R - \gamma_S = \text{cod}_S R.$$

*Then the same conclusion as in (i) holds but with  $\tilde{S}^+ \supset \tilde{R}^+$  instead of  $\tilde{S}^+ \subset \tilde{R}^+$ .*

*Proof.* Consider

$$R \subset S_1 \subset S_2 \subset \cdots \subset S_m = S, \quad \text{cod}_{S_{i+1}}(S_i) = 1.$$

We have

$$\begin{aligned}
(6) \quad \mathbb{Z} &= \mu \operatorname{hom}(\mathbb{Z}_{S_{i+1}}, \mathbb{Z}_{S_i})_p \\
&= \mu \operatorname{hom}(\mathbb{Z}_{\tilde{S}_{i+1}}, \mathbb{Z}_{\tilde{S}_i})_q [(d_{\tilde{S}_{i+1}} - d_{S_{i+1}}) - (d_{\tilde{S}_i} - d_{S_i})] \\
&= R\Gamma_{\tilde{S}_{i+1}^+}(\mathbb{Z}_{\tilde{S}_i^+})_{\tilde{z}} [(d_{\tilde{S}_{i+1}} - d_{S_{i+1}}) - (d_{\tilde{S}_i} - d_{S_i})].
\end{aligned}$$

Note now that

$$(7) \quad s_{\tilde{S}_i}^-(q) = s_{\tilde{S}_i}^-(p) \quad \forall i.$$

In fact we have

$$(8) \quad \begin{cases} \operatorname{Ker}(L_{S_i}) \xleftarrow[\pi']{\sim} T_p T_{S_i}^* X \cap iT_p T_{S_i}^* X \xrightarrow[\chi']{\sim} T_q T_{\tilde{S}_i}^* X \cap iT_q T_{\tilde{S}_i}^* X \xrightarrow[\pi']{\sim} \operatorname{Ker}(L_{\tilde{S}_i}), \\ \dim T^{\mathbb{C}} \tilde{S}_i - \dim T^{\mathbb{C}} S_i = \operatorname{cod}_X S_i - 1 - \gamma_{S_i}, \end{cases}$$

i.e.

$$(9) \quad \operatorname{rank}(L_{\tilde{S}_i}) = \operatorname{rank}(L_{S_i}) + (\operatorname{cod}_{\mathbb{C}^n} T^{\mathbb{C}} S_i - 1).$$

On the other hand it is easily seen that

$$(10) \quad s_{\tilde{S}_i}^+ \geq s_{S_i}^+ + (\operatorname{cod}_{\mathbb{C}^n} T^{\mathbb{C}} S_i - 1).$$

Thus (9), (10) give (7). It follows from (7):

$$(11) \quad (d_{\tilde{S}_{i+1}} - d_{S_{i+1}}) - (d_{\tilde{S}_i} - d_{S_i}) = \operatorname{cod}_{T^{\mathbb{C}} S_{i+1}}(T^{\mathbb{C}} S_i).$$

(i): Assume (4). Note that

$$(12) \quad \begin{aligned} \gamma_R = \gamma_S &\iff \gamma_{S_{i+1}} = \gamma_{S_i} \quad \forall i \\ &\iff \operatorname{cod}_{T^{\mathbb{C}} S_{i+1}} T^{\mathbb{C}} S_i = 1 \quad \forall i. \end{aligned}$$

Thus in this case (6) gives:

$$(13) \quad \mathbb{Z} \simeq R\Gamma_{\tilde{S}_{i+1}^+}(\mathbb{Z}_{\tilde{S}_i^+})_{\tilde{z}}[1].$$

We know on the other hand from Proposition 2 that  $\tilde{S}_i$  and  $\tilde{S}_{i+1}$  intersect at the order 2 along a 1-codimensional manifold (namely  $\pi(T_{\tilde{S}_{i+1}}^* X \cap T_{\tilde{S}_i}^* X)$ ) with either  $\tilde{S}_{i+1}^+ \subset \tilde{S}_i^+$  or  $\tilde{S}_{i+1}^+ \supset \tilde{S}_i^+$ . But (13) says that  $\tilde{S}_{i+1}^+ \subset \tilde{S}_i^+ \forall i$ . Iteration of this inclusion gives the conclusion.

(ii): Assume (5). We have

$$(14) \quad \begin{aligned} \gamma_R - \gamma_S = \text{cod}_S(R) &\iff \gamma_{S_i} - \gamma_{S_{i+1}} = 1 \quad \forall i \\ &\iff \text{cod}_{T^c S_{i+1}}(T^c S_i) = 0 \quad \forall i. \end{aligned}$$

Thus we have in this case

$$\mathbb{Z} \simeq R\Gamma_{\tilde{S}_{i+1}^+}(\mathbb{Z}_{\tilde{S}_i^+})\tilde{z}$$

which obviously implies  $\tilde{S}_{i+1}^+ \supset \tilde{S}_i$ .  $\square$

### §3. Application to analytic discs and symplectic transformations

Let  $R$  be a real submanifold of codimension  $l$  of a complex manifold  $X$  of dimension  $n$  in a neighborhood of a point  $z_o$ . Let us choose complex coordinates such that  $T_R^* X_{z_o}$  is the plane  $\mathbb{C}_{z_1, \dots, z_\gamma}^\gamma \times i\mathbb{R}_{y_{\gamma+1}, \dots, y_{l-\gamma}}^{l-2\gamma}$  and write  $z = (z', z'')$ ,  $z' = z_1, \dots, z_{l-\gamma}$ . Let us introduce a new complex symplectic transformation, that we still call  $\chi$ :

$$\chi : (z; \zeta) \longmapsto \left( z + \frac{\zeta'}{\|\zeta'\|}; \zeta \right)$$

from a neighborhood of a conormal  $p_o = (z_o, \zeta_o)$  with  $\zeta_o \in (\mathbb{C}^l \times i\mathbb{R}^{l-2\gamma}) \setminus (\mathbb{C}^l \times \{0\})$  to a neighborhood of  $p_o = \chi(p_o)$ . For this transformation  $\chi$  all conclusions of §2 hold without modifications. In particular

$$\tilde{R} := \pi\chi(T_R^* X) \text{ is a hypersurface.}$$

We shall deal with analytic discs in  $X$  and denote  $A = \{A(\tau) ; \tau \in \Delta\}$  (where  $\Delta$  is the unit disc in  $\mathbb{C}$ ). We shall say that  $A$  is “attached” to  $R$  if  $\partial A \subset R$ . The transformation above defined has the great advantage of giving a rule to interchange analytic discs “attached” to  $R$  and  $\tilde{R}$  respectively. Assume that  $R$  is defined by a system of equations  $r = 0$  ( $r = r_1, \dots, r_{l-\gamma}$ ) with  $\partial r_j|_{z_o} = dz_j$ ,  $j = 1, \dots, \gamma$ ,  $\partial r_j|_{z_o} = -i dy_j$ ,  $j = \gamma + 1, \dots, l - \gamma$  and that  $\zeta_o = (\dots, 0, -i, 0, \dots)$  where  $-i$  is in the  $(l - \gamma)$ -th position. We write  $z = (z', z'')$ ,  $z' = (z_1, \dots, z_{l-\gamma})$ ; we similarly write  $\zeta = (\zeta', \zeta'')$ ,  $\partial = (\partial', \partial'')$  and so on. Let  $A$  be a “small” analytic disc attached to  $R$  with  $A(1) = z_o$ . It is easy to prove existence of an  $(l - \gamma) \times (l - \gamma)$  matrix  $G$ , real on  $\partial\Delta$  with  $G(z_o) = id$  such that

$$G\partial'r \text{ extends holomorphically from } \partial\Delta \text{ to } \Delta.$$

To this end it is enough to solve the Bishop equation

$$(15) \quad G \Im \partial' r - T_1(G \Re \partial' r) = id_{l-\gamma \times l-\gamma} \quad \text{on } \partial \Delta$$

where  $T_1$  is the Hilbert transform with  $T_1(\cdot)|_1 = 0$ . Note that (15) is solvable, in suitable Banach spaces, by the implicit function theorem, due to  $|\Re \partial' r| \ll 1$ . Let  $\lambda = (\dots, 0, 1, 0, \dots)G$  and define

$$A^* = (A(\tau); \lambda \partial' r|_{A(\tau)}), \quad \tilde{A} = \left\{ A(\tau) + \lambda \frac{\partial' r(A(\tau))}{\|\lambda \partial' r(A(\tau))\|} \right\}.$$

It is clear that, if  $\pi : T^*X \rightarrow X$  is the canonical projection, then

$$(16) \quad \tilde{A} = \pi \chi A^*.$$

It is also obvious that  $A^*$ , and hence  $\tilde{A}$  are holomorphic discs and that

$$\partial \tilde{A} \subset \tilde{R}$$

due to  $\lambda \partial r|_{\partial A} \hookrightarrow T_R^*X|_{\partial A}$ . If we apply Proposition 1 to  $A^* \subset T^*X^{\mathbb{R}}$  we get  $\Re \langle \partial_\tau \tilde{A}, \zeta_o \rangle = 0$ ; if we apply it to  $iA^* \hookrightarrow T^*X^{\mathbb{R}}$  we get  $\Im \langle \partial_\tau \tilde{A}, \zeta_o \rangle = 0$  which implies  $\partial_\tau \tilde{A} \in T^{\mathbb{C}}\tilde{R}$ .

Let  $W$  be a “wedge” with edge  $R$  (cf. [8]). For an open cone  $\Gamma \subset (T_S X)_{z_o}$  the so called “profile” of  $W$ , in an identification by coordinates  $X \simeq \mathbb{C}^n = T_{z_o}R \oplus (T_R X)_{z_o}$ , and for a neighborhood  $B$  of  $z_o$ ,  $W$  has the form

$$W = ((B \cap R) + \Gamma) \cap B.$$

Let  $\mathcal{O}_X$  be the sheaf of holomorphic functions on  $X$ . Let  $S$  be a submanifold of  $X$  which contains  $R$  and which has  $\zeta_o$  among its conormals at  $z_o$ . Let  $\mathcal{C}_{S|X}$ ,  $\mathcal{B}_{S|X}$  be the complexes of  $CR$  microfunctions and  $CR$  hyperfunctions along  $S$  respectively. Let  $sp : H^0(\pi^{-1}\mathcal{B}_{S|X}) \rightarrow H^0(\mathcal{C}_{S|X})$  be the spectral morphism, and define

$$SS(u) = \text{supp } sp(u), \quad u \in \mathcal{B}_{S|X}.$$

Note that  $SS(u)_{z_o} = \{0\}$  if and only if  $u$  is a holomorphic function in a neighborhood of  $z_o$ . Let  $\zeta_o \in T_S^*X_{z_o}$ , take  $\Gamma \subset \{\Re \langle z, \zeta_o \rangle > 0\}$ , and set  $W^\pm = ((B \cap R) \pm \Gamma) \cap B$ .

**THEOREM 5.** *Assume*

- (i)  $A \subset R \cup W^-$  (resp.  $A \subset R \cup W^+$ ),
- (ii)  $\gamma_S = \gamma_R$  (resp.  $\gamma_R - \gamma_S = \text{cod}_S R$ ),
- (iii)  $T_{z_o} A \perp \zeta_o$ ,
- (iv)  $A \not\subset R$  in any neighborhood of  $z_o$ .

Then for  $f \in (\mathcal{B}_{S|X})_{z_o}$  we have  $p_o \notin SS(b(f))$  (resp.  $-p_o \notin SS(b(f))$ ).

*Remark 6.* It is not necessary to assume  $A \not\subset R$  in order to get an analytic disc  $\tilde{A} \subset \tilde{S}^\mp \setminus \tilde{S}$  which is the only fact we really need in the proof. Here again  $\tilde{S}^\mp$  are the closed half spaces with boundary  $\tilde{S}$  and inward conormal  $\mp \zeta_o$ . Thus let  $S : r' = 0$ ,  $R : r' = 0, r'' = 0$ . Assume for instance there is an analytic “lift”  $A^*$  i.e. a holomorphic section of  $T^*X$  over  $A$  such that:

$$A^*|_{\partial A} \subset T_R^*X \setminus T_S^*X$$

i.e.  $A^* = (A; \theta \partial r)$  with  $\theta \partial r$  extending holomorphically,  $\theta$  real over  $\partial A$ ,  $\theta'' \neq 0$ . Then

$$\partial \tilde{A} \subset \tilde{R} \subset \tilde{S}^- \quad \text{but} \quad \partial \tilde{A} \not\subset \tilde{S}.$$

*Proof.* Let  $\{B_r\}$  (resp.  $\{\tilde{B}_r\}$ ) be the family of spheres with center  $z_o$  (resp.  $\tilde{z}_o$ ) and radius  $r$ . We can find a sequence of subdiscs  $A_\nu$  such that

$$A_\nu \subset A \cap B_{r_\nu}, \quad \partial A_\nu \not\subset R$$

(for a sequence  $r_\nu \rightarrow 0$ ). Suppose we are proving the statement for  $p_o$ . By the discussion above, these are interchanged to analytic discs  $\tilde{A}_\nu$  such that

$$\partial \tilde{A}_\nu \subset (\tilde{R}^- \cap \tilde{B}_{s_\nu}) \subset (\tilde{S}^- \cap \tilde{B}_{s_\nu}) \quad \text{but} \quad \partial \tilde{A}_\nu \not\subset \tilde{S},$$

(since  $\tilde{R}^- \subset \tilde{S}^-$  due to  $\gamma_R = \gamma_S$ ) for a new sequence  $s_\nu \rightarrow 0$ ). By Proposition 1 we also have

$$(17) \quad T_{\tilde{z}_o} \tilde{A}_\nu \subset T_{\tilde{z}_o}^{\mathbb{C}} \tilde{S}.$$

We then enter [3, Theorem 1] and conclude that holomorphic functions  $\tilde{f}$  in  $\overset{\circ}{\tilde{S}^-} \cap \tilde{B}_\nu$  extend to a full neighborhood of  $\tilde{z}_o$ ; thus germs of holomorphic functions on  $\overset{\circ}{\tilde{S}^-}$  extend to  $\mathbb{C}^n$  at  $\tilde{z}_o$ . Now we introduce a quantization  $\phi_K$  of  $\chi$  by a kernel  $K$ . This induces a “microlocal” transformation of  $\mathcal{O}_X$ .  $CR$  hyperfunctions  $u$  at  $z_o$  are transformed into sums of boundary values  $b(\tilde{f}^+) + b(\tilde{f}^-)$  on  $\tilde{S}$  of germs  $\tilde{f}^\pm \in \mathcal{O}_X(\overset{\circ}{\tilde{S}^\pm})_{\tilde{z}_o}$  in such a way that  $p_o \notin SSb(f)$  if and only if  $\tilde{f}^-$  extends at  $\tilde{z}_o$ . The proof is complete.  $\square$



If we take  $R = S$  in Theorem 5 and consider wedges  $W^\pm$  with edge  $S$  we regain [2, Proposition 7] by a new method of “reduction to a hypersurface”. If moreover we assume that  $A$  is orthogonal to any conormal  $\zeta \in T_S^*X_{z_o}$  (instead of the only  $\zeta_o$ ) we get:

COROLLARY 7. *Assume*

- (i)  $A \subset W^\mp \cup S$  but  $A \not\subset S$  in any neighborhood of  $z_o$ ,
- (ii)  $T_{z_o}A \subset T_{z_o}^{\mathbb{C}}S$ .

Then any  $f \in \mathcal{O}_X(W^\mp)_{z_o}$  extends holomorphically to a full neighborhood of  $z_o$ .

*Proof.* We apply Theorem 5 to all  $p \in \pm\Gamma^*$  and conclude that  $\pm\Gamma^* \cap SSb(f) = \{0\}$ . On the other hand recall that there is an elementary estimate of microsupport; for  $f \in \mathcal{O}_X(W^\mp)_{z_o}$  we have  $SSb(f)_{z_o} \subset \pm\Gamma^*$ . Hence we can conclude  $SSb(f)_{z_o} = \{0\}$ .  $\square$

EXAMPLE 8. In  $\mathbb{C}^4$  let

$$\begin{aligned} S &= \{y_3 = z_1 z_2 + \bar{z}_1 \bar{z}_2, y_4 = 0\}, \quad \mp p = \mp dy_3 + \lambda dy_4, \\ R &= \{y_2 = 0, y_3 = z_1 z_2 + \bar{z}_1 \bar{z}_2, y_4 = 0\}, \\ A &= \mathbb{C}_{z_1} \times \{0\} \times \{0\} \times \{0\}. \end{aligned}$$

We can find a section  $\lambda \partial r \in (T_R^*X \setminus T_S^*X)|_{\partial A}$  which extends holomorphically. For that just notice that the tangent direction  $u = (1, 0, \dots)$  to  $A$  verifies  $u \in \text{Ker}(L_R)(\lambda \partial r)$ . Hence Remark 6 applies and yields  $\pm p \notin WF(f)$ .

#### §4. Appendix. Positivity of Lagrangians (cf. [4])

We shall further exploit here the techniques of §2 to give an extension of the results of [4].

Let  $X$  be a complex manifold,  $R$  and  $S$  real submanifolds of  $X$  with  $R \subset S$ . Recall that  $T_R^*X \cap T_S^*X$  is clean and that (3) of §2 holds. Let  $p \in R \times_S T_S^*X$ ,  $ip \notin T_R^*X$ .

THEOREM 9. (i) *Suppose*

$$(18) \quad d_R - d_S = \text{cod}_S R.$$

Then there exists a germ of a homogeneous complex symplectic transformation  $\chi$  of  $T^*X$  from a neighborhood of  $p_o$  to a neighborhood of  $q_o = \chi(p_o)$  which interchanges

$$T_R^*X \xrightarrow{\sim} T_{\tilde{R}}^*X, \quad T_S^*X \xrightarrow{\sim} T_{\tilde{S}}^*X,$$

for a pair of hypersurfaces  $\tilde{R}, \tilde{S}$  with  $s_{\tilde{R}}^-(q_o) = 0, s_{\tilde{S}}^-(q_o) = 0$  and such that  $\tilde{R}, \tilde{S}$  intersect at the order 2 along  $\pi(T_{\tilde{R}}^*X \cap T_{\tilde{S}}^*X)$  with  $\tilde{R}^+ \supset \tilde{S}^+$ .

(ii) Suppose

$$(19) \quad d_R = d_S.$$

Then there exists  $\chi$  such that the same conclusion as in (i) holds but with  $\tilde{S}^+ \supset \tilde{R}^+$  instead of  $\tilde{S}^+ \subset \tilde{R}^+$ .

*Remark 10.* Generally, the transformation  $\chi$  of §2 does not suffice for the conclusion of Theorem 9.

*Proof.* Consider

$$R = S_1 \subset S_2 \subset \cdots \subset S_m = S, \quad \text{cod}_{S_{i+1}} S_i = 1.$$

Put  $\tilde{a}_i = d_{\tilde{S}_i} - d_{\tilde{S}_{i+1}}, a_i = d_{S_i} - d_{S_{i+1}}$ . By the result of §2 we have

$$\mathbb{Z} \simeq R\Gamma_{\tilde{S}_{i+1}^+}(\mathbb{Z}_{\tilde{S}_i^+})_{\bar{z}}[a_i - \tilde{a}_i].$$

Recall that  $0 \leq a_i \leq 1$ . Thus (18) and (19) are equivalent to  $a_i = 1 \forall i$  and  $\tilde{a}_i = 1 \forall i$  respectively.

We recall that if a submanifold  $\Lambda \subset T^*X$  is  $\mathbb{R}$  Lagrangian (i.e. Lagrangian for  $\sigma^{\mathbb{R}}$  the real part of  $\sigma$ ) and verifies

$$(20) \quad \dim(T_{p_o}\Lambda \cap \mathbb{C}H(\zeta_o dz)) = 1,$$

then  $\Lambda$  is symplectically equivalent to the conormal bundle to a hypersurface. (Note here that if  $\Lambda = T_R^*X$ , then (20) is equivalent to  $ip_o \notin T_R^*X$ , hence this latter condition characterizes the higher codimensional manifolds  $R$  which are ‘‘symplectically equivalent’’ to a hypersurface.) In particular for any family of Lagrangian manifolds  $\Lambda_i, i = 1, \dots, m$  which satisfy (20) we can find  $\chi$  such that

$$(21) \quad \Lambda_i \xrightarrow[\chi]{\sim} T_{M_i}^*X, \quad \text{cod}(M_i) = 1 \quad \forall i.$$

Also we can arrange (cf. [6]) that

$$(22) \quad s_{M_i}^-(q_o) = 0 \quad \text{for at least one } i.$$

We shall apply the above remarks for  $\Lambda_i = T_{S_i}^* X$ .

(i): We take in this case  $\chi$  such that

$$T_{S_i}^* X \xrightarrow{\sim} T_{\tilde{S}_i}^* X, \quad \text{cod}(\tilde{S}_i) = 1 \quad \forall i, \quad s_{\tilde{R}}^-(q_o) = 0$$

Assume  $s_{\tilde{S}_i}^-(q_o) = 0$ ; we show that

$$(23) \quad \begin{cases} \tilde{S}_{i+1}^+ \subset \tilde{S}_i^+, \\ s_{\tilde{S}_{i+1}}^-(q_o) = 0 \quad \forall i. \end{cases}$$

In fact we are in the situation

$$\begin{cases} \tilde{a}_i = -s_{\tilde{S}_{i+1}}^-(q_o), \\ a_i = 1, \end{cases}$$

whence  $a_i - \tilde{a}_i = s_{\tilde{S}_{i+1}}^-(p_o) + 1$  and

$$(24) \quad \mathbb{Z} = R\Gamma_{\tilde{S}_{i+1}}(\mathbb{Z}_{\tilde{S}_i}^{\tilde{z}}[1 + s_{\tilde{S}_{i+1}}^-]).$$

But since we know from Proposition 2 that  $\tilde{S}_i, \tilde{S}_{i+1}$  intersect at the order 2 along a 1-codimensional submanifold with either of the inclusions  $\tilde{S}_{i+1}^+ \subset \tilde{S}_i^+$  or  $\tilde{S}_{i+1}^+ \supset \tilde{S}_i^+$ , then (24) implies (23).

Hence induction applies and gives the conclusion

$$\begin{cases} \tilde{S}^+ \subset \tilde{R}^+, \\ s_{\tilde{S}}^-(q_o) = 0. \end{cases}$$

(ii): We take now  $\chi$ :

$$\chi : T_{S_i}^* X \xrightarrow{\sim} T_{\tilde{S}_i}^* X, \quad \text{cod}(\tilde{S}_i) = 1 \quad \forall i, \quad s_{\tilde{S}}^-(q_o) = 0.$$

Assume  $s_{\tilde{S}_{i+1}}^-(q_o) = 0$ ; we show that

$$(25) \quad \begin{cases} s_{\tilde{S}_i}^-(q_o) = 0, \\ \tilde{S}_i^+ \subset \tilde{S}_{i+1}^+. \end{cases}$$

In fact we have

$$\begin{cases} \tilde{a}_i = +s_{\tilde{S}_i}^-(q_o), \\ a_i = 0 \end{cases}$$

Thus  $a_i - \tilde{a}_i = -s_{\tilde{S}_i}^-$  and therefore

$$\mathbb{Z} = R\Gamma_{\tilde{S}_{i+1}^+}(\mathbb{Z}_{\tilde{S}_i^+})_{\tilde{z}}[-s_{\tilde{S}_i}^-],$$

which implies (25). The conclusion will follow again by induction.  $\square$

*Remark 11.* Recall the semiorder relation of positivity “ $\succ$ ” between Lagrangians in the sense of [5]. Thus we have in fact proved that  $T_R^*X \succ T_S^*X$  in case (i) (resp.  $T_S^*X \succ T_R^*X$  in case (ii)).

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