H. Render and A. Sauer
Nagoya Math. J.
Vol. 159 (2000), 167–178

MULTIPLIERS ON VECTOR SPACES OF HOLOMORPHIC FUNCTIONS

HERMANN RENDER AND ANDREAS SAUER

Abstract. Let G be a domain in the complex plane containing zero and $H(G)$ be the set of all holomorphic functions on G. In this paper the algebra $M(H(G))$ of all coefficient multipliers with respect to the Hadamard product is studied. Central for the investigation is the domain $\widehat{G}$ introduced by Arakelyan which is by definition the union of all sets $\frac{1}{w}G$ with $w \in G^c$. The main result is the description of all isomorphisms between these multipliers algebras. As a consequence one obtains: If two multiplier algebras $M(H(G_1))$ and $M(H(G_2))$ are isomorphic then $\widehat{G_1}$ is equal to $\widehat{G_2}$. Two algebras $H(G_1)$ and $H(G_2)$ are isomorphic with respect to the Hadamard product if and only if G_1 is equal to G_2. Further the following uniqueness theorem is proved: If G_1 is a domain containing 0 and if $M(H(G))$ is isomorphic to $H(G_1)$ then G_1 is equal to $\widehat{G}$.

Introduction

The concept of multipliers is a very powerful and widely used tool in mathematical analysis. In this paper we consider coefficient multipliers with respect to the Hadamard product of holomorphic functions. Recall that the *Hadamard product* of two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined by $f * g(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. Throughout the paper we assume that G_1 and G_2 are domains in the complex plane containing zero and $H(G_i)$ denotes the set of all holomorphic functions on G_i for $i = 1, 2$. A power series $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is called a *coefficient multiplier* if $g * f \in H(G_2)$ for all $f \in H(G_1)$, i.e., that $T_g(f) := g * f$ defines a linear mapping $T_g\colon H(G_1) \to H(G_2)$, cf. e.g. [2, 6]. For the case $G_1 = G_2$ one obtains that the set $M(H(G))$ of all coefficient multipliers is an algebra with respect to composition. We consider the following questions: is it possible to identify the coefficient multiplier algebra $M(H(G))$ with a certain vector space of holomorphic functions? What does it mean that two coefficient multiplier algebras $M(H(G_1))$ and $M(H(G_2))$ are isomorphic?

An important characterization of coefficient multipliers has been given

Received January 18, 1999.

in Theorem 1 in [6]: a power series $g(u) := \sum_{n=0}^{\infty} b_n u^n$ is a coefficient multiplier if and only if for every $w \in G_1^c$ the power series g has an analytic continuation to the domain $\frac{1}{w}G_2$. It follows that each function g holomorphic on the domain

$$\widehat{G_1 G_2} := \bigcup_{w \in G_1^c} \frac{1}{w} G_2 \tag{1}$$

induces a coefficient multiplier $T_g\colon H(G_1) \to H(G_2)$. In the case $G_1 = G_2$ we simply write $\widehat{G_1}$ instead of $\widehat{G_1 G_1}$. The above characterization leads to a linear embedding of $H(\widehat{G})$ into the algebra $M(H(G))$ of all coefficient multipliers. Up to now there is no general easy criterion on the domain under which conditions this embedding is actually an isomorphism. However, in [11] it is shown that for a simply connected domain G the above embedding $L\colon H(\widehat{G}) \to M(H(G))$ is surjective if and only if G is an α-starlike domain which means that $\{t^{1+i\alpha} z : t \in [0,1], z \in G\} \subset G$ with respect to $\alpha \in \mathbb{R}$.

Our main result states that the isomorphy of two multiplier algebras $M(H(G_1))$ and $M(H(G_2))$ implies that $\widehat{G_1}$ is necessarily equal to $\widehat{G_2}$. Indeed, we are able to describe all isomorphisms between two coefficient multiplier algebras, cf. Theorem 3.3. This result has an interesting consequence for a given multiplier algebra $M(H(G))$: assume that there exists a domain $\widetilde{G}$ in the complex plane such that $M(H(G))$ is isomorphic to $H(\widetilde{G})$. Then $\widetilde{G}$ is necessarily equal to $\widehat{G}$ and the natural embedding $L\colon H(\widehat{G}) \to M(H(G))$ is already an isomorphism.

The paper is divided in three sections. In the first one we give equivalent operator-theoretic characterizations for multipliers which may be interesting in its own right. The second section shows that $M(H(G))$ possesses a so-called strongly orthogonal sequence. It follows that an isomorphism on $M(H(G))$ permutes the Taylor coefficients of the power series. The third section contains the above-mentioned main results.

Finally we fix some notations. By $\mathbb{D}$ we denote the open unit disk. More generally $\mathbb{D}_r$ denotes the open disk with center 0 and radius $r > 0$. Further γ is the geometric series $\gamma(z) = 1/(1-z)$. Note that $\gamma \in H(G)$ if and only if $1 \in G^c$. For simplicity we identify z^n with the function $z \mapsto z^n$ on a domain G. In order to avoid pathologies (e.g. in the definition of $\widehat{G}$) we assume that the domains are different from $\mathbb{C}$. This is not a real restriction since the multipliers $T\colon H(\mathbb{C}) \to H(G_2)$ correspond to the power series with positive radius of convergence (see [6, p. 79]).

Recall that a domain G containing 0 is *admissible* if the set $H(G)$ of all holomorphic functions on G is an algebra with respect to the Hadamard product. By the Hadamard multiplication theorem G is admissible if and only if the complement G^c is a multiplicative semigroup. An important observation due to N. Arakelyan is the fact that $\widehat{G}$ is admissible, cf. Lemma 2.1 in [1]. Hence $H(\widehat{G})$ is an algebra with unit element γ and $H(G)$ is a module over the ring $H(\widehat{G})$ by the Hadamard multiplication theorem, see e.g. [11].

§1. Characterizations of coefficient multipliers

Let G be a domain containing 0. Then $H(G)$ is a Fréchet space, i.e. a completely metrizable locally convex vector space where the (semi)-norms are given by $|f|_K := \sup_{z \in K} |f(z)|$ for an arbitrary compact subset K of G. The functionals $\delta_n \colon H(G) \to \mathbb{C}$ defined by $\delta_n(f) := a_n$ (where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ locally) are called the *dirac functionals.* The proof of the following lemma is omitted.

LEMMA 1.1. *The functional $\delta_n \colon H(G) \to \mathbb{C}$ is continuous with respect to the topology of compact convergence.*

Observe that for any entire function f and for $g \in H(G_1)$ the function $f * g$ is an entire function which can also be considered as an element of $H(G_1)$ and $H(G_2)$. In particular condition c) and e) in Theorem 1.2 are meaningful where exp denotes the exponential function.

THEOREM 1.2. *Let $T \colon H(G_1) \to H(G_2)$ be a linear operator. Then the following statements are equivalent:*

a) *T is a coefficient multiplier.*

b) *$\delta_n \circ T = b_n \delta_n$ for all $n \in \mathbb{N}_0$ and suitable $b_n \in \mathbb{C}$.*

c) *T is continuous and $T(f * \exp) = T(f) * \exp$ for all $f \in H(G_1)$.*

d) *There exist $b_n \in \mathbb{C}, n \in \mathbb{N}_0$, such that $T(f)(z) = \sum_{n=0}^{\infty} b_n a_n z^n$ in a neighborhood of zero for all $f \in H(G_1)$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ locally.*

e) *$T(f * z^n) = T(f) * z^n$ for all $f \in H(G_1)$ and $n \in \mathbb{N}_0$.*

Proof. For a) $\Rightarrow$ b) let $g(z) = \sum_{n=0}^{\infty} b_n z^n$ and $T_g(f) = g * f$ be a coefficient multiplier. Then $\delta_n(T_g(f)) = \delta_n(f * g) = b_n \delta_n(f)$ which proves b).

For b) $\Rightarrow$ c) we show at first the continuity of T by applying the closed graph theorem: Let $f_k \to 0$ in $H(G_1)$ and assume that $T(f_k)$ converges to some $g \in H(G_2)$. It suffices to show that $g = 0$. By b) and Lemma 1.1 each functional $\delta_n \circ T$ is continuous. Hence $\delta_n(T)(f_k)$ converges to 0. On the other hand $\delta_n(T)(f_k)$ converges to $\delta_n(g)$ since $T(f_k) \to g$ and δ_n is continuous. Hence $\delta_n(g) = 0$ for all $n \in \mathbb{N}_0$ and therefore $g = 0$ by the identity theorem. Thus T is continuous. For $T(f) \in H(G_2)$ we have $T(f)(z) = \sum_{k=0}^{\infty} c_k z^k$ in a neighborhood of 0. Then $\delta_n(T(f) * \exp) = c_n/n!$. On the other hand $T(f * \exp)(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} T(z^k)$ by continuity. Hence $\delta_n(T(f * \exp)) = \sum_{k=0}^{\infty} \frac{a_k}{k!} \delta_n \circ T(z^k) = \sum_{k=0}^{\infty} \frac{a_k}{k!} b_n \delta_n(z^k) = a_n b_n/n!$. Hence $c_n = a_n b_n$ and c) is proved.

For c) $\Rightarrow$ d) we show at first that there exist $b_n \in \mathbb{C}$ with $T(z^n) = b_n z^n$ for all $n \in \mathbb{N}_0$. Let $T(z^n) = \sum_{k=0}^{\infty} c_k z^k$ in a neighborhood of 0. Then $\frac{1}{n!} T(z^n) = T(z^n * \exp(z)) = T(z^n) * \exp(z) = \sum_{k=0}^{\infty} \frac{c_k}{k!} z^k$ in a neighborhood of 0. Hence $c_k/n! = c_k/k!$ which implies $c_k = 0$ for all $k \neq n$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(G_1)$. We claim that $T(f)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$ in some neighborhood of 0. Let $T(f)(z) = \sum_{n=0}^{\infty} c_n z^n$. Since $f * \exp$ is an entire function the continuity of T implies $T(f * \exp)(z) = \sum_{n=0}^{\infty} \frac{a_n b_n}{n!} z^n$. On the other hand $(T(f) * \exp)(z) = \sum_{n=0}^{\infty} \frac{c_n}{n!} z^n$. By c) we obtain $c_n = a_n b_n$ for all $n \in \mathbb{N}_0$.

d) $\Rightarrow$ e) is easy. For e) $\Rightarrow$ a) note that $T(z^n) = T(z^n) * z^n$. Thus there exists $b_n \in \mathbb{C}$ with $T(z^n) = b_n z^n$. Put $g(z) = \sum_{n=0}^{\infty} b_n z^n$. Let $T(f)(z) = \sum_{n=0}^{\infty} c_n z^n$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in a neighborhood of 0. Now e) implies that $a_k b_k = c_k$. Hence $T(f) = f * g$ for all $f \in H(G_1)$. ∎

§2. Orthogonal families and multiplier algebras

Let A be an algebra over the field K of real or complex numbers. A family of distinct points $z_i \in A, i \in I$ is called *strongly orthogonal* if $z_i z_i = z_i \neq 0$ for all $i \in I$ and $a z_i \in K \cdot z_i$ for all $a \in A, i \in I$. Note that a linear functional $\delta_i : A \to K$ is induced via the formula $a z_i = \delta_i(a) z_i$. We call $(z_i)_{i \in I}$ *separating* if $a z_i = 0$ for all $i \in I$ implies that $a = 0$ for each $a \in A$. Obviously this is equivalent to say that the functionals $\delta_i, i \in I$ separate the points. Algebras with a strongly orthogonal family have been discussed in [12] where further references and examples can be found, cf. also [3] for algebras with an orthogonal basis.

THEOREM 2.1. *Let $L_{z^n}: H(G) \to H(G)$ be defined by $L_{z^n}(f) = z^n * f$. Then $(L_{z^n})_{n \in \mathbb{N}_0}$ is a strongly orthogonal and separating sequence in $M(H(G))$.*

Proof. It is easy that $L_{z^n} \circ L_{z^n} = L_{z^n} \neq 0$ for all $n \in \mathbb{N}_0$. Now let T_g, defined by $T_g(f) = g * f$, be a coefficient multiplier. Since $g * z^n = \lambda z^n$ for some $\lambda \in \mathbb{C}$ we obtain $T_g \circ L_{z^n}(f) = g * (z^n * f) = \lambda \cdot f * z^n = \lambda L_{z^n}(f)$. Hence $T_g \circ L_{z^n} \in \mathbb{C} \cdot L_{z^n}$ for all $n \in \mathbb{N}_0$. For the second statement assume that $T_g \circ L_{z^n} = 0$ for all $n \in \mathbb{N}_0$. It follows that the Taylor coefficients of g are zero and therefore T_g is zero. ∎

THEOREM 2.2. *Let A and B be algebras with strongly orthogonal families $(a_i)_{i\in I}$ and $(b_j)_{j\in J}$ respectively and suppose that $(b_j)_j$ is separating. If $\Psi: A \to B$ is an isomorphism then for each $i \in I$ there exists $\psi(i) := j \in J$ such that $\Psi(a_i) = b_j = b_{\psi(i)}$ and $\psi: I \to J$ is bijective.*

Proof. Let $i \in I$. Since $(b_j)_{j\in J}$ is separating and $\Psi(a_i) \neq 0$ there exists $j \in J$ such that $\delta_j(\Psi(a_i)) \neq 0$. Choose $a \in A$ such that $\Psi(a) = b_j$. Then

$$\delta_i(a)\Psi(a_i) = \Psi(\delta_i(a)a_i) = \Psi(aa_i) = \Psi(a)\Psi(a_i) = b_j\Psi(a_i) = b_j\delta_j(\Psi(a_i)). \tag{2}$$

Since $\delta_j(\Psi(a_i)) \neq 0$ we infer $\delta_i(a) \neq 0$ and therefore $\Psi(a_i) = \lambda b_j$ for some $\lambda \neq 0$. Since $a_i^2 = a_i$ it is easy to see that $\lambda = 1$. Further it is easy to see that ψ is a bijection. ∎

The next result will not be used in the sequel but it might be interesting in its own right. Recall that a topological algebra is a B_0-*algebra* if the topology is locally convex and completely metrizable.

THEOREM 2.3. *Let A and B be B_0-algebras with strongly orthogonal families $(a_i)_{i\in I}$ and $(b_j)_{j\in J}$ respectively. If $(b_j)_j$ is separating then every isomorphism $\Psi: A \to B$ is topological.*

Proof. By the open mapping theorem it suffices to show that Ψ is continuous. Note that the multiplicative functionals $\delta_j: B \to \mathbb{C}$ separate the points of B. Moreover $h_j := \delta_j \circ \Psi$ is multiplicative. We show that h_j is continuous: by Theorem 2.2 there exists $i \in I$ such that $\Psi(a_i) = b_j$. Then $h_j(a_i) = 1$ and therefore

$$h_j(a) = h_j(aa_i) = h_j(\delta_i(a)a_i) = \delta_i(a)h_j(a_i) = \delta_i(a). \tag{3}$$

Hence we have proved that $h_j = \delta_i$. By Lemma 3.1 in [8] the functionals δ_i are continuous. An application of the closed graph theorem yields the continuity of Ψ, cf. the proof of Theorem 13.2 in [13]. ∎

§3. Isomorphisms of $M(H(G))$

Let G_1, G_2 be domains containing 0. We call a linear map $\Phi\colon H(G_1) \to H(G_2)$ a *permutation operator* if there exists an injective map $\varphi\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that for each function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $H(G_1)$ the function $\Phi(f)$ is locally of the form

$$\Phi(f)(z) = \sum_{n=0}^{\infty} a_n z^{\varphi(n)}. \tag{4}$$

A permutation operator Φ is continuous with respect to the topology of compact convergence on $H(G_i)$ for $i = 1, 2$. This rests on the observation that each functional $\delta_n\colon H(G_i) \to \mathbb{C}$ is continuous (Lemma 1.1) and that $\delta_n \circ \Phi$ is equal to δ_m with $m := \varphi^{-1}(n)$ or to the zero functional. An appeal to the closed graph theorem yields the continuity of Φ.

Isomorphisms between *algebras* of holomorphic functions with Hadamard multiplication are permutation operators (in particular continuous), see [9], [10]. We need a slight generalization of this result:

PROPOSITION 3.1. *Let $\Phi\colon H(G_1) \to H(G_2)$ be an injective linear map. If G_1 is admissible and $\delta_n \circ \Phi$ is a multiplicative functional for each $n \in \mathbb{N}_0$ then Φ is a permutation operator.*

Proof. First we show that $\Phi(z^n) = z^{\varphi(n)}$ for some $n \in \mathbb{N}_0$. We know that $\Phi(z^n)$ is locally of the form $\sum_{k=0}^{\infty} a_k z^k \neq 0$ (note that Φ is injective). By assumption each $h_l := \delta_l \circ \Phi$ is multiplicative. Note that $z^n * z^n = z^n$ and $z^n * z^m = 0$. Hence $h_l(z^n)$ is equal to 0 or 1 and there exists exactly one $l_n \in \mathbb{N}_0$ with $h_{l_n}(z^n) = 1$. Since $h_l(\Phi(z^n)) = a_l$ we infer $\Phi(z^n) = z^{\varphi(n)}$ with $\varphi(n) := l_n$. Since Φ is injective it follows that φ is injective. Let us prove that Φ is continuous: the multiplicative functional $\delta_n \circ \Phi$ is continuous by the results in [8]. An appeal to the closed graph theorem yields the continuity of Φ. In order to show that Φ is a permutation operator let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $H(G_1)$. Then $\Phi(f)$ can be expanded in a power series, say $\sum_{n=0}^{\infty} b_n z^n$. Since $f(z) * \exp(z)$ is an entire function the continuity of Φ implies that

$$\Phi(f(z) * \exp(z)) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^{\varphi(n)}.$$

It follows that $\delta_{\varphi(n)}(\Phi(f(z) * \exp(z))) = a_n/n!$ and similarly $\delta_{\varphi(n)}(\Phi(\exp(z))) = 1/n!$. Since $\delta_{\varphi(n)} \circ \Phi$ is multiplicative we have $\delta_{\varphi(n)} \circ \Phi(f * \exp) = [\delta_{\varphi(n)} \circ \Phi(f)] \cdot [\delta_{\varphi(n)} \circ \Phi(\exp)]$. Comparison of the coefficients shows that $a_n/n! =$

$b_{\varphi(n)} \cdot 1/n!$ and $b_m = 0$ for all $m \in \mathbb{N}_0 \setminus \{\varphi(n) : n \in \mathbb{N}_0\}$. Hence $\Phi(f)(z) = \sum_{n=0}^{\infty} a_n z^{\varphi(n)}$. ∎

The number k_G in the next definition will be a characteristic of the domain G.

Definition 3.2. Let G be a domain containing 0. For $k \in \mathbb{N}$ we denote by A_k the set of all k-th roots of unity. If there exists a largest natural number $k \in \mathbb{N}$ such that

$$\xi w \in G^c \text{ for all } \xi \in A_k, w \in G^c \tag{5}$$

this number is denoted by k_G. Note that for $k = 1$ the condition is always satisfied.

Suppose that there does not exist a largest number. Then we can find a sequence $(k_n)_n$ satisfying (5). Let $w_0 \in G^c$ with $|w_0| \leq |w|$ for all $w \in G^c$. Then $\{w_0\xi : \xi \in A_{k_n}, n \in \mathbb{N}\} \subset G^c$ is dense in the circle of radius $|w_0|$. It follows that G is equal to $\{z \in \mathbb{C} : |z| < |w_0|\}$. Hence $k_G \in \mathbb{N}$ if and only if G is different from $\mathbb{D}_r$ for all $r > 0$. Moreover $\widehat{G}$ is equal to $\mathbb{D}$ if and only if G is equal to some $\mathbb{D}_r$.

Lemma 3.3. *The number k_G is equal to the cardinality of $M := \{z \in \widehat{G}^c : |z| = 1\}$ which is denoted by $k_{\widehat{G}}$.*

Proof. By Lemma 2.1 in [1] $\widehat{G}^c$ is a multiplicative semi-group with unit element. Now it is not hard to see that M is either equal to A_k with suitable $k \in \mathbb{N}$ or it is the boundary of the unit disk. Hence $k_G \in \mathbb{N}$ if and only if $k_{\widehat{G}} \in \mathbb{N}$. Let $k \in \mathbb{N}$ with $\xi w \in G^c$ for all $\xi \in A_k, w \in G^c$. Suppose that $\xi \in \widehat{G}$. Then there exists $w \in G^c$ and $z \in G$ with $\xi = z/w$, i.e., that $w\xi \in G$, a contradiction. Hence $A_k \subset \widehat{G}^c$ and $k_G \leq k_{\widehat{G}}$. For the other inequality assume that $A_k \subset \widehat{G}^c$. For $\xi \in A_k$ we infer that $w\xi \in \widehat{G}^c$ for all $w \in G^c$. Since k_G is the largest number with this property we obtain $k_{\widehat{G}} \leq k_G$. ∎

For the next result note that by Theorem 2.2 there exists a permutation $\psi: \mathbb{N}_0 \to \mathbb{N}_0$ with $\Psi(L_{z^n}) = L_{z^{\psi(n)}}$.

THEOREM 3.4. *Let G_1, G_2 be domains containing* 0 *and different from* $\mathbb{D}_r$ *for all* $r > 0$. *Let* $\Psi: M(H(G_1)) \to M(H(G_2))$ *be an isomorphism. Then* $k := k_{G_1} = k_{G_2}$ *and there exist* $n_0 \in \mathbb{N}_0$ *and* $b_0, \ldots, b_{k-1} \in \mathbb{Z}$ *such that* $\psi(kn + j) = kn + b_j$ *for all* $nk + j \geq n_0$ *and for all* $j = 0, \ldots, k-1$ *where* $\psi: \mathbb{N}_0 \to \mathbb{N}_0$ *is given by the formula* $\Psi(L_{z^n}) = L_{z^{\psi(n)}}$ *for all* $n \in \mathbb{N}_0$.

Proof. In order to apply the techniques of complex analyis used in [9] it is of advantage to associate to Ψ a permutation operator Φ in the following way: Choose $w_0 \in G_2^c$ with $|w_0| = \min\{|w| : w \in G_2^c\}$ and put $G_3 := \frac{1}{w_0} G_2$. Note that G_3 contains the open unit disk strictly by our assumptions. To each multiplier $T \in M(H(G_2))$ there exists a holomorphic function T_{w_0} defined on $\frac{1}{w_0} G_2$, cf. the introduction. We define $\rho(T)$ as the holomorphic function T_{w_0} on G_3. Then $\rho: M(H(G_2)) \to H(G_3)$ is a linear map satisfying

$$\rho(S \circ T)(z) = (\rho(S) * \rho(T))(z) \text{ for all } z \in G_3. \tag{6}$$

Let $L: H(\widehat{G_1}) \to M(H(G_1))$ be the canonical injection. Then $\Phi := \rho \circ \Psi \circ L: H(\widehat{G_1}) \to H(G_3)$ is a linear map with the property that $\delta_n \circ \Phi$ is multiplicative on the algebra $H(\widehat{G_1})$ by (6) for each $n \in \mathbb{N}_0$. Proposition 3.1 shows that Φ is a permutation operator. We now use arguments which we have already used in the proof of Theorem 3.2 in [9]: define $\gamma_1(z) := \gamma(z/\xi)$ and $\xi := \exp(2\pi i / k_{G_1})$. Note that $\Phi(\gamma) = \Phi(\gamma) * \Phi(\gamma)$. It follows that the Taylor coefficients of $\Phi(\gamma)$ are either 0 or 1. Let $\Phi(\gamma_1) = \sum_{n=0}^{\infty} b_n z^n$. Since $\gamma_1^{k_{G_1}} = \gamma$ we infer $\Phi(\gamma) = (\Phi(\gamma_1))^{k_{G_1}}$. Hence $b_n^{k_{G_1}}$ are either equal to 0 or 1 for all $n \in \mathbb{N}_0$, i.e. that the coefficients b_n are either 0 or k_{G_1}-roots of unity. Since $G_3 \neq \mathbb{D}$ a theorem of Szegö [7, p. 227] shows that there exist $r \in \mathbb{N}$ and a polynomial $p(z)$ such that $\Phi(\gamma_1) = p(z)/(1 - z^r) =: g(z)$. We can assume that $r \in \mathbb{N}$ is minimal with this property. Now consider the multiplier $T := \Psi(L(\gamma_1))$: for each $w \in G_2^c$ the corresponding holomorphic function $T_w: \frac{1}{w} G_2 \to \mathbb{C}$ is an extension of $g(z)$. Since $g(z)$ is a rational function it follows that the poles of $g(z)$ (which are simple and of absolute value 1) must be contained in $\frac{1}{w} G_2^c$ for all $w \in G_2^c$. Hence the poles of $g(z)$ are contained in $\widehat{G_2}^c = \cap_{w \in G_2^c} \frac{1}{w} G_2^c$. Consequently there exists a polynomial $q(z)$ with $g(z) = q(z)/(1 - z^{k_{G_2}})$. By minimality we obtain $r \leq k_{\widehat{G_2}} = k_{G_2}$.

By polynomial divison there exist polynomials p_1, p_2 with $p(z) = p_1(z)(1 - z^r) + p_2(z)$ and the degree of p_2 is at most $r - 1$. Let $p_2(z) = c_0 + c_1 z + \ldots + c_{r-1} z^{r-1}$. Since $\Phi(\gamma_1) = p_1(z) + p_2(z)/(1 - z^r)$ there exists $n_0 \in \mathbb{N}$ such that the Taylor expansion of $\Phi(\gamma_1)$ is periodic for all $n \geq n_0$

and the coefficients are given by $c_0, \dots, c_{r-1}$. In particular, $c_0, \dots, c_{r-1}$ are k_{G_1}-roots of unity. We claim that

$$\{1, \xi, \dots, \xi^{k_{G_1}-1}\} = \{c_0, \dots, c_{r-1}\} \tag{7}$$

For this we consider $f_N(z) := \sum_{n=N}^{\infty} \xi^{-n} z^n$ for large $N \in \mathbb{N}$. Then the Taylor coefficients of $\Phi(f_N)$ are either zero or equal to some c_j for $j = 0, \dots, r-1$ since φ only permutes the Taylor coefficients of f_N. Now (7) implies $k_{G_1} \leq r \leq k_{G_2}$. The same argument applied to Φ^{-1} yields $k_{G_2} \leq k_{G_1}$. Hence we have proved that $k_{G_2} = k_{G_1} =: k$.

By Theorem 1.3 in [9] applied to Φ we infer that $\psi(n)/n$ is bounded. By repeating this argument to the inverse homomorphism Ψ^{-1} it follows that $\psi^{-1}(n)/n$ is bounded. Following the proof of Theorem 4.3 in [9] (applied to the above defined permutation operator Φ) we conclude that there exist $n_0 \in \mathbb{N}, a_j \geq 0$ and $b_j \in \mathbb{Z}$ such that $\psi^{-1}(nk + j) = a_j(nk) + b_j$ for all $nk + j \geq n_0$ and for all $j = 0, \dots, k-1$. Moreover we have $a_j k \in \mathbb{Z}$. It remains to prove that $a_j = 1$. Define $\Phi_2 := \rho_2 \circ \Psi^{-1} \circ L_2$ analogously to the construction of Φ. Then $T := \Psi^{-1}(L_2(z^j/1 - z^k))$ defines a multiplier on G_1 such that $T_w(z)$ is equal to $z^m \frac{1}{1-z^{a_j k}} + r(z)$ for a suitable $m \in \mathbb{N}_0$ and a suitable polynomial $r(z)$. As before it follows that the zeros of $1 - z^{a_j k}$ must be contained in each $\frac{1}{w} G_2^c$ for all $w \in G_2^c$. As in [9] it follows that $a_j = 1$. The proof is complete. □

Theorem 3.5. *Suppose that* $\Phi \colon M(H(G_1)) \to M(H(G_2))$ *is an isomorphism. Then* $\widehat{G_1} = \widehat{G_2}$.

Proof. Let $\Phi := \rho \circ \Psi \circ L$ and G_3 as in the last proof. In the first case assume that $G_1 = \mathbb{D}_r$. If $G_2 \neq \mathbb{D}_s$ then G_3 is strictly larger than $\mathbb{D}$. Theorem 1.3 in [9] shows that $r_3 := \max\{|z| : z \in G_3\} \leq 1$, a contradiction. It follows that $\widehat{G_1} = \widehat{G_2}$. If $G_2 = \mathbb{D}_s$ the same argument applied to Φ^{-1} yields $\widehat{G_1} = \widehat{G_2}$. In the second case assume that both G_1 and G_2 are not open disks. Let $\psi(nk + j) = nk + b_j$ as in Theorem 3.4. For $a \in \widehat{G_1}^c$ the function $f_j := z^j/(1-(z/a)^k)$ is in $H(\widehat{G_1})$. As in the proof of Theorem 5.1 in [9] it follows that $\Phi(f_j)$ is of the form $r(z) + (z^m/(1 - (z/a)^k)$ for a suitable polynomial $r(z)$ and $m \in \mathbb{N}_0$. Hence $T := \Psi(L(f_j))$ is a multiplier such that T_w is holomorphic on $\frac{1}{w} G_2$ for each $w \in G_2^c$. It follows that $a \in \widehat{G_2}^c$. Hence $\widehat{G_1}^c \subset \widehat{G_2}^c$ and by symmetry we infer equality. □

THEOREM 3.6. *Let $\Phi: H(G_1) \to H(G_2)$ be a bijective permutation operator. Then there exists an isomorphism $\widehat{\Phi}: M(H(G_1)) \to M(H(G_2))$ extending Φ, i.e., that $\widehat{\Phi}(L_{z^n}) = L_{\Phi(z^n)} = L_{z^{\varphi(n)}}$.*

Proof. Let $T \in M(H(G_1))$ and define $\widehat{\Phi}(T)(f) := \Phi(T(\Phi^{-1}(f)))$ for $f \in H(G_2)$. We claim that $\widehat{\Phi}(T): H(G_2) \to H(G_2)$ is a coefficient multiplier: for $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we have locally $\Phi^{-1}(f) = \sum_{n=0}^{\infty} a_n z^{\varphi^{-1}(n)}$. Theorem 1.2 implies that $T(\Phi^{-1}(f)) = \sum_{n=0}^{\infty} a_{\varphi(n)} b_n z^n$ for suitable $b_n \in \mathbb{C}$. Thus

$$\widehat{\Phi}(T)(f) = \Phi(T(\Phi^{-1}(f))) = \sum_{n=0}^{\infty} a_{\varphi(n)} b_n z^{\varphi(n)} = \sum_{n=0}^{\infty} a_n b_{\varphi^{-1}(n)} z^n. \tag{8}$$

Thus $\widehat{\Phi}(T)$ is a multiplier. It is straightforward to check that $\widehat{\Phi}$ is linear and multiplicative. Note that $\widehat{\Phi}(L_{z^n}) = a_{\varphi(n)} z^{\varphi(n)} = L_{z^{\varphi(n)}}(f)$ by formula (8). Further $\widehat{\Phi}$ is a bijection since the inverse function is given by $\widehat{\Phi^{-1}}$. ∎

THEOREM 3.7. *Let $\Phi: H(G_1) \to H(G_2)$ be a bijective permutation operator. Then $G_1 = G_2$.*

Proof. In the first case assume that $G_1 = \mathbb{D}_r$ for some $r > 0$. Then $H(G_1)$ is an algebra (with respect to the Hadamard product) and it is not very difficult to see that $H(G_2)$ is an algebra since Φ is an isomorphism. By Theorem 5.2 in [9] it follows that $G_1 = G_2 = \mathbb{D}_r$.
In the second case assume that $G_1 \neq \mathbb{D}_r$. Clearly we can assume that $G_2 \neq \mathbb{D}_s$. According to Theorem 3.6 Φ can be lifted to an isomorphism $\widehat{\Phi} : M(H(G_1)) \to M(H(G_2))$. By Theorem 3.4 there exist $n_0 \in \mathbb{N}_0$ and $b_0, \dots, b_{k-1} \in \mathbb{Z}$ such that $\varphi(kn+j) = kn+b_j$ for all $nk+j \geq n_0$ and for all $j = 0, \dots, k-1$ where $k := k_{G_1} = k_{G_2}$. The rest of the proof follows the lines of the proof of Theorem 5.1 in [9]: Let $a \in G_1^c$. Then $a/(a-z) \in H(G_1)$ and $1/(1-z^k) \in H(\widehat{G_1})$. By the Hadamard multiplication theorem (see e.g. Theorem 1.3 in [11]) $(1/(1-z^k)) * (a/(a-z)) = 1/(1-(z/a)^k) \in H(G_1)$. Hence $f(z) := (z^j/(1-(z/a)^k)$ defines a function in $H(G_1)$ for $j = 0, \dots, k-1$. Now put $p(z) := \sum_{n=0}^{n_0-1} \Phi(z^{kn+j}/a^{nk})$. Then

$$\Phi(f) - p(z) = \Phi(\sum_{n=n_0}^{\infty} \frac{z^{nk+j}}{a^{nk}}) = \sum_{n=n_0}^{\infty} \frac{z^{nk+b_j}}{a^{nk}} = z^{b_j+n_0k} \frac{1}{1-(\frac{z}{a})^k}. \tag{9}$$

It follows that $a \in G_2^c$ since otherwise $\Phi(f)$ would have a pole in $z = a$. Hence $G_1^c \subset G_2^c$ and equality follows by symmetry. ∎

In [9] we proved that two admissible Hadamard-isomorphic domains G_1, G_2 are equal if and only if $H(G_1)$ and $H(G_2)$ are isomorphic *provided that* $H(G_1)$ and $H(G_2)$ possess a unit element. It was left as an open question whether this result remains true in the non-unital case. The foregoing Theorem immediately gives a positive answer using the fact that Hadamard isomorphisms are permutation operators.

COROLLARY 3.8. *Let* G_1, G_2 *be admissible domains such that* $H(G_1)$ *and* $H(G_2)$ *are isomorphic with respect to the Hadamard product. Then* $G_1 = G_2$.

THEOREM 3.9. *Suppose that there exists a domain* $\widetilde{G} \subset \mathbb{C}$ *containing* 0 *such that* $H(\widetilde{G})$ *is isomorphic to* $M(H(G))$. *Then* $\widetilde{G} = \widehat{G}$ *and the canonical injection* $L\colon H(\widehat{G}) \to M(H(G))$ *is already an isomorphism.*

Proof. Since $M(H(G))$ possesses a unit element the algebra $H(\widetilde{G})$ is unital. Hence $H(\widetilde{G})$ is isomorphic to $M(H(\widetilde{G}))$. It follows that $\widetilde{G}$ is admissible and $1 \in \widetilde{G}^c$. Hence $\widetilde{G} = \widehat{\widetilde{G}}$ and Theorem 3.5 yields $\widehat{\widetilde{G}} = \widehat{G}$. For the second statement let $\Psi\colon H(\widehat{G}) \to M(H(G))$ be an isomorphism. In the case $\widehat{G} \neq \mathbb{D}$ let $\psi\colon \mathbb{N}_0 \to \mathbb{N}_0$ be the induced bijection described in Theorem 3.4. Then $\varphi := \psi^{-1}$ is of the same form. By Theorem 4.4 in [9] there exists an isomorphism $\Phi\colon H(\widehat{G}) \to H(\widehat{G})$ with $\Phi(z^n) = z^{\varphi(n)}$. Then $\Psi \circ \Phi\colon H(\widehat{G}) \to M(H(G))$ is an isomorphism with $\Psi \circ \Phi(z^n) = \Psi(z^{\varphi(n)}) = L_{z^n}$. It follows that the isomorphism $\Psi \circ \Phi$ is identical to L. In the case $\widehat{G} = \mathbb{D}$ note that G is equal to some $\mathbb{D}_r$. By Theorem 1.3 in [11] the natural injection $L\colon H(\mathbb{D}) \to M(H(G))$ is a bijection. ∎

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Hermann Render
Universität Duisburg
Fachbereich Mathematik
Lotharstr. 65
D-47057 Duisburg
Federal Republic of Germany
render@math.uni-duisburg.de

Andreas Sauer
Universität Duisburg
Fachbereich Mathematik
Lotharstr. 65
D-47057 Duisburg
Federal Republic of Germany
sauer@math.uni-duisburg.de