H. Matsumoto and M. Yor
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AN ANALOGUE OF PITMAN'S $2M - X$ THEOREM FOR EXPONENTIAL WIENER FUNCTIONALS

PART I: A TIME-INVERSION APPROACH

HIROYUKI MATSUMOTO AND MARC YOR

Abstract. Let $\{B_t^{(\mu)}, t \geqq 0\}$ be a one-dimensional Brownian motion with constant drift $\mu \in \mathbf{R}$ starting from 0. In this paper we show that

$$Z_t^{(\mu)} = \exp(-B_t^{(\mu)}) \int_0^t \exp(2B_s^{(\mu)}) ds$$

gives rise to a diffusion process and we explain how this result may be considered as an extension of the celebrated Pitman's $2M - X$ theorem. We also derive the infinitesimal generator and some properties of the diffusion process $\{Z_t^{(\mu)}, t \geqq 0\}$ and, in particular, its relation to the generalized Bessel processes.

§1. Introduction and main results

1.a. Let $B = \{B_t, t \geqq 0\}$ denote a one-dimensional Brownian motion starting from 0 and define $M_t = \sup_{s \leqq t} B_s, t \geqq 0$. The family of stochastic processes $\Sigma^{(k)} = \{\Sigma_t^{(k)} = kM_t - B_t, t \geqq 0\}$, indexed by $k > 0$, has attracted a lot of interest in the probabilistic literature (The main results about these processes are recalled below in this introduction).

In the present paper we undertake a systematic study of the "generalizations" of the processes $\Sigma^{(k)}$, which are "extended" as follows:

$$L_t^{(k)} \equiv \log\Big(\int_0^t \exp(kB_s) ds\Big) - B_t, \qquad t > 0.$$

Indeed, the scaling property of Brownian motion and a simple Laplace method argument yield that $\{cL_{t/c^2}^{(k)}, t \geqq 0\}$ converges in law towards $\Sigma^{(k)}$ as $c \downarrow 0$ (cf. Subsections 1.d and 1.e below).

1.b. The most striking results about the processes $\Sigma^{(k)}$ concern the cases $k = 1$, $k = 2$ and $k = 3$.

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THEOREM 1.1. (Lévy) $\{\Sigma_t^{(1)} \equiv M_t - B_t, t \geqq 0\}$ *and* $\{|B_t|, t \geqq 0\}$ *have the same probability law.*

THEOREM 1.2. (Pitman [30]) $\{\Sigma_t^{(2)} \equiv 2M_t - B_t, t \geqq 0\}$ *is distributed as the three-dimensional Bessel process* $BES(3)$.

THEOREM 1.3. (Le Gall–Yor [23]) *The family* $\{L_\infty^a(\Sigma^{(3)}), a \geqq 0\}$ *of the total local times of* $\Sigma^{(3)}$*, up to time* ∞*, is distributed as the square of a one-dimensional Brownian motion.*

1.c. One obtains interesting extensions of Theorems 1.1 and 1.2 when replacing B by $B^{(\mu)} = \{B_t^{(\mu)} = B_t + \mu t, t \geqq 0\}$, a Brownian motion with constant drift $\mu \in \mathbf{R}$.

THEOREM 1.1$_\mu$. *Let* $\mu \in \mathbf{R}$. *Then the identity in law*

$$\{(M_t^{(\mu)} - B_t^{(\mu)}, M_t^{(\mu)}), t \geqq 0\} \overset{\text{(law)}}{=} \{(|X_t^{(\mu)}|, \lambda_t^{(\mu)}), t \geqq 0\}$$

holds, where $X^{(\mu)} = \{X_t^{(\mu)}, t \geqq 0\}$ *is the bang-bang process with parameter* μ*, that is, the diffusion process with infinitesimal generator*

$$\frac{1}{2}\frac{d^2}{dx^2} - \mu \operatorname{sgn}(x)\frac{d}{dx}$$

and $\{\lambda_t^{(\mu)}, t \geqq 0\}$ *is the local time of* $X^{(\mu)}$ *at* 0. *Furthermore, the natural filtration of* $\{M_t^{(\mu)} - B_t^{(\mu)}, t \geqq 0\}$ *is identical to that of* $B^{(\mu)}$.

THEOREM 1.2$_\mu$. (Rogers–Pitman [36]) (i) *Let* $\mu \geqq 0$. *Then the identity in law*

$$\{(2M_t^{(\mu)} - B_t^{(\mu)}, M_t^{(\mu)}), t \geqq 0\} \overset{\text{(law)}}{=} \{(\rho_t^{(\mu)}, j_t^{(\mu)}), t \geqq 0\}$$

holds, where $\{\rho_t^{(\mu)}, t \geqq 0\}$ *is the diffusion process with infinitesimal generator*

$$\frac{1}{2}\frac{d^2}{dx^2} + \mu \coth(\mu x)\frac{d}{dx}$$

and $j_t^{(\mu)} = \inf_{s \geqq t} \rho_s^{(\mu)}$. *Furthermore, the natural filtration of* $\{2M_t^{(\mu)} - B_t^{(\mu)}, t \geqq 0\}$ *is strictly contained in that of* $B^{(\mu)}$*, that is, for every* $t > 0$*, one has*

$$\mathcal{G}_t^{(\mu)} \equiv \sigma\{2M_s^{(\mu)} - B_s^{(\mu)}; s \leqq t\} \subsetneqq \sigma\{B_s^{(\mu)}; s \leqq t\}.$$

(ii) *The diffusion processes* $\{2M_t^{(\mu)} - B_t^{(\mu)}, t \geqq 0\}$ *and* $\{2M_t^{(-\mu)} - B_t^{(-\mu)}, t \geqq 0\}$ *have the same distribution.*

Theorem 1.2_μ is easily derived from Theorem 1.1_μ once the following result about the conditional law of $B_t^{(\mu)}$ given $\mathcal{G}_t^{(\mu)}$ is known.

PROPOSITION 1.4. ([36], [34]) *Let* $\mu \in \mathbf{R}$. *Then the conditional distribution of* $B_t^{(\mu)}$ *given* $\mathcal{G}_t^{(\mu)}$ *is*

$$P(B_t^{(\mu)} \in dx | \mathcal{G}_t^{(\mu)}, 2M_t^{(\mu)} - B_t^{(\mu)} = r) = \frac{\mu \cosh(\mu x)}{2\sinh(\mu r)} 1_{[-r,r]}(x)dx.$$

Further extensions of Pitman's theorem than Theorem 1.2_μ and a number of related discussions have been made by Saisho-Tanemura [37], following Tanaka [41], [42], Rauscher [33], Rogers [35], Takaoka [40], and also Bertoin [3] for certain Lévy processes. Another example of such a situation is Biane's work [4] on Brownian motion in a cone, which is to Le Gall's study [22] what Pitman's theorem is to Lévy's.

1.d. Before giving our main results, we introduce a family of generalized Ornstein–Uhlenbeck processes. In [7], it was remarked that, if $\{\delta_t, t \geqq 0\}$ and $\{C_t, t \geqq 0\}$ are two independent Lévy processes, then

$$X_x = \Big\{ X_x(t) \equiv \exp(-\delta_t) \Big\{ x + \int_0^t \exp(\delta_s) dC_s \Big\}, t \geqq 0 \Big\}$$

defines a Markovian family of processes starting from x, that is, if P_x is the law of X_x on the canonical path space, then $\{P_x\}$ is Markovian. See also de Haan–Karandikar [9], which preceeds [7].

Undoubtedly, the best known example concerns the case where $\delta_t = \lambda t, \lambda \in \mathbf{R}$, and $\{C_t\}$ is a Brownian motion, which yields the standard Ornstein–Uhlenbeck process with drift λ. But the case where $\{\delta_t\}$ is a Brownian motion with drift and $C_t \equiv \lambda t, \lambda \in \mathbf{R}$, is also very interesting. The more general case where both $\{\delta_t\}$ and $\{C_t\}$ are Brownian motions with drifts has been considered in [1], [2], [54].

Our extensions of Lévy's and Pitman's theorems to exponential Brownian functionals shall concern the case where $\delta_t = aB_t^{(\mu)}$ and $C_t = t$. This explains the following notations and discussions.

To a Brownian motion $B^{(\mu)} = \{B_t^{(\mu)}, t \geqq 0\}$ with drift μ, we associate the corresponding geometric Brownian motion defined by $e_t^{(\mu)} =$

$\exp(B_t^{(\mu)}), t \geqq 0$, as well as its quadratic variation process

$$A_t^{(\mu)} = \int_0^t (e_s^{(\mu)})^2 ds$$

and the two stochastic processes given by

$$\xi_t^{(\mu)} = (e_t^{(\mu)})^{-2} A_t^{(\mu)} \qquad \text{and} \qquad Z_t^{(\mu)} = (e_t^{(\mu)})^{-1} A_t^{(\mu)}.$$

The main purpose of this and the next subsections is to exhibit these two processes as diffusions and to show that their semigroups enjoy an intertwining relationship. As will be explained below, the following two theorems may be understood as extensions, respectively, of Theorems 1.1_μ and 1.2_μ.

Fix $\mu \in \mathbf{R}$ and consider the stochastic process

$$\xi_t^{a,b} = \exp(-aB_t^{(\mu)}) \Big\{ \xi_0 + \int_0^t \exp(bB_s^{(\mu)}) ds \Big\}$$

for $\xi_0, a, b \in \mathbf{R}$. Itô's formula yields

$$\xi_t^{a,b} = \xi_0 - a \int_0^t \xi_s^{a,b} dB_s^{(\mu)} + \frac{a^2}{2} \int_0^t \xi_s^{a,b} ds + \int_0^t \exp((b-a)B_s^{(\mu)}) ds. \tag{1.1}$$

In particular, if $a = b$, $\{\xi_t^{a,a}, t \geqq 0\}$ is a diffusion on $\mathbf{R}_+$ generated by the second order differential operator

$$\frac{a^2}{2} x^2 \frac{d^2}{dx^2} + \Big(\Big(\frac{a^2}{2} - a\mu \Big) x + 1 \Big) \frac{d}{dx}. \tag{1.2}$$

For the sake of clarity in our subsequent discussions, we write precisely the preceding statement for $a = 2$.

THEOREM 1.5. *Let $\mu \in \mathbf{R}$.* (i) *$\{\xi_t^{(\mu)}, t \geqq 0\}$ is a diffusion process whose natural filtration is that of $B^{(\mu)}$ and whose infinitesimal generator is given by* (1.2) *with $a = 2$, in which case $\xi_t^{a,a} = \xi_t^{(\mu)}$.*
(ii) *For any given $t > 0$, one has*

$$\xi_t^{(\mu)} \overset{\text{(law)}}{=} \int_0^t \exp(-2B_s^{(\mu)}) ds \overset{\text{(law)}}{=} \int_0^t \exp(2B_s^{(-\mu)}) ds.$$

We can recover Theorem 1.1_μ from Theorem 1.5 in the following manner. At first we note that, by the scaling property of Brownian motion, $\{\xi^{(c\mu)}_{t/c^2}, t \geqq 0\}$ is identical in law with $\{\xi^{(\mu)}_t, t \geqq 0\}$ for every $c > 0$. Therefore

$$c \log(\xi^{(c\mu)}_{t/c^2}) \overset{\text{(law)}}{=} c \log\Big(\int_0^t \exp(\frac{2}{c} B^{(\mu)}_s) ds\Big) - 2B^{(\mu)}_t - c \log c^2, \quad t > 0,$$

gives rise to a diffusion process and, letting $c \downarrow 0$, we get $2(M^{(\mu)}_t - B^{(\mu)}_t)$ from the right hand side by an elementary Laplace method argument.

1.e. Here is a (Pitman type) companion to Theorem 1.5 concerning the stochastic process $Z^{(\mu)} = \{Z^{(\mu)}_t, t \geqq 0\}$.

THEOREM 1.6. *Let $\mu \in \mathbf{R}$.* (i) *$Z^{(\mu)}$ is a diffusion process whose natural filtration $\{\mathcal{Z}^{(\mu)}_t, t \geqq 0\}$ is strictly contained in that of $B^{(\mu)}$, that is, for any $t > 0$,*

$$\mathcal{Z}^{(\mu)}_t \subsetneqq \sigma\Big\{B^{(\mu)}_s; s \leqq t\Big\}.$$

Furthermore, the infinitesimal generator of $Z^{(\mu)}$ is given by

$$\frac{1}{2} z^2 \frac{d^2}{dz^2} + \Big\{\Big(\frac{1}{2} - \mu\Big) z + \Big(\frac{K_{1+\mu}}{K_\mu}\Big)\Big(\frac{1}{z}\Big)\Big\} \frac{d}{dz}. \tag{1.3}$$

(ii) *The diffusion processes $Z^{(\mu)}$ and $Z^{(-\mu)}$ have the same distribution.*

Note that the second statement of Theorem 1.6 agrees with the fact that, in (1.3), the two drifts corresponding to μ and $-\mu$ are identical, which follows from the fact that $K_{-\nu} = K_\nu$ and the recurrence property

$$K_{\nu-1}(z) - K_{\nu+1}(z) = -\frac{2\nu}{z} K_\nu(z)$$

of the Macdonald functions $K_\nu, \nu \in \mathbf{R}$ (cf. Lebedev [20], p.110).

Here is a key result which allows to derive the Markov property of $Z^{(\mu)}$ from that of $\xi^{(\mu)}$.

PROPOSITION 1.7. *Let $\mu \in \mathbf{R}$. For any $t > 0$, the conditional distribution of $B^{(\mu)}_t$ given $\mathcal{Z}^{(\mu)}_t$ is*

$$P\Big(B^{(\mu)}_t \in dx | \mathcal{Z}^{(\mu)}_t, Z^{(\mu)}_t = z\Big) = \frac{\exp(\mu x)}{2K_\mu(1/z)} \exp\Big(-\frac{\cosh(x)}{z}\Big) dx. \tag{1.4}$$

The same formula holds when t is replaced by any $(\mathcal{Z}^{(\mu)}_t)$-stopping time T.

Now let $P^{(\mu)} = \{P_t^{(\mu)}, t \geqq 0\}$ and $Q^{(\mu)} = \{Q_t^{(\mu)}, t \geqq 0\}$ denote the semigroups of $\xi^{(\mu)}$ and $Z^{(\mu)}$, respectively. As a consequence of Proposition 1.7, one obtains the following intertwining property between $P^{(\mu)}$ and $Q^{(\mu)}$. We set

$$k_z^{(\mu)}(u) = \frac{u^{\mu-1}}{2K_\mu(z)} \exp\Big(-\frac{z}{2}\Big(u + \frac{1}{u}\Big)\Big)$$

and define a Markov kernel $\mathbb{K}^{(\mu)}$ by

$$\mathbb{K}^{(\mu)}\varphi(z) = \int_0^\infty k_{1/z}^{(\mu)}(u)\varphi(z/u)du.$$

$k_z^{(\mu)}$ is the density function of a particular generalized inverse Gaussian distribution (cf. [38]; a detailed discussion of the deep relations between our work and generalized inverse Gaussian laws is made in [27]).

THEOREM 1.8. *For every $\mu \in \mathbf{R}$, the semigroups $P^{(\mu)}$ and $Q^{(\mu)}$ are intertwined as*

$$Q_t^{(\mu)}\mathbb{K}^{(\mu)} = \mathbb{K}^{(\mu)} P_t^{(\mu)}. \tag{1.5}$$

Likewise, as a consequence of Proposition 1.7, we obtain an intertwining relationship between $Q^{(\mu)}$ and the semigroup of the geometric Brownian motion $\{e_t^{(\mu)}, t \geqq 0\}$. For fairly general discussions of such intertwinings between probability semigroups, see [8], [52]; Biane [5] presents further examples.

1.f. In this subsection we show how Theorem 1.6 implies Theorem 1.2_μ. Indeed, a simple change of variables allows to rephrase Theorem 1.6 in the following manner.

THEOREM 1.5′. *Let $\mu \in \mathbf{R}$. Then the stochastic process*

$$\Big\{ \log\Big(\int_0^t \exp(2B_s^{(\mu)})ds\Big) - B_t^{(\mu)}, t > 0\Big\} \tag{1.6}$$

is a Brownian motion [1] *with "drift" $b^{(\mu)}$ given by*

$$b^{(\mu)}(x) = -\mu + e^{-x}\Big(\frac{K_{1+\mu}}{K_\mu}\Big)(e^{-x}) = \frac{d}{dx}(\log K_\mu(e^{-x})). \tag{1.7}$$

[1] Here, and below, we call a diffusion process whose infinitesimal generator is of the form $d^2/2dx^2 + b(x)d/dx$ a Brownian motion with drift b.

More generally, for any $c > 0$, the process

$$\left\{c\log\Big(\int_0^t \exp\Big(\frac{2}{c}B_s^{(\mu)}\Big)ds\Big) - B_t^{(\mu)} - c\log c^2, t > 0\right\} \tag{1.8}$$

is a Brownian motion with "drift"

$$b^{(\mu),c}(x) = -\mu + \frac{1}{c}e^{-x/c}\Big(\frac{K_{1+\mu c}}{K_{\mu c}}\Big)(e^{-x/c}) = \frac{d}{dx}(\log K_{\mu c}(e^{-x/c})). \tag{1.9}$$

Note that the second part of the statement of Theorem 1.5′ follows from the first one with the help of the scaling property of Brownian motion and that the second equalities in (1.7) and (1.9) may be verified from the recurrence formula (cf. [20], p.110)

$$\frac{d}{dz}(z^{-\nu}K_\nu(z)) = -z^{-\nu}K_{\nu+1}(z).$$

Now the reason why Theorem 1.5′ may be considered as a generalization of the Rogers–Pitman result (Theorem 1.2_μ) is that the latter theorem is recovered by letting $c \to 0$ in (1.8); indeed, one recovers $\{2M_t^{(\mu)} - B_t^{(\mu)}, t \geqq 0\}$ again from an elementary Laplace method argument (as we already pointed out after the statement of Theorem 1.5). Moreover it is not difficult to show

$$\lim_{c\downarrow 0} b^{(\mu),c}(x) = \mu\coth(\mu x) \tag{1.10}$$

by using the integral representation

$$K_\nu(x) = \frac{2^\nu\Gamma(\nu+1/2)}{x^\nu\sqrt{\pi}}\int_0^\infty \frac{\cos xt}{(1+t^2)^{\nu+1/2}}dt$$

of K_ν (cf. [20], p.140).

It should be noted that the convergence in (1.10) is uniform in x on a compact interval, hence that (1.10) implies the convergence of the corresponding scale functions and the speed measures; thus, following Ogura [29], this would also ensure the convergence of the diffusion process given by (1.8). However, here, we obtain this convergence directly in a pathwise manner.

1.g. The rest of the present paper is organized as follows. In Section 2 we recall the basic properties of Bessel processes and, more generally, of the so-called generalized Bessel processes, which were introduced by Watanabe [43] and developed in Pitman–Yor [32]. In Section 3 we prove the above Theorems 1.6 and 1.8 by considering the well-known Lamperti's representation (cf. [34], Chapter XI, Exercise (1.28), p.452)

$$\exp(B_t^{(\mu)}) = R^{(\mu)}_{A_t^{(\mu)}}$$

with the transformation by time-inversion of these Bessel processes into generalized Bessel processes. In Section 4 we discuss some distributional properties of the diffusion $Z^{(\mu)}$. In Section 5 we study the laws of first hitting times of $\xi^{(\mu)}$, whereas in Section 6 we study certain quantities involving $\exp(B_t^{(\mu)})$ and its integral. In Section 7 we present a number of open questions and propose some natural generalizations. We have also gathered in an Appendix some useful formulae about Bessel functions.

1.h. Finally, let us indicate that our original proofs of Theorems 1.6 and 1.8 used a generalization, which we detail in [26], of an identity in law due to Dufresne [13] together with the resolution of a particular "stochastic distribution equation". This alternative approach to Theorems 1.6 and 1.8 shall be discussed almost independently of the present paper in our subsequent Part II ([27]).

§2. Preliminaries: Generalized Bessel processes

In this section we recall the definitions and the basic properties of the generalized Bessel processes studied by Pitman–Yor [31], [32] and Watanabe [43].

For $\mu \geqq 0$ we denote by G_μ the infinitesimal generator of the Bessel process with index μ:

$$G_\mu = \frac{1}{2}\frac{d^2}{dx^2} + \frac{2\mu+1}{2x}\frac{d}{dx}.$$

For $\delta > 0$ we set

$$\psi_{\mu,\delta\uparrow}(x) = \delta\frac{I_{\mu+1}(\delta x)}{I_\mu(\delta x)} \equiv \frac{d}{dx}\log(\tilde{I}_\mu(\delta x))$$

and

$$\psi_{\mu,\delta\downarrow}(x) = -\delta\frac{K_{\mu+1}(\delta x)}{K_\mu(\delta x)} \equiv \frac{d}{dx}\log(\tilde{K}_\mu(\delta x)),$$

where $\tilde{I}_\mu(z) = z^{-\mu} I_\mu(z)$ and $\tilde{K}_\mu(z) = z^{-\mu} K_\mu(z)$.

We call the diffusion processes on $[0,\infty)$ with the infinitesimal generators

$$G_\mu^{\delta\uparrow} = G_\mu + \psi_{\mu,\delta\uparrow}\frac{d}{dx} \qquad \text{and} \qquad G_\mu^{\delta\downarrow} = G_\mu + \psi_{\mu,\delta\downarrow}\frac{d}{dx}$$

an upward and a downward $BES(\mu)$ process with parameter δ, respectively.

These are particular cases of the conditioned diffusion processes described in Section 3 of [32]. They play an essential role in the study of the class of $\mathbf{R}_+$-valued processes which is globally stable under time inversion. This study was made by Watanabe [43]. See also Watanabe [44] and [45] for the more general study of time inversion for $\mathbf{R}$-valued diffusion processes. For the moment we gather the following important properties. For further details, see Section 4 of Pitman–Yor [32].

Proposition 2.1. *Let $\mu \geqq 0, \delta > 0$ and $x > 0$, and denote by $P_x^{(\mu,\delta\uparrow)}, P_x^{(\mu,\delta\downarrow)}$ and $P_x^{(\mu)}$ the probability laws of $BES(\mu,\delta\uparrow), BES(\mu,\delta\downarrow)$ and $BES(\mu)$ processes, respectively, started at x on the canonical path space $C(\mathbf{R}_+;\mathbf{R}_+)$, where $R_t(w) = w(t)$. We also denote by $\mathcal{R}_t = \sigma\{R_s; s \leqq t\}$ the natural filtration of R. Then*

(i) *The $BES(\mu,\delta\downarrow)$ process reaches 0 in finite time and remains there.*

(ii) *The $BES(\mu,\delta\downarrow)$ process started at x is the time reversal of the $BES(\mu, \delta\uparrow)$ process which starts at 0 and runs until its last exit time L_x at x.*

(iii) *The following absolute continuity relations take place:*

$$P_x^{(\mu,\delta\uparrow)}|_{\mathcal{R}_t} = \frac{\tilde{I}_\mu(\delta R_t)}{\tilde{I}_\mu(\delta x)} \exp\Big(-\frac{\delta^2 t}{2}\Big) \cdot P_x^{(\mu)}|_{\mathcal{R}_t},$$

$$P_x^{(\mu,\delta\downarrow)}|_{\mathcal{R}_t \cap \{t<T_0\}} = \frac{\tilde{K}_\mu(\delta R_t)}{\tilde{K}_\mu(\delta x)} \exp\Big(-\frac{\delta^2 t}{2}\Big) \cdot P_x^{(\mu)}|_{\mathcal{R}_t},$$

where T_0 is the first hitting time of the $BES(\mu,\delta\downarrow)$ process. As a consequence, one has

$$P_x^{(\mu,\delta\uparrow)}|_{\mathcal{R}_t} = \Big(\frac{I_\mu}{K_\mu}\Big)(\delta R_t) \Big/ \Big(\frac{I_\mu}{K_\mu}\Big)(\delta x) \cdot P_x^{(\mu,\delta\downarrow)}|_{\mathcal{R}_t \cap \{t<T_0\}}. \tag{2.1}$$

Remark 2.1. The relation (2.1) confirms that the function

$$\Big(\frac{I_\mu}{K_\mu}\Big)(\delta x), \qquad x \geqq 0,$$

is a scale function for $BES(\mu, \delta\downarrow)$ and, equivalently, the function

$$-\Big(\frac{K_\mu}{I_\mu}\Big)(\delta x), \qquad x \geqq 0,$$

is a scale function for the transient diffusion $BES(\mu, \delta\uparrow)$, which takes the value $-\infty$ for $x = 0$ and 0 for $x = \infty$. This is a particular case of the formula (3.n) on p.303 of Pitman-Yor [32].

§3. Proofs of Theorems 1.6 and 1.8, and Proposition 1.7

3.a. In order to facilitate the reader's understanding, we adopt the following "abstract" Markovian presentation: in short, proving Theorems 1.6 and 1.8 amounts to showing that, if a Markov process $\{R(t), t \geqq 0\}$ retains the (homogeneous) Markov property under the time-inversion operation

$$R = \{R(t), t \geqq 0\} \mapsto \hat{R} = \{\hat{R}(u) \equiv uR(1/u), u > 0\}, \tag{3.1}$$

then a certain related stochastic process is Markovian with respect to its own filtration. More precisely, let us assume that $\{R(t), t \geqq 0\}$ is a $\mathbf{R}_+$-valued diffusion process and that

$$\lim_{t\to\infty} \frac{1}{t} R(t) = 0 \tag{3.2}$$

holds, so that $\hat{R}$ may be assumed to start at 0, i.e., $\hat{R}(0) = 0$. We also assume $R(0) > 0$, so that

$$\lim_{t\to\infty} \hat{R}(t) = \infty. \tag{3.3}$$

Hence the last passage time

$$\hat{L}_z = \sup\{t; \hat{R}(t) = z\}$$

is well defined and finite and, by the time reversal theory (cf. Nagasawa [28]), there exists a Markov process $\tilde{R} = \{\tilde{R}(u), u < \tilde{T}_0\}$ such that, for any $z > 0$, one has

$$\{\hat{R}(\hat{L}_z - u), u < \hat{L}_z\} \overset{\text{(law)}}{=} \{\tilde{R}_z(u), u < \tilde{T}_0\}, \tag{3.4}$$

where $\tilde{T}_0$ is the first hitting time of $\tilde{R}_z$ at 0. Here and below, we use the notation $\{X_x(u), u \geqq 0\}$ to denote a stochastic process $\{X(u), u \geqq 0\}$ starting at x.

We now define the stochastic processes of direct interest to us with the help of $\{R(u), u \geqq 0\}$. First, we consider a diffusion process $\{U_t, t \geqq 0\}$ defined implicitly in terms of $\{R(u)\}$ as follows:

$$U_t = R(A_t) \tag{3.5}$$

where A_t is given by

$$A_t = \int_0^t U_u^2 du,$$

and we assume that $A_\infty = \infty$ almost surely.

Now define $Y = \{Y_t, t > 0\}$ and $Z = \{Z_t, t \geqq 0\}$ by

$$Y_t = \frac{1}{A_t} U_t \qquad \text{and} \qquad Z_t = \frac{1}{Y_t}, \tag{3.6}$$

respectively. While Z is our main object of study, it is convenient to deal with Y in the following argument. We first show that, under our hypothesis, $\{Y_t, t > 0\}$ is a Markov process whose main characteristics may be described as follows.

Proposition 3.1. (i) *Set* $\tau_y = \inf\{t; Y_t = y\}$. *Then,* $\{Y_{\tau_y+t}, t \geqq 0\}$ *is represented in terms of* $\tilde{R}$ *by*

$$Y_{\tau_y+t} = \tilde{R}_y\Big(\int_0^t Y_{\tau_y+u}^2 du\Big), \qquad t \geqq 0. \tag{3.7}$$

(ii) *Set* $\lambda_y = \sup\{u; Y_u = y\}$. *Then, the time reversed process* $\{Y_{\lambda_y-u}, u < \lambda_y\}$ *is represented as*

$$Y_{\lambda_y-t} = \hat{R}_y\Big(\int_0^t Y_{\lambda_y-u}^2 du\Big), \qquad t < \lambda_y. \tag{3.8}$$

Remark 3.1. We find the implicit representations (3.5), (3.7) and (3.8) quite convenient; of course, they may also be considered as closed formulae, defining, for example, U in terms of R, since

$$A_t = \inf\Big\{u; \alpha_u \equiv \int_0^u \frac{ds}{R(s)^2} > t\Big\}. \tag{3.9}$$

Proof. (i) We first note

$$Y_t \equiv \hat{R}(1/A_t) \quad \text{and} \qquad \frac{1}{A_t} = \int_t^\infty Y_u^2 du, \qquad t > 0.$$

Consequently, we obtain

$$\frac{1}{A_{\tau_y}} \equiv \hat{L}_y = \int_{\tau_y}^{\infty} Y_u^2 du.$$

Hence we get

$$Y_{\tau_y+t} = \hat{R}\Big(\int_{\tau_y+t}^{\infty} Y_u^2 du\Big),$$

which can be written as

$$Y_{\tau_y+t} = \hat{R}\Big(\hat{L}_y - \int_0^t Y_{\tau_y+u}^2 du\Big)$$

and yields (3.7).
(ii) The proof of (3.8) is similar. Indeed, it is easy to show

$$Y_{\lambda_y-t} = \hat{R}\Big(\int_{\lambda_y-t}^{\infty} Y_u^2 du\Big) = \hat{R}\Big(\hat{T}_y + \int_0^t Z_{\lambda_y-v}^2 dv\Big),$$

since

$$\inf\{t; \hat{R}_t = y\} \equiv \hat{T}_y = \frac{1}{A_{\lambda_y}} = \int_{\lambda_y}^{\infty} Y_u^2 du.$$

□

The next proposition states that we can replace the stopping time τ_y by any time $s > 0$.

PROPOSITION 3.2. *Let $s > 0$. Then, conditionally on $\mathcal{Y}_s \equiv \sigma\{Y_u, u \leqq s\}$ and $Y_s = y$, there exists a process $\{\tilde{R}_y(u), u \geqq 0\}$ such that*

$$Y_t = \tilde{R}_y\Big(\int_s^t Y_u^2 du\Big), \qquad t \geqq s.$$

Proof. It suffices to write the Markov property for $\{Y_{\tau_y+u}, u \geqq 0\}$,

$$E[f(Y_{\tau_y+t})|\mathcal{Y}_{\tau_y+s}] = Q_{t-s} f(Y_{\tau_y+s}), \qquad s < t$$

and to let $y \to \infty$, using the continuity of $Q_u f(\,\cdot\,)$ for any bounded continuous function f. □

We now state, in this framework, an intertwining result.

PROPOSITION 3.3. *Denote by* $\{P_t, t \geqq 0\}$ *and* $\{Q_t, t \geqq 0\}$*, respectively, the semigroups of* $\{U_t\}$ *and* $\{Y_t\}$*. Then, these semigroups are intertwined, that is,*

$$Q_t\mathbb{K} = \mathbb{K}P_t,$$

where $\mathbb{K}$ *is the Markov kernel defined by*

$$\mathbb{K}f(y) = E[f(y\sigma_y)] = E[f(y/\hat{L}_y)] = E[f(y/\tilde{T}_0)]$$

with $\sigma_y = \inf\{t; R(t) = ty\}, \hat{L}_y = \sup\{t; \hat{R}(t) = y\}$ *and* $\tilde{T}_0 = \inf\{u; \tilde{R}(u) = 0\}$.

Proof. As is explained in general in [7], the intertwining property between $\{Q_t\}$ and $\{P_t\}$ is obtained by projecting $f(U_t)$, for a generic function f, in two different manners on the σ-field $\mathcal{Y}_s$ for $s < t$, and the intertwining kernel $\mathbb{K}$ is obtained as

$$\mathbb{K}f(y) = E[f(U_t)|\mathcal{Y}_t, Y_t = y] = E[f(yA_t)|\mathcal{Y}_t, Y_t = y] = E[f(y/\tilde{T}_0)].$$

□

3.b. In order to give — in the general scheme adopted so far — all the ingredients of the proof of Theorem 1.6, it now remains to deduce the (extended) infinitesimal generator of $\{Y_t, t \geqq 0\}$ from that of $\tilde{R}$, which, using formula (3.7), is a particular case of Volkonski's time-substitution formula (see, e.g., Williams [47], p.150).

In fact, we prefer to give an SDE presentation of $\{Y_t\}$, assuming that $\{U_t\}$ satisfies:

$$U_t = u + \int_0^t \sigma(U_s)dB_s + \int_0^t b(U_s)ds. \tag{3.10}$$

Then the quadratic variation process of $Y_t = (A_t)^{-1}U_t, t \geqq s$, is given by

$$\int_s^t \frac{\sigma(U_h)^2}{(A_h)^2}dh.$$

Therefore, in order that $\{Y_t\}$ be indeed a diffusion process, the process

$$\frac{\sigma(U_s)^2}{(A_s)^2} = (Y_s)^2\frac{\sigma(U_s)^2}{(U_s)^2}$$

should be a function of Y_s only; hence, we may assume that the coefficient σ is a linear function. In fact we take $\sigma(z) = z$.

We then use Itô's formula to obtain,

$$Y_t = \frac{1}{A_t} U_t = Y_s + \int_s^t \frac{1}{A_u}(U_u dB_u + b(U_u)du) - \int_s^t U_u \frac{dA_u}{A_u^2}, \quad s < t. \tag{3.11}$$

Hence, the "drift" of $\{Y_t\}$ in the original filtration is

$$\frac{b(U_u)}{A_u} - Y_u^2 U_u. \tag{3.12}$$

In order to find the drift $\hat{b}(y)$ of $\{Y_t\}$ in its own filtration, it now remains to project the previous variable on $\mathcal{Y}_u$. Thus, with the help of Proposition 3.3, we obtain

$$\begin{aligned} \hat{b}(y) &= yE\Big[\frac{b(U_u)}{U_u}\Big| Y_u = y\Big] - y^2 E[U_u | Y_u = y] \\ &= E[\tilde{T}_0 b(y/\tilde{T}_0)] - y^3 E[1/\tilde{T}_0]. \end{aligned}$$

3.c. We apply the "abstract" results given above in this section to prove Theorems 1.6 and 1.8, and Proposition 1.7. For this purpose we take

$$U_t \equiv \exp(a + B_t^{(\mu)}), \qquad t \geqq 0.$$

Hence, from Lamperti's representation, we deduce

$$U_t = R_\delta^{(\mu)}(A_t^{(\mu)}), \qquad \delta = \exp(a)$$

for a $BES(\mu)$ process $\{R_\delta^{(\mu)}(u), u \geqq 0\}$ starting from δ and, using the results presented in Section 2, we obtain

$$\hat{R}(u) \equiv R_0^{(\mu,\delta\uparrow)}(u) \quad \text{and} \quad \tilde{R}(u) \equiv R^{(\mu,\delta\downarrow)}(u), \qquad u \geqq 0.$$

Moreover, in this case, by using the explicit form of the distribution of the last hitting time of $BES(\mu, \delta\uparrow)$ process given in Pitman–Yor ([32] (7.d), p.330), we see that the Markov kernel $\mathbb{K}$ is given by

$$\mathbb{K}f(y) = \int_0^\infty f(y/u) k_{y,\delta}^{(\mu)}(u)du,$$

where $k_{y,\delta}^{(\mu)}(u)$ is the probability density of the generalized inverse Gaussian distribution ([27], [38]) given by

$$k_{y,\delta}^{(\mu)}(u) = \Big(\frac{y}{\delta}\Big)^\mu \frac{u^{-\mu-1}}{2K_\mu(\delta y)} \exp\Big(-\frac{1}{2}\Big(\frac{y^2}{u} + \delta^2 u\Big)\Big).$$

Therefore, setting $a = 0$, we get Theorem 1.8. Moreover, since the drift b is given by $b(y) = (\mu + 1/2)y$ in this special case, we obtain

$$\hat{b}(y) = (\mu + 1/2)y - y^3 \int_0^\infty \frac{1}{u} k_{y,\delta}^{(\mu)}(u)du = (\mu + 1/2)y - \delta y^2 \frac{K_{1+\mu}(\delta y)}{K_\mu(\delta y)}.$$

Finally, a simple change of variable yields Theorem 1.6. Also, Proposition 1.7 may be seen as a particular case of Proposition 3.3.

Remark 3.2. It really suffices to consider the case $a = 0$ in the above discussion, since the process $Z^{(\mu)}$, defined as in Theorem 1.6 for $a = B_0^{(\mu)} \neq 0$, is nothing else but the process $Z^{(\mu)}$ for $a = 0$, multiplied by $\delta = \exp(a)$.

3.d. It will be convenient when developing the next section to have the following partial summary at disposal. We will consider the special case where the parameter δ is 1 and denote $BES(\mu, 1\uparrow), R^{(\mu,1\uparrow)}$ and so on simply by $BES(\mu,\uparrow), R^{(\mu,\uparrow)}$, respectively.

THEOREM 3.4. (i) *Let $\mu \geqq 0$. Then there exists a $BES(\mu,\uparrow)$ process $R^{(\mu,\uparrow)} = \{R_u^{(\mu,\uparrow)}, u \geqq 0\}$ starting from 0 such that*

$$\frac{1}{Z_t^{(\mu)}} = R_{1/A_t^{(\mu)}}^{(\mu,\uparrow)} = R_{\alpha_t}^{(\mu,\uparrow)} \tag{3.13}$$

holds for every $t > 0$, where

$$\frac{1}{A_t^{(\mu)}} \equiv \alpha_t = \int_t^\infty \frac{du}{(Z_u^{(\mu)})^2}.$$

(ii) *Let $s > 0$ be fixed and $\mu \geqq 0$. Then there exists a $BES(\mu,\downarrow)$ process $R^{(\mu,\downarrow)} = \{R_u^{(\mu,\downarrow)}, u \geqq 0\}$ such that*

$$\frac{1}{Z_t^{(\mu)}} = R_{C(s,t)}^{(\mu,\downarrow)} \tag{3.14}$$

holds for $t \geqq s$, where

$$C(s,t) = \int_s^t \frac{du}{(Z_u^{(\mu)})^2}.$$

Remark 3.3. For "forward" and "backward" representations similar to (3.13) and (3.14) involving perturbed Bessel processes, see Doney–Warren–Yor [12], which is closely related to Le Gall [21] who has developed and used similar results for Bessel processes.

§4. A Markovian study of $Z^{(\mu)}$

4.a. We first consider the first hitting and the last exit times of the transient process $Z^{(\mu)}$. We set

$$T_z^{(\mu)} = \inf\{t; Z_t^{(\mu)} = z\} \qquad \text{and} \qquad L_z^{(\mu)} = \sup\{t; Z_t^{(\mu)} = z\}.$$

We obtain the explicit forms of the Laplace transforms of these random variables, which are closely related to the Hartman laws (cf. [32], [50]) recalled in (4.6) below.

THEOREM 4.1. (i) *$B^{(\mu)}_{T_z^{(\mu)}}$ is independent of $\mathcal{Z}^{(\mu)}_{T_z^{(\mu)}}$, hence, in particular, of $T_z^{(\mu)}$, and it holds that*

$$E\Big[\exp\Big(-\frac{\lambda^2}{2}T_z^{(\mu)}\Big)\Big] = \Big(\frac{K_\mu}{K_\nu}\Big)\Big(\frac{1}{z}\Big) \tag{4.1}$$

and

$$E\Big[\exp\Big(\alpha B^{(\mu)}_{T_z^{(\mu)}}\Big)\Big] = \Big(\frac{K_{\alpha+\mu}}{K_\mu}\Big)\Big(\frac{1}{z}\Big) \tag{4.2}$$

for every $\mu, \lambda \geqq 0$ and $\alpha \in \mathbf{R}$, where $\nu = \sqrt{\mu^2+\lambda^2}$.
(ii) *$B^{(\mu)}_{L_z^{(\mu)}}$ is independent of $\mathcal{Z}^{(\mu)}_{L_z^{(\mu)}}$, hence, in particular, of $L_z^{(\mu)}$, and it holds that*

$$E\Big[\exp\Big(-\frac{\lambda^2}{2}L_z^{(\mu)}\Big)\Big] = \Big(\frac{I_\nu}{I_\mu}\Big)\Big(\frac{1}{z}\Big) \tag{4.3}$$

and

$$E\Big[\exp\Big(\alpha B^{(\mu)}_{L_z^{(\mu)}}\Big)\Big] = \frac{1}{\Gamma(\alpha)2^{\alpha-1}}\int_0^\infty \frac{s^{2\alpha-1}(1+zs^2)^{\mu/2}I_\mu(1/z)}{I_\mu(\sqrt{1+zs^2}/z)}ds. \tag{4.4}$$

for every $\mu, \lambda, \alpha \geqq 0$.

Proof. From (3.13), it follows that

$$R_t^{(\mu,\uparrow)} = \frac{1}{Z^{(\mu)}_{C(t,\infty)}} \qquad \text{and} \qquad C(t,\infty) = \int_t^\infty \frac{du}{(R_u^{(\mu,\uparrow)})^2}.$$

Therefore, setting

$$T_x^{(\mu,\uparrow)} = \inf\Big\{t; R_t^{(\mu,\uparrow)} = x\Big\} \qquad \text{and} \qquad L_x^{(\mu,\uparrow)} = \sup\Big\{t; R_t^{(\mu,\uparrow)} = x\Big\},$$

we obtain

$$T_z^{(\mu)} = C\left(L_{1/z}^{(\mu,\uparrow)}, \infty\right) \qquad \text{and} \qquad L_z^{(\mu)} = C\left(T_{1/z}^{(\mu,\uparrow)}, \infty\right).$$

Conversely we also obtain

$$T_z^{(\mu,\uparrow)} = \frac{1}{A^{(\mu)}_{L^{(\mu)}_{1/z}}} \qquad \text{and} \qquad L_z^{(\mu,\uparrow)} = \frac{1}{A^{(\mu)}_{T^{(\mu)}_{1/z}}}. \tag{4.5}$$

Now, using the explicit forms of the Laplace transforms of $C(L_y^{(\mu,\uparrow)}, \infty)$ and $C(T_y^{(\mu,\uparrow)}, \infty)$ given in Corollary (4.5) in [32], we get (4.1) and (4.3).

(4.2) follows easily from Proposition 1.7, using $T \equiv T_z^{(\mu)}$ and recalling:

$$K_\nu(x) = \frac{1}{2}\int_0^\infty u^{\nu-1} \exp\Big(-\frac{x}{2}\Big(u + \frac{1}{u}\Big)\Big) du.$$

For the proof of (4.4), we recall

$$\frac{1}{z} = Z^{(\mu)}_{L^{(\mu)}_{1/z}} = \exp\Big(-B^{(\mu)}_{L^{(\mu)}_{1/z}}\Big) A^{(\mu)}_{L^{(\mu)}_{1/z}}.$$

Hence, from (4.5):

$$\exp\Big(-B^{(\mu)}_{L^{(\mu)}_{1/z}}\Big) = \frac{1}{z} T_z^{(\mu,\uparrow)}.$$

Then we get

$$\begin{aligned} E\Big[\exp\Big(\alpha B^{(\mu)}_{L^{(\mu)}_z}\Big)\Big] &= E\Big[\Big(z T^{(\mu,\uparrow)}_{1/z}\Big)^{-\alpha}\Big] \\ &= \frac{1}{2^{\alpha-1}\Gamma(\alpha)}\int_0^\infty s^{2\alpha-1} E\Big[\exp\Big(-\frac{s^2 z}{2} T^{(\mu,\uparrow)}_{1/z}\Big)\Big] ds. \end{aligned}$$

Now, applying Corollary (4.5) in [32] again, we obtain (4.4). ∎

(4.1) can also be shown by using Proposition 1.7 and we give such an alternative proof, since it will be useful in the next section.

Alternative Proof of (4.1). Let F_t be a non-negative $\mathcal{Z}_t$-measurable functional. Thanks to the Cameron–Martin theorem and Proposition 1.7, it is easy to show

$$E^{Q^{(\nu)}}[F_t] = E^{Q^{(\mu)}}\Big[F_t\Big(\frac{K_\nu}{K_\mu}\Big)\Big(\frac{1}{Z_t}\Big)\exp((\mu^2 - \nu^2)t/2)\Big]$$

for every $\mu, \nu \geqq 0$, where $E^{Q^{(\nu)}}$ denotes the expectation with respect to the probability law $Q^{(\nu)}$ of $\{Z_t^{(\nu)}, t \geqq 0\}$. In particular, setting $\nu = \sqrt{\mu^2 + \lambda^2}$,

$$\Big\{ \Big(\frac{K_\nu}{K_\mu}\Big)\Big(\frac{1}{Z_t}\Big) e^{-\lambda^2 t/2}, t > 0 \Big\}$$

is a martingale. Moreover, since

$$K_\alpha(x) = \sqrt{\frac{\pi}{2x}} e^{-x}(1 + o(1)) \qquad \text{as} \qquad x \to \infty,$$

we recover that this martingale tends to 1 as $t \downarrow 0$ a.s., as it ought to be since the germ σ-field is trivial.

Now, applying the optional stopping theorem, we get (4.1). □

Next we give some immediate consequences of Theorem 4.1. In particular, it provides some realization of the Hartman laws (see [32] and [50]), the definition of which we first recall. Let $r > 0$ and denote by $H_r^{(1)}(du)$ and $H_r^{(2)}(du)$ the first and the second Hartman laws, respectively, which are characterized by

$$\begin{aligned} \int_0^\infty \exp(-\lambda^2 u/2) H_r^{(1)}(du) &= \Big(\frac{I_\lambda}{I_0}\Big)(r), \\ \int_0^\infty \exp(-\lambda^2 u/2) H_r^{(2)}(du) &= \Big(\frac{K_0}{K_\lambda}\Big)(r). \end{aligned} \tag{4.6}$$

Then (4.1) and (4.3) imply the following.

Corollary 4.2. *Set*

$$T_{1/r}^{(0)} = \inf\{t; Z_t^{(0)} = 1/r\} = \inf\{t; \exp(B_t) = rA_t^{(0)}\}$$

and

$$L_{1/r}^{(0)} = \sup\{t; Z_t^{(0)} = 1/r\} = \sup\{t; \exp(B_t) = rA_t^{(0)}\}.$$

Then the distributions of $T_{1/r}^{(0)}$ *and* $L_{1/r}^{(0)}$ *are* $H_r^{(2)}$ *and* $H_r^{(1)}$, *respectively.*

The following corollary will be our main tool in the next subsection to derive an explicit expression for $\{Q_t^{(\mu)}\}$.

Corollary 4.3. *Let $a < b$ and denote by $E_a^{Q^{(\mu)}}$ the expectation with respect to the law of $\{Z_t^{(\mu)}, t \geqq 0\}$ starting from a. Then one has*

$$E_a^{Q^{(\mu)}}\Big[\exp\Big(-\frac{\lambda^2}{2}L_b^{(\mu)}\Big)\Big] = \Big(\frac{K_\nu}{K_\mu}\Big)(1/a)\Big(\frac{I_\nu}{I_\mu}\Big)(1/b), \tag{4.7}$$

where $\nu = \sqrt{\mu^2+\lambda^2}$.

Proof. This is immediate from the strong Markov property of $Z^{(\mu)}$ at the first hitting time $T_a^{(\mu)}$ of a, and the formulae (4.1) and (4.3). ∎

4.b. In this subsection we aim to show how our result in Proposition 1.7, which gives the conditional law of $B_t^{(\mu)}$ given $\mathcal{Z}_t^{(\mu)} = \sigma\{Z_u^{(\mu)}; u \leqq t\}$, translates itself in previously obtained formulae.

At first we discuss its relation to the joint law of $(B_t^{(\mu)}, A_t^{(\mu)})$. In the case $\mu = 0$, the explicit expression is given by (cf. Yor [53])

$$P(B_t \in dx, A_t \in du) = \frac{1}{u}\exp\Big(-\frac{1}{2u}(1+e^{2x})\Big)\theta_{e^x/u}(t)dxdu, \tag{4.8}$$

where $\theta_r(t), r > 0$, $t > 0$, is defined from the first Hartman law (cf. (4.6)) by

$$I_0(r)H_r^{(1)}(dt) = \theta_r(t)dt. \tag{4.9}$$

An explicit formula for $\theta_r(u)$ is given in [50], p.85. See also the formula (6.b) in [53].

Using the definition of $Z_t^{(\mu)} = \exp(-B_t^{(\mu)})A_t^{(\mu)}$, the Cameron–Martin theorem and a simple change of variables from (4.8) yield the following.

Proposition 4.4. *The joint law of $(Z_t^{(\mu)}, B_t^{(\mu)})$ is given by*

$$\begin{aligned} &P(Z_t^{(\mu)} \in dz, B_t^{(\mu)} \in dx) \\ &\quad = z^{-1}\theta_{1/z}(t)e^{-\mu^2 t/2}e^{\mu x}\exp\Big(-\frac{\cosh(x)}{z}\Big)dxdz. \end{aligned} \tag{4.10}$$

Equivalently one has

$$P(Z_t^{(\mu)} \in dz) = 2z^{-1}\theta_{1/z}(t)K_\mu(1/z)e^{-\mu^2 t/2}dz \tag{4.11}$$

and

$$P(B_t^{(\mu)} \in dx | Z_t^{(\mu)} = z) = \frac{\exp(\mu x)}{2K_\mu(1/z)}\exp\Big(-\frac{\cosh(x)}{z}\Big)dx. \tag{4.12}$$

Thus the disintegration of the formula (4.10) gives (4.12), which is precisely the formula already obtained in Proposition 1.7.

We now deduce from Proposition 4.4 an interesting Laplace transform formula for $\theta_r(t)$ with respect to the argument r, as well as the following result found in [24]: for any given $t>0$,

$$(\mathbf{e}Z_t^{(\mu)}, B_t^{(\mu)}) \stackrel{(\text{law})}{=} (\cosh(|B_t^{(\mu)}| + L_t^{(\mu)}) - \cosh(B_t^{(\mu)}), B_t^{(\mu)}), \tag{4.13}$$

where $\mathbf{e}$ on the left hand side denotes an exponential random variable with mean 1, independent of $B^{(\mu)}$ and $L_t^{(u)}$ on the right hand side is the local time at level 0 and time t of $B^{(\mu)}$.

Proposition 4.5. (i) *Set* $a(y)=\arg\cosh(y)$. *Then it holds that*

$$\int_0^\infty \theta_r(t) e^{-yr}\frac{dr}{r} = \frac{1}{\sqrt{2\pi t}}\exp(-(a(y))^2/2t). \tag{4.14}$$

Consequently, one has

$$\ell_\theta(y,t) \equiv \int_0^\infty \theta_r(t) e^{-yr}\,dr = \frac{1}{\sqrt{2\pi t^3}} a'(y)a(y)\exp(-(a(y))^2/2t). \tag{4.15}$$

(ii) *Let* $t>0$. *Then the following identities hold*:

$$\begin{aligned} &P(\cosh(|B_t|+L_t) - \cosh(B_t) \in dx, B_t\in db) \\ &\qquad = E[(Z_t)^{-1}\exp(-x/Z_t)|B_t=b]\frac{\exp(-b^2/2t)}{\sqrt{2\pi t}}dxdb \\ &\qquad = \ell_\theta(x+\cosh(b),t)dxdb, \end{aligned} \tag{4.16}$$

where $Z_t = Z_t^{(0)}$.

Proof. (i) We take $\mu=0$ in (4.10) and (4.11). Then we get

$$\begin{aligned} P(B_t\in dx) &= \int_0^\infty P(B_t\in dx|Z_t=z)P(Z_t\in dz) \\ &= \Big(\int_0^\infty \exp(-\cosh(x)r)\theta_r(t)\frac{dr}{r}\Big)dx. \end{aligned}$$

Therefore, making the change of variables $y=\cosh(x)$ gives (4.14). Formula (4.15) follows from (4.14) by differentiation with respect to y.

(ii) The first equality in (4.16) is equivalent to (4.13). To prove the second equality, we deduce from (4.10) that

$$P(Z_t \in dz | B_t = b) \frac{\exp(-b^2/2t)}{\sqrt{\sqrt{2\pi t}}} = \theta_{1/z}(t) \exp\Big(-\frac{\cosh(b)}{z}\Big)\frac{dz}{z}.$$

Thus we obtain

$$\begin{aligned} & E[(Z_t)^{-1}\exp(-x/Z_t)|B_t = b]\frac{\exp(-b^2/2t)}{\sqrt{2\pi t}} \\ &= \int_0^\infty \frac{1}{z}\exp(-x/z)\theta_{1/z}(t)\exp\Big(-\frac{\cosh(b)}{z}\Big)\frac{dz}{z} \\ &= \int_0^\infty \exp(-(x+\cosh(b))r)\theta_r(t)dr \\ &= \ell_\theta(x+\cosh(b), t). \end{aligned}$$

□

Remark 4.1. Of course, we could also deduce the equality between the first and third expressions in (4.16) from the following well known formula (which, indeed, has a lot to do with Pitman's theorem):

$$P(|B_t| + L_t \in dx, B_t \in db) = \frac{x\exp(-x^2/2t)}{2\sqrt{2\pi t^3}} 1_{\{|b| \leqq x\}} dx db.$$

4.c. We now note how formula (4.11) is coherent with the expression of the law of $L_z^{(\mu)}$ presented in formula (4.3) above. Indeed, on one hand, (4.3) shows that [2]

$$P(L_z^{(\mu)} \in dt) = \frac{1}{I_\mu(1/z)}\theta_{1/z}(t)e^{-\mu^2 t/2}dt.$$

In particular, $L_z^{(0)}$ is distributed as the first Hartman law $H_{1/z}^{(1)}$. On the other hand, for a transient $\mathbf{R}_+$-valued diffusion which we take here to be $Z^{(\mu)}$, there is the formula

$$P_a(L_z^{(\mu)} \in dt) = C_z p_t^{(\mu)}(a, z)dt, \tag{4.17}$$

[2] $L_z^{(\mu)}$ denotes here a last passage time at z; these should not be any confusion with the local time $L_z^{(\mu)}$ appearing (4.13).

where $p_t^{(\mu)}(a,z)$ is the transition probability density of $Z^{(\mu)}$ with respect to the Lebesgue measure dz and, letting s be the scaling function for $Z^{(\mu)}$ such that $s(0)=-\infty$ and $s(\infty)=0$, C_z is given by

$$C_z = -\frac{1}{2}\Big(\frac{s'}{s}\Big)(z)\sigma^2(z) \equiv -\frac{1}{2}\Big(\frac{s'}{s}\Big)(z)z^2,$$

where $\sigma^2(z)=z^2$ is the diffusion coefficient of $Z^{(\mu)}$. See, e.g., Pitman–Yor [32], p.326, and Revuz–Yor [34], Chapter VII, Exercise (4.16), p.321.

Thus we obtain the following two formulae for $p_t^{(\mu)}(0,z)$:

$$\begin{aligned} p_t^{(\mu)}(0,z) &= 2z^{-1}e^{-\mu^2 t/2}\theta_{1/z}(t)K_\mu(1/z), \\ C_z p_t^{(\mu)}(0,z) &= \frac{1}{I_\mu(1/z)}\theta_{1/z}(t)e^{-\mu^2 t/2}, \end{aligned} \tag{4.18}$$

the comparison of which yields

$$\Big(\frac{s'}{s}\Big)(z) = -\frac{1}{zI_\mu(1/z)K_\mu(1/z)}. \tag{4.19}$$

As a consequence, we may state the following.

Proposition 4.6. *Scale functions s_μ and $s^{(\mu,\downarrow)}$ for the transient diffusion processes $Z^{(\mu)}$ and $R^{(\mu,\downarrow)}$, respectively, are given by*

$$s_\mu(z) = -\Big(\frac{I_\mu}{K_\mu}\Big)(1/z) \qquad \textit{and} \qquad s^{(\mu,\downarrow)}(r) = -\Big(\frac{I_\mu}{K_\mu}\Big)(r). \tag{4.20}$$

Proof. By the Wronskian formula

$$W(I_\mu, K_\mu)(z) = I_\mu(z)K_\mu'(z) - I_\mu'(z)K_\mu(z) = -\frac{1}{z},$$

it is easy to show that s_μ satisfies (4.19). Moreover, by using the asymptotics of I_μ and K_μ near 0 and ∞, it is also easy to check $s_\mu(0)=-\infty$ and $s_\mu(\infty)=0$.

From the formula (3.14), we know that $\{s_\mu((R_u^{(\mu,\downarrow)})^{-1}), u>0\}$ is a local martingale, so that we can deduce $s^{(\mu,\downarrow)}(r)=s_\mu(1/r)$. ∎

Remark 4.2. Of course, the formula (4.20) may also be confirmed directly by checking that $\mathcal{L}^\mu(s_\mu)=0$ and $\mathcal{L}^{(\mu,\downarrow)}(s^{(\mu,\downarrow)})=0$ for the respective infinitesimal generators $\mathcal{L}^\mu$ and $\mathcal{L}^{(\mu,\downarrow)}$ of $Z^{(\mu)}$ and $R^{(\mu,\downarrow)}$.

We are now able to give an explicit expression for the Laplace transform of $p_t^{(\mu)}(a,b)$. Combining (4.7) with (4.17), we obtain the following:

Proposition 4.7. *Let $a < b$. Then the Laplace transform of the transition density $p_t^{(\mu)}(a,b)$ of the diffusion process $Z^{(\mu)}$ with respect to the Lebesgue measure dt is given by*

$$\int_0^\infty p_t^{(\mu)}(a,b)\exp(-\frac{\lambda^2}{2}t)dt = \frac{2K_\mu(1/b)}{bK_\mu(1/a)}K_\nu(1/a)I_\nu(1/b),$$

where $\nu = \sqrt{\lambda^2+\mu^2}$.

4.d. We now take advantage of our knowledge of the Markovian features of $Z^{(\mu)}$ to identify the law of its local time at time ∞; in fact, we phrase the result as follows, in terms of the process

$$L_t^{(\mu)} \equiv \log\Big(\int_0^t \exp(2B_s^{(\mu)})ds\Big) - B_t^{(\mu)}, \qquad t > 0,$$

whose local time we denote by $\{\ell_t^{(\mu)}(a), a \in \mathbf{R}\}$.

Theorem 4.8. (i) *Let $\{\ell_\infty^{(\mu,\uparrow)}(r), r \geqq 0\}$ be the local time at time ∞ of a $BES(\mu,\uparrow)$ process $R^{(\mu,\uparrow)}$. Then one has*

$$\{\ell_\infty^{(\mu)}(x), x \in \mathbf{R}\} \overset{\text{(law)}}{=} \{e^x \ell_\infty^{(\mu,\uparrow)}(e^{-x}), x \in \mathbf{R}\}. \tag{4.21}$$

(ii) *Consider the increasing function $k(y) = (I_\mu/K_\mu)(y)$. Then one has*

$$\{\ell_\infty^{(\mu,\uparrow)}(y), y > 0\} \overset{\text{(law)}}{=} \Big\{\frac{1}{k'(y)}(R^{(2)}(k(y)))^2, y > 0\Big\}, \tag{4.22}$$

where $R^{(2)} = \{R^{(2)}(u), u \geqq 0\}$ denotes a $BES(2)$ process starting from 0.

Proof. (i) The assertion follows easily from the representation (3.13) of $Z^{(\mu)}$ in terms of $R^{(\mu,\uparrow)}$, using time and space change.
(ii) Let $\sigma(y) = 1/k(y)$ and recall from Section 2 that σ is a non-negative scale function of $R^{(\mu,\uparrow)}$. Then, setting $R_0^{(\mu,\uparrow)} = r$, we have

$$\sigma(R_t^{(\mu,\uparrow)}) = \beta(\int_0^t (\sigma'(R_u^{(\mu,\uparrow)}))^2 du) \quad \text{and} \quad \int_0^\infty (\sigma'(R_u^{(\mu,\uparrow)}))^2 du = T_0(\beta),$$

where $\beta = \{\beta(u), u \geqq 0\}$ is a Brownian motion starting from $\sigma(r)$ and $T_0(\beta)$ is its first hitting time at 0. Using the classical result due to Ray (cf. Ito–McKean [18], Section 2.8) that the local time of β on the time interval $[0, T_0(\beta)]$ and level a between 0 and $\sigma(r)$ is distributed as $\{(R^{(2)}(a))^2, 0 \leqq a \leqq \sigma(r)\}$, we easily obtain, by letting $r \downarrow 0$, that

$$\{\ell_\infty^{(\mu,\uparrow)}(y), y > 0\} \stackrel{\text{(law)}}{=} \Big\{\frac{1}{|\sigma'(y)|}(R^{(2)}(\sigma(y)))^2, y > 0\Big\}.$$

Formula (4.22) now follows by using time inversion on $(R^{(2)}(\,\cdot\,))^2$. □

§5. Hitting times of $\xi^{(\mu)}$

In this section we relate the laws of the first hitting times of $\xi^{(\mu)}$ to the calculations made in Yor [51], where the primary aim was to obtain (the Laplace transform of) the laws of the first hitting times of square root boundaries for Bessel processes.

We first show that Bessel processes with "naive" drift, which have been discussed in [51] following Kendall [19] (see also Brillinger [6]), are related to the diffusion process $\xi^{(\mu)}$ the same way $BES(\mu, \downarrow)$ are related to $Z^{(\mu)}$.

${}^{\delta}BES(\mu)$, the Bessel process with "naive" drift δ and index μ, is a $\mathbf{R}_+$-valued diffusion process with infinitesimal generator

$$\frac{1}{2}\frac{d^2}{dx^2} + \Big(\frac{2\mu+1}{2x} - \delta\Big)\frac{d}{dx}.$$

Again we take $\delta = 1$ and denote ${}^{(n)}BES(\mu)$ for ${}^{1}BES(\mu)$.

We denote by $Y^{(\mu)}(y) = \{Y_t^{(\mu)}(y)\}_{t \geqq 0}$ the diffusion process defined by

$$\exp(-B_t^{(\mu)})\Big\{y + \int_0^t \exp(B_s^{(\mu)})ds\Big\}, \quad t \geqq 0,$$

which, in the case $y = 0$, we denoted by $\xi^{1,1}$ in Section 1. We also denote by $\xi^{(\mu)}(h) = \{\xi_t^{(\mu)}(h), t \geqq 0\}$ the (diffusion) process defined by

$$\xi_t^{(\mu)}(h) = \exp(-2B_t^{(\mu)})\Big\{h + \int_0^t \exp(2B_s^{(\mu)})ds\Big\}, \quad t \geqq 0.$$

$\xi^{(\mu)}(0)$ is the diffusion process $\xi^{(\mu)}$ which we considered in the previous sections.

We note the following, which is implicitly contained in [51] and can be obtained by using Itô's formula and the scaling property of Brownian motion.

PROPOSITION 5.1. (i) *To a process $Y^{(\mu)}(y)$ starting from $y \neq 0$, one can associate a ${}^{(n)}BES(\mu)$ process $\{\rho_u, u \geqq 0\}$ such that*

$$\frac{1}{Y_t^{(\mu)}(y)} = \rho_{C_t^{(Y^{(\mu)}(y))}} \tag{5.1}$$

or, equivalently,

$$\frac{1}{\rho_u} = Y^{(\mu)}_{C_u^{(\rho)}}(y), \tag{5.2}$$

where, for any measurable process $X = \{X_t, t \geqq 0\}$,

$$C_t^{(X)} = \int_0^t \frac{du}{X_u^2}.$$

(ii) *The diffusion processes $Y^{(\mu)}(y)$ and $\xi^{(\mu)}(h)$ are related by*

$$\{Y_{4t}^{(\mu)}(y)\}_{t\geqq 0} \overset{\text{(law)}}{=} \{4\xi_t^{(2\mu)}(y/4)\}_{t\geqq 0}. \tag{5.3}$$

(5.2) implies that $C^{(\rho)}_{T_b(\rho)} = T_{1/b}(Y^{(\mu)}(y))$ holds, where, for a stochastic process X, $T_x(X)$ denotes the first hitting time of x by X. Therefore, combining with the formula (3.e) in [51], we obtain the following.

COROLLARY 5.2. *The Laplace transform of $T_z(Y^{(\mu)}(y))$ is given by*

$$E\Big[\exp\Big(-\frac{\lambda^2}{2}T_z(Y^{(\mu)}(y))\Big)\Big] = \Big(\frac{z}{y}\Big)^{\nu-\mu}\frac{\Lambda(\nu-\mu, 1+2\nu; 2/y)}{\Lambda(\nu-\mu, 1+2\nu; 2/z)}, \tag{5.4}$$

where $\nu = \sqrt{\lambda^2+\mu^2}$, $\Lambda = \Phi$ if $y > z$ and $\Lambda = \Psi$ if $y < z$ for the confluent hypergeometric functions Φ and Ψ.

Remark 5.1. For the confluent hypergeometric functions, we follow the notations in Lebedev [20], Chapter 5.

By using (5.3) and (5.4), we obtain the following.

THEOREM 5.3. *The Laplace transform of the hitting time $T_k(\xi^{(\mu)}(h))$ of k by $\xi^{(\mu)}(h)$ is given by*

$$E\Big[\exp\Big(-\frac{\lambda^2}{2}T_k(\xi^{(\mu)}(h))\Big)\Big] = \Big(\frac{k}{h}\Big)^{\sigma}\frac{\Lambda(\sigma, 1+\nu, 1/2h)}{\Lambda(\sigma, 1+\nu; 1/2k)},$$

where $\nu = \sqrt{\lambda^2+\mu^2}$ and $\sigma = (\nu-\mu)/2$. In particular, if $h = 0$, it holds that

$$E\Big[\exp\Big(-\frac{\lambda^2}{2}T_k(\xi^{(\mu)})\Big)\Big] = \frac{(2k)^{\sigma}}{\Psi(\sigma, \nu+1; 1/2z)}. \tag{5.5}$$

We now show how the intertwining relation (1.5) allows to connect the results obtained in Theorems 4.1 and 5.3. If we set

$$\psi(z) = \Big(\frac{K_\nu}{K_\mu}\Big)\Big(\frac{1}{z}\Big),\ z>0,\ \psi(0)=1,$$

then (4.1) amounts to the fact that

$$\psi_t = \psi(Z_t^{(\mu)})\exp(-\lambda^2 t/2)$$

is a martingale. On the other hand, if we set

$$\varphi(x) = \frac{1}{(2x)^\sigma}\Psi(\sigma, 1+\nu; 1/2x),$$

then (5.5) amounts to the fact that

$$\varphi_t = \varphi(\xi_t^{(\mu)})\exp(-\lambda^2 t/2)$$

is a martingale. In fact, we can directly prove the martingale property of $\{\varphi_t\}$ by using Itô's formula and the properties of the confluent hypergeometric function Ψ; for closely related computations, see Donati-Martin–Ghomrasni–Yor, [11].

However, it is easily seen, as a consequence of the intertwining relation (1.5), that

$$(\mathbb{K}^{(\mu)}\varphi)(Z_t^{(\mu)})\exp(-\lambda^2 t/2)$$

is a $(\mathcal{Z}_t^{(\mu)})$-martingale. But this fact jibes with the martingale property of ψ_t, since there is the relationship

$$\mathbb{K}^{(\mu)}\varphi = \psi \tag{5.6}$$

as we now show in the following.

Proof of (5.6). At first we rewrite $\mathbb{K}^{(\mu)}\varphi$ as follows:

$$\begin{aligned}\mathbb{K}^{(\mu)}\varphi(z) &= \frac{1}{2K_\mu(1/z)}\int_0^\infty u^{-\mu+1}\exp\Big(-\frac{1}{2z}\Big(u+\frac{1}{u}\Big)\Big)\varphi(zu)\frac{du}{u^2} \\ &= \frac{z^\mu}{2K_\mu(1/z)}\int_0^\infty \frac{1}{t^{\mu+1}}\exp\Big(-\frac{1}{2}\Big(\frac{t}{z^2}+\frac{1}{t}\Big)\Big)\varphi(t)dt.\end{aligned}$$

Therefore, setting $x = 1/z$ and

$$\varphi_\mu(t) = \frac{1}{t^{\mu+1}} \exp(-1/2t)\varphi(t)$$

and denoting its Laplace transform by $\widehat{\varphi}_\mu$,

$$\widehat{\varphi}_\mu(\alpha) = \int_0^\infty e^{-\alpha t}\varphi_\mu(t)dt,$$

we obtain

$$\mathbb{K}^{(\mu)}\varphi(z) = \frac{1}{2x^\mu K_\mu(1/z)}\widehat{\varphi}_\mu(x^2/2).$$

Thus we need to find a function φ_μ satisfying

$$\widehat{\varphi}_\mu(x^2/2) = 2x^\mu K_\nu(x).$$

This can be done as follows. By noting the elementary identities

$$K_\nu(x) = \frac{1}{2}x^{-\nu}\int_0^\infty u^{1+\nu}\exp\Big(-\frac{1}{2}\Big(\frac{1}{u}+x^2u\Big)\Big)du$$

and

$$x^{-2\sigma} = \frac{1}{\Gamma(\sigma)2^\sigma}\int_0^\infty t^{\sigma-1}e^{-x^2t/2}dt,$$

we obtain

$$\begin{aligned} 2x^\mu K_\nu(x) &= \frac{1}{\Gamma(\sigma)2^\sigma}\int_0^\infty dt\int_0^\infty t^{\sigma-1}u^{1+\nu}\exp\Big(-\frac{1}{2u}-\frac{x^2}{2}(t+u)\Big)du \\ &= \frac{1}{\Gamma(\sigma)2^\sigma}\int_0^\infty e^{-x^2z/2}dz\int_0^z (z-u)^{\sigma-1}u^{1+\nu}e^{-1/2u}du. \end{aligned}$$

Therefore we get

$$\begin{aligned} \varphi(z) &= \frac{z^{\mu+1}e^{1/2z}}{\Gamma(\sigma)2^\sigma}\int_0^z u^{-1-\nu}e^{-1/2u}(z-u)^{\sigma-1}du \\ &= \frac{1}{\Gamma(\sigma)(2z)^\sigma}\int_0^1 x^{-1-\nu}\exp\Big(-\frac{1}{2z}\Big(\frac{1}{x}-1\Big)\Big)(1-x)^{\sigma-1}dx \\ &= \frac{1}{\Gamma(\sigma)(2z)^\sigma}\int_0^\infty e^{-y/2z}y^{\sigma-1}(1+y)^{\nu-\sigma}dy \\ &= \frac{1}{(2z)^\sigma}\Psi(\sigma,\nu+1;1/2z), \end{aligned}$$

where we have used the integral representation of the confluent hypergeometric functions (cf. [20]). The proof is completed. ∎

§6. Hitting times related to geometric Brownian motion and its integral

Given the importance of geometric Brownian motions in a number of domains, it seems of some interest to present clearly the Laplace transforms of both

$$S^{(\mu)}_{\alpha,\beta} = \inf\Big\{t; \exp(B^{(\mu)}_t) < \alpha + \beta \int_0^t \exp(B^{(\mu)}_u)du\Big\}$$

and

$$\tilde{S}^{(\mu)}_{\alpha,\beta} = \inf\Big\{t; \exp(B^{(\mu)}_t) < \alpha + \beta \int_0^t \exp(2B^{(\mu)}_u)du\Big\}.$$

We now make the elementary observations that $S^{(\mu)}_{\alpha,\beta}$, resp. $\tilde{S}^{(\mu)}_{\alpha,\beta}$, may be expressed in terms of first hitting times of $Y^{(\mu)}(y)$, resp. $Z^{(\mu)}(z)$, where $Y^{(\mu)}(y)$ is a diffusion process introduced in Section 5 and $Z^{(\mu)}(z) = \{Z^{(\mu)}_t(z), t \geqq 0\}$ is defined by

$$Z^{(\mu)}_t(z) = \exp(-B^{(\mu)}_t)\Big(z + \int_0^t \exp(2B^{(\mu)}_u)du\Big).$$

Indeed, we have obviously

$$S^{(\mu)}_{\alpha,\beta} = T_{1/\beta}\Big(Y^{(\mu)}\Big(\frac{\alpha}{\beta}\Big)\Big), \qquad \tilde{S}^{(\mu)}_{\alpha,\beta} = T_{1/\beta}\Big(Z^{(\mu)}\Big(\frac{\alpha}{\beta}\Big)\Big).$$

Consequently, we deduce from formula (5.4) that

$$\text{(6.1)} \qquad E\Big[\exp\Big(-\frac{\lambda^2}{2}S^{(\mu)}_{\alpha,\beta}\Big)\Big] = \alpha^{\mu-\nu}\frac{\Lambda(\nu-\mu, 1+2\nu, \frac{2\beta}{\alpha})}{\Lambda(\nu-\mu, 1+2\nu, 2\beta)}$$

holds for any $\lambda > 0$, where $\nu = \sqrt{\lambda^2+\mu^2}$, $\Lambda = \Phi$ if $\alpha > 1$ and $\Lambda = \Psi$ if $\alpha < 1$.

These results yield in particular the Laplace transforms of the first hitting time of the martingale

$$N^{(\mu)}_t = \int_0^t e^{(\mu)}_s dB_s,$$

since, from Itô's formula, one has

$$e^{(\mu)}_t \equiv \exp(B^{(\mu)}_t) = 1 + N^{(\mu)}_t + \Big(\mu + \frac{1}{2}\Big)\int_0^t \exp(B^{(\mu)}_s)ds.$$

Therefore, setting

$$T_a(N^{(\mu)}) = \inf\{s; N_s^{(\mu)} = a\},$$

we have $T_a(N^{(\mu)}) = S_{\alpha,\beta}^{(\mu)}$ with $\alpha = 1 + a, \beta = \mu + 1/2$. Hence, from (6.1), we deduce the following.

THEOREM 6.1. *Let $\lambda > 0$ and set $\nu = \sqrt{\lambda^2 + \mu^2}$. Then it holds that*

$$E\Big[\exp\Big(-\frac{\lambda^2}{2}T_a(N^{(\mu)})\Big)\Big] = \frac{1}{(a+1)^{\nu-\mu}}\frac{\Phi(\nu-\mu, 2\nu+1; \frac{2\mu+1}{a+1})}{\Phi(\nu-\mu, 2\nu+1; 2\mu+1)}. \tag{6.2}$$

In the rest of this section, we shall show how formula (6.2) may be derived from the computations made in [51]. However, before giving this proof, we present some consequences of (6.2). Set

$$\Sigma_t^{(\mu)} = \sup_{0\leqq s\leqq t} N_s^{(\mu)}.$$

Then, letting S_λ be an exponential random variable with parameter $\lambda^2/2$ independent of $B^{(\mu)}$, it is easy to show

$$\begin{aligned} E\Big[\exp\Big(-\frac{\lambda^2}{2}T_a(N^{(\mu)})\Big)\Big] \\ = \frac{\lambda^2}{2}\int_0^\infty \exp(-\lambda^2 t/2)P(\Sigma_t^{(\mu)} > a)dt = P(\Sigma_{S_\lambda}^{(\mu)} > a). \end{aligned} \tag{6.3}$$

Moreover, if $\mu < 0$,

$$\Sigma_\infty^{(\mu)} = \sup_{t\geqq 0}\int_0^t e_s^{(\mu)} dB_s$$

is finite almost surely. Therefore, letting $\lambda \downarrow 0$ in (6.2), we obtain the following.

COROLLARY 6.2. *Let $\mu < 0$. Then*

$$P(\Sigma_\infty^{(\mu)} > a) = (a+1)^{2\mu}\frac{\Phi(-2\mu, 1-2\mu; \frac{2\mu+1}{a+1})}{\Phi(-2\mu, 1-2\mu; 2\mu+1)}. \tag{6.4}$$

holds for every $a > 0$.

Remark 6.1. If $\mu = -1/2$, we get

$$P(\Sigma_\infty^{(-1/2)} > a) = \frac{1}{1+a}.$$

That is, we recover the well known identity:

$$\sup_{t \geqq 0} (\exp(B_t - t/2)) \overset{\text{(law)}}{=} \frac{1}{U},$$

where U is a uniform random variable on $(0,1)$.

For the proof of Theorem 6.1, we need the explicit form of the negative moments associated to first hitting times of square root boundaries for Bessel processes studied in Yor [51] following the method due to Shepp [39].

Let $\{R_t^{(\eta)}\}_{t \geqq 0}$ be a Bessel process with index η starting from ρ and define the first hitting times of c-square root boundaries for $\{R_t^{(\eta)}\}$ by

$$T_c^{(\eta,+)} = \inf\{t; R_t^{(\eta)} = c\sqrt{1+t}\}, \qquad T_c^{(\eta,-)} = \inf\{t; R_t^{(\eta)} = c\sqrt{1-t}\}$$

for $c > 0$. Then we show the following theorem, a part of which is given in [51].

THEOREM 6.3. *For every $m > 0$, it holds that*

$$E\Big[(R^{(\eta)}_{T_c^{(\eta,+)}})^{-2m}\Big] = c^{-2m} E[(1+T_c^{(\eta,+)})^{-m}] = c^{-2m} \frac{\Phi(m; \eta+1; \rho^2/2)}{\Phi(m, \eta+1; c^2/2)} \tag{6.5}$$

and

$$E\Big[(R^{(\eta)}_{T_c^{(\eta,-)}})^{-2m}\Big] = c^{-2m} E[(1-T_c^{(\eta,-)})^{-m}] = c^{-2m} \frac{\Phi(m; \eta+1; -\rho^2/2)}{\Phi(m, \eta+1; -c^2/2)}. \tag{6.6}$$

Proof. We give a proof of (6.6), since (6.5) was shown in [51]. A slight modification of the following arguments also proves (6.5).

At first we set

$$\tilde{J}_\eta(z) = z^{-\eta} J_\eta(z)$$

for the Bessel function J_η of the first kind of order η. Then, by Itô's formula, it is easy to show that $\tilde{J}_\eta(\theta R_t^{(\eta)})e^{\theta^2 t/2}$ is a martingale if one note the differential equation for J_η:

$$J_\eta''(z) + \frac{1}{z}J_\eta'(z) + \Big(1 - \frac{\eta^2}{z^2}\Big)J_\eta(z) = 0.$$

Therefore the optimal sampling theorem implies

$$\begin{aligned} &E\Big[\tilde{J}_\eta(\theta R^{(\eta)}_{T_c^{(\eta,-)}} \exp(\theta^2 T_c^{(\eta,-)}/2)\Big] \\ &\qquad = E\Big[\tilde{J}_\eta\Big(\theta c\sqrt{1 - T_c^{(\eta,-)}}\Big)\exp\Big(\theta^2 T_c^{(\eta,-)}/2\Big)\Big] = \tilde{J}_\eta(\theta\rho). \end{aligned}$$

Now we integrate the both hand sides in θ with respect to $\theta^{2m-1}e^{-\theta^2/2}d\theta$. Then we obtain

$$\begin{aligned} &\int_0^\infty E\Big[\tilde{J}_\eta\Big(\theta c\sqrt{1 - T_c^{(\eta,-)}}\Big)\exp\Big(\theta^2 T_c^{(\eta,-)}/2\Big)\Big]\theta^{2m-1}e^{-\theta^2/2}d\theta \\ &= E\Big[\int_0^\infty \tilde{J}_\eta\Big(\theta c\sqrt{1 - T_c^{(\eta,-)}}\Big)\exp\Big(-\Big(1 - T_c^{(\eta,-)}\Big)\theta^2/2\Big)\theta^{2m-1}d\theta\Big] \\ &= \varphi(c)E\Big[\Big(1 - T_c^{(\eta,-)}\Big)^{-m}\Big] \end{aligned}$$

and

$$E\Big[\Big(1 - T_c^{(\eta,-)}\Big)^{-m}\Big] = \frac{\varphi(\rho)}{\varphi(c)},$$

where

$$\varphi(z) = \int_0^\infty \tilde{J}_\eta(zb)b^{2m-1}e^{-b^2/2}db.$$

Moreover, by using the identity

$$\int_0^\infty J_\nu(at)e^{-p^2t^2}t^{\mu-1}dt = \frac{\Gamma((\nu+\mu)/2)(a/2p)^\nu}{2p^\mu\Gamma(\nu+1)}\Phi\Big(\frac{\nu+\mu}{2}, \nu+1; -a^2/4p^2\Big)$$

(cf. [46], page 392), we can easily show

$$\varphi(z) = \frac{\Gamma(m)}{2^{2\eta+1-m}\Gamma(\eta+1)}\Phi(m, \eta+1; -z^2/2).$$

□

We give a proof of Theorem 6.1 in the rest of this section. By Lamperti's relation, there exists a Bessel process with index μ such that

$$e_t^{(\mu)} = R^{(\mu)}_{A_t^{(\mu)}}.$$

We set

$$C_u^{(\mu)} = \int_0^u \frac{ds}{(R_s^{(\mu)})^2},$$

which is the inverse of $A_t^{(\mu)}$. Moreover we set

$$\beta_u^{(\mu)} = R_u^{(\mu)} - 1 - (\mu+1/2)\int_0^u \frac{ds}{(R_s^{(\mu)})^2}$$

and

$$T_a^{(\mu),*} = \inf\{u; \beta_u^{(\mu)} > a\}.$$

Then we show the following.

Lemma 6.1. $P(\Sigma_{S_\lambda}^{(\mu)} > a) = E[\exp(-\lambda^2 C^{(\mu)}_{T_a^{(\mu),*}}/2)]$.

Proof. We note

$$\begin{aligned} \beta^{(\mu)}_{A_t^{(\mu)}} &= R^{(\mu)}_{A_t^{(\mu)}} - 1 - (\mu+1/2)\int_0^t \frac{1}{R^{(\mu)}_{A_u^{(\mu)}}} \exp(2B_u^{(\mu)})du \\ &= e_t^{(\mu)} - 1 - (\mu+1/2)\int_0^t e_u^{(\mu)} du \\ &= N_t^{(\mu)}. \end{aligned}$$

Then we obtain

$$\beta_s^{(\mu),*} \equiv \sup_{0\leqq s\leqq u} \beta_u^{(\mu)} = \Sigma_t^{(\mu)} \qquad \text{and} \qquad \Big\{\Sigma_t^{(\mu)} > a\Big\} = \Big\{t > C^{(\mu)}_{T_a^{(\mu),*}}\Big\}.$$

The rest of the proof is easy and left to the reader. ∎

Next we note

$$E\Big[\exp\Big(-\frac{1}{2}\lambda^2 C^{(\mu)}_{T_a^{(\mu),*}}\Big)\Big] = E\Big[\Big(\frac{\rho}{R^{(\nu)}_{T_a^{\nu,*}}}\Big)^{\nu-\mu}\Big]. \tag{6.7}$$

This follows from [34], Chapter XI, Exercise (1.22), page 450, which says that the probability laws of two Bessel processes with different indices are absolutely continuous when these laws are restricted to $\mathcal{R}_T = \sigma\{R_u; 0 \leqq u \leqq T\}$ for any stopping time T and that the density is given in terms of C_T and R_T. Then, combining (6.7) with (6.3) and Lemma 6.1, we obtain (6.2) from the following lemma.

LEMMA 6.2. *For $\alpha > 0$ and $\beta \in \mathbf{R}$, set*

$$T^{(\nu)}_{\alpha,\beta} = \inf\Big\{u; R^{(\nu)}_u > \alpha + \beta \int_0^u \frac{ds}{R^{(\nu)}_s}\Big\}.$$

Then it holds that

$$E\Big[\Big(R^{(\nu)}_{T^{(\nu)}_{\alpha,\beta}}\Big)^{-m}\Big] = \alpha^{-m} \frac{\Phi(m, 2\nu+1; \frac{2\beta\rho}{\alpha})}{\Phi(m, 2\nu+1; 2\beta)} \tag{6.8}$$

for any $m > 0$, where $\rho = R^{(\nu)}_0$.

Proof. We note that there exists a Bessel process $\{\hat{R}^{(2\nu)}_u, u \geqq 0\}$ with index 2ν starting from $2\sqrt{\rho}$ such that

$$2\Big(R^{(\nu)}_u\Big)^{1/2} = \hat{R}^{(2\nu)}_{\int_0^u (R^{(\nu)}_s)^{-1} ds}$$

(cf. [34], Chapter XI, Proposition (1.11), p. 446). Therefore, setting

$$\hat{T}_{\alpha,\beta} = \inf\Big\{u; \hat{R}^{(2\nu)}_u = 2\sqrt{\alpha + \beta u}\Big\},$$

we obtain

$$R^{(\nu)}_{T^{(\nu)}_{\alpha,\beta}} = \frac{1}{4}\Big(\hat{R}^{(2\nu)}_{\hat{T}_{\alpha,\beta}}\Big)^2.$$

Moreover, thanks to the Brownian scaling property of Bessel processes, there exists a Bessel process $\{\tilde{R}^{(2\nu)}_u, u \geqq 0\}$ with index 2ν starting from $2\sqrt{|\beta|\rho/\alpha}$ such that

$$\hat{R}^{(2\nu)}_{\hat{T}_{\alpha,\beta}} = \begin{cases} \dfrac{\alpha}{\beta} \tilde{R}^{(2\nu)}_{\tilde{T}^{(2\nu,+)}_{2\sqrt{\beta}}} = 2\sqrt{\alpha}\sqrt{1 + \tilde{T}^{(2\nu,+)}_{2\sqrt{\beta}}}, & \text{if} \quad \beta > 0, \\ \dfrac{\alpha}{|\beta|} \tilde{R}^{(2\nu)}_{\tilde{T}^{(2\nu,-)}_{2\sqrt{|\beta|}}} = 2\sqrt{\alpha}\sqrt{1 - \tilde{T}^{(2\nu,-)}_{2\sqrt{|\beta|}}}, & \text{if} \quad \beta < 0, \end{cases}$$

where $\tilde{T}^{(2\nu,\pm)}_c$ is the first hitting times of the c-square root boundary for $\tilde{R}^{(2\nu)}$ defined above Theorem 6.3.

Now, by applying (6.5) and (6.6), we obtain (6.8). ∎

§7. Possible extensions and open questions

In this section we consider a one-dimensional standard Brownian motion $\{B_t, t \geqq 0\}$ and its filtration $\{\mathcal{B}_t, t \geqq 0\}$ for simplicity.

7.a. For any $\lambda \in \mathbf{R} \setminus \{0\}$, the stochastic process

$$\Big\{ \exp(-\lambda B_t) \int_0^t \exp(\lambda B_s) ds, t \geqq 0 \Big\}$$

has the same filtration $\{\mathcal{B}_t, t \geqq 0\}$ and is a Markov process with respect to this filtration. This result extends easily when λ is replaced by a vector $\vec{\lambda} = (\lambda_1, ...\lambda_k) \in \mathbf{R}^k$ for any $k \in \mathbf{N}$. More generally, the measure-valued process $\{\pi_t, t \geqq 0\}$ defined by

$$\pi_t(f) = \int_0^t f(B_t - B_s) ds, \qquad t \geqq 0,$$

is a measure-valued diffusion with respect to $\{\mathcal{B}_t, t \geqq 0\}$. It is the unique solution to the following equation:

$$\pi_t(f) = tf(0) + \int_0^t \pi_s(f') dB_s + \frac{1}{2} \int_0^t \pi_s(f'') ds$$

for any $f \in C_b^2$, where C_b^2 is the set of all C^2 functions on $\mathbf{R}$ with bounded derivatives of orders $\leqq 2$. Proofs of these assertions have been presented in Donati-Martin–Yor [10].

It is now natural, given our study of $Z^{(\mu)}$ in this paper, to consider the family of processes

$$\Big\{ \exp(-\lambda B_t) \int_0^t \exp(2\lambda B_s) ds, t \geqq 0 \Big\} \tag{7.1}$$

which are parametrized by $\lambda \in \mathbf{R} \setminus \{0\}$. For each λ, we know that the process (7.1) is a Markov process with respect to its own filtration $\{{}^{(\lambda)}\mathcal{Z}_t, t \geqq 0\}$ as a consequence of the Brownian scaling and our result for $Z^{(0)}$.

Then the following appears to be a natural question.

Question 1. *Replace λ by $\vec{\lambda} = (\lambda_1, ..., \lambda_k) \in \mathbf{R}^k$ for any $k \in \mathbf{N}$ or consider the measure-valued process $\hat{\pi} = \{\hat{\pi}_t, t \geqq 0\}$ defined by*

$$\hat{\pi}_t(f) = \int_0^t f(B_t - 2B_s) ds, \quad t \geqq 0, \quad f \in C_b^2.$$

Then, is there a finite dimensional extension of the above mentioned results? Is $\hat{\pi}$ a diffusion process with respect to its natural filtration $(\Pi_t,\ t \geqq 0)$ *which turns out to be* $\{\mathcal{B}_t, t \geqq 0\}$?

In fact, the answer to this question is negative, as Emery [14] remarked the following: elementary Laplace method arguments show that both $\{2S_t - B_t, t \geqq 0\}$ and $\{B_t - 2I_t, t \geqq 0\}$, where $S_t = \sup_{s \leqq t} B_s$ and $I_t = \inf_{s \leqq t} B_s$, are adapted to $\hat{\Pi}_t \equiv \sigma\{\hat{\pi}_s; s \leqq t\}$ and that, as a consequence, $\hat{\Pi}_t = \mathcal{B}_t$. This rules out the Markov property of $\{\hat{\pi}_t, t \geqq 0\}$. Nonetheless, we feel that some investigations about this process are worth while.

7.b. We know that the filtration $\{\mathcal{G}_t^{(c)}\}_{t \geqq 0}$ of the stochastic process

$$\Big\{ \exp(-B_t) \int_0^t \exp(cB_s) ds, t \geqq 0 \Big\}$$

is equal to that of the original Brownian motion $\{\mathcal{B}_t\}$ if $c = 1$, but $\{\mathcal{G}_t^{(2)}\}$ is strictly contained in $\{\mathcal{B}_t\}$. By analogy with the result of Emery–Perkins [15], it is natural to ask the following.

Question 2. *Is* $\{\mathcal{G}_t^{(c)}, t \geqq 0\}$ *equal to* $\{\mathcal{B}_t, t \geqq 0\}$ *for any* $c > 0, c \neq 2$?

We conjecture that the answer is yes; here is a partial confirmation.

Proposition 7.1. ([17]) *For* $c < 3/2$, *one has* $\mathcal{G}_t^{(c)} = \mathcal{B}_t$.

Proof. We set

$$A_{(c),t} = \int_0^t \exp(cB_s) ds \qquad \text{and} \qquad Z_{(c),t} = \exp(-B_t) \int_0^t \exp(cB_s) ds.$$

Then we have

$$\frac{dA_{(c),t}}{(A_{(c),t})^c} = \frac{dt}{(Z_{(c),t})^c}.$$

Therefore, if $0 < c < 1$, we get

$$\frac{1}{1-c} (A_{(c),t})^{1-c} = \int_0^t \frac{ds}{(Z_{(c),s})^c}. \tag{7.2}$$

If $c = 1$, it is easy to show

$$\log A_{(c),t} = \log A_{(c),\varepsilon} + \int_\varepsilon^t \frac{ds}{Z_{(c),s}} = \lim_{\varepsilon \downarrow 0} \Big\{ \int_\varepsilon^t \frac{ds}{Z_{(c),s}} - \log \frac{1}{\varepsilon} \Big\}. \tag{7.3}$$

For the case $1 < c < 3/2$, we note

$$\lim_{\varepsilon \downarrow 0} (Z_{(c),\varepsilon})^{1-c} (\exp((c-1)B_\varepsilon) - 1) = 0, \qquad \text{a.s.}$$

Then we obtain

$$(A_{(c),t})^{1-c} = (A_{(c),\varepsilon})^{1-c} - (c-1) \int_\varepsilon^t \frac{ds}{(Z_{(c),s})^c} = \lim_{\varepsilon \downarrow 0} \Big\{ (Z_{(c),\varepsilon})^{1-c} - (c-1) \int_\varepsilon^t \frac{ds}{(Z_{(c),s})^c} \Big\}. \tag{7.4}$$

Now the proposition follows from (7.2)–(7.4). ∎

7.c. We now recall the works of Rogers [35] and Fitzsimmons [16]. Roughly speaking, they have shown the following: let $X = \{X_t, t \geqq 0\}$ be a diffusion process and assume that $\max_{s \leqq t} X_s - X_t$ or $2(\max_{s \leqq t} X_s) - X_t$ defines a diffusion. Then X must be essentially a Brownian motion with constant drift.

Concerning our diffusion process $Z^{(\mu)}$, the following question is now natural.

Question 3. *Can we characterize the* $\mathbf{R}$*-valued diffusion processes* $X = \{X_t, t \geqq 0\}$ *such that*

$$\Big\{ \exp(-X_t) \int_0^t \exp(2X_s) ds, t \geqq 0 \Big\}$$

is again a diffusion?

7.d. Concerning Theorem 3.4, we may ask the following which should be closely related to problems involving time inversion.

Question 4. *Let* $\{X_t, t \geqq 0\}$ *be a* $\mathbf{R}$*-valued diffusion process. Then* $\{tX_{1/t}, t > 0\}$ *is, quite generally, a time-inhomogeneous diffusion. Can we determine explicitly this diffusion?*

Question 5. *Can we characterize the diffusion processes X such that*

$$\left\{\frac{1}{X_t} = Y_{\int_t^\infty X_u^{-2}du}, t \geqq 0\right\}$$

for some diffusion $\{Y_t, t \geqq 0\}$?

7.e. The next question is somewhat vague. Pitman [30] and Rogers–Pitman [36] dealt with $2(\max X) - X$, whereas, in this paper, we deal with $\log(\int \exp(2X)ds) - X$.

Question 6. *Are there other such "natural" transforms of a Brownian motion for which the transformed processes are diffusions with respect to their own filtrations, which are strict subfiltrations of the original Brownian filtration?*

For example, we recall (see [49], Chapter 17) that

$$\hat{B}_t = B_t - \int_0^t \frac{ds}{B_s}, \qquad t \geqq 0,$$

generates a strictly smaller filtration than that of $\{B_t\}$, where we take the Hilbert principal value for the integral on the right hand side. But, we do not know whether $\{\hat{B}_t\}$ is a diffusion in its own filtration.

7.f. Most likely, the stochastic process

$$L_t^{(\lambda)} \equiv \log\Big(\int_0^t \exp(\lambda B_s)ds\Big) - B_t, \quad t > 0,$$

is not a diffusion for $\lambda \neq 1, 2$; this is similar to

$$\Sigma^{(\lambda)} \equiv \{\lambda M_t - B_t, t \geqq 0\} \overset{\text{(law)}}{=} \{(\lambda - 1)L_t + |B_t|, t \geqq 0\}.$$

Nonetheless, for $\lambda > 1$, the local times of $\Sigma^{(\lambda)}$, at time ∞, are distributed as a BES squared process with dimension $\delta = 2/(\lambda - 1)$ (see Le Gall–Yor [23], Yor [48]); we presented the case $\lambda = 3$ as Theorem 1.3 above. Thus we may ask the following question concerning the local times of $\{L_t^{(\lambda)}\}$.

Question 7. *Is the family of local times of $\{L_t^{(\lambda)}\}$, taken at time $t = \infty$, Markov?*

Appendix. Some useful facts about the Bessel functions

We recall for the reader's convenience the definitions and some formulae for the Bessel and the modified Bessel functions J_ν, I_ν and K_ν which have been used in the present paper. In the following we assume that the variable z is not negative. For details we refer to Lebedev [20] and Watson [46].

(1) Definitions and Differential Equations. The Bessel function J_ν of the first kind of order ν is defined by

$$J_\nu(z) = \Big(\frac{z}{2}\Big)^\nu \sum_{k=1}^\infty \frac{(-1)^k (z/2)^{2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}$$

and it solves the second order linear differential equation

$$u'' + \frac{1}{z}u' + \Big(1 - \frac{\nu^2}{z^2}\Big)u = 0.$$

The modified Bessel functions I_ν and K_ν (K_ν is also called the Macdonald function) are given by

$$I_\nu(z) = e^{-\nu\pi\sqrt{-1}/2} J_\nu(e^{\pi\sqrt{-1}/2}z) = \Big(\frac{z}{2}\Big)^\nu \sum_{k=0}^\infty \frac{(z/2)^{2k}}{k!\Gamma(k+\nu+1)}, \quad \nu \in \mathbf{R},$$

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi}, \quad \nu \notin \mathbf{Z},$$

and

$$K_n(z) = \lim_{\nu\to n} K_\nu(z), \quad n \in \mathbf{Z}.$$

I_ν and K_ν are solutions of

$$u'' + \frac{1}{z}u' - \Big(1 + \frac{\nu^2}{z^2}\Big)u = 0.$$

(2) Recurrence relations.

$$\frac{d}{dz}(z^\nu I_\nu(z)) = z^\nu I_{\nu-1}(z), \qquad \frac{d}{dz}(z^\nu K_\nu(z)) = -z^\nu K_{\nu-1}(z),$$

$$\frac{d}{dz}(z^{-\nu} I_\nu(z)) = z^{-\nu} I_{\nu+1}(z), \qquad \frac{d}{dz}(z^{-\nu} K_\nu(z)) = -z^{-\nu} K_{\nu+1}(z),$$

$$I_{\nu-1}(z) - I_{\nu+1}(z) = \frac{2\nu}{z} I_\nu(z), \qquad K_{\nu-1}(z) - K_{\nu+1}(z) = -\frac{2\nu}{z} K_\nu(z).$$

(3) Indices with opposite sign. For $n \in \mathbf{Z}$ and $\nu \in \mathbf{R}$,

$$I_{-n}(z) = I_n(z), \qquad K_{-\nu}(z) = K_\nu(z).$$

(4) Explicit formulae for Modified Bessel functions of half-integer order.

$$\begin{aligned}
&I_{1/2}(z) = \sqrt{2/\pi z}\sinh z, \qquad I_{3/2}(z) = \sqrt{2/\pi z}(\cosh z - z^{-1}\sinh z),\\
&I_{5/2}(z) = \sqrt{2/\pi z}((1+3z^{-2})\sinh z - 3z^{-1}\cosh z),\\
&I_{-1/2}(z) = \sqrt{2/\pi z}\cosh z, \qquad I_{-3/2}(z) = \sqrt{2/\pi z}(\sinh z - z^{-1}\cosh z),\\
&I_{-5/2}(z) = \sqrt{2/\pi z}((1+3z^{-2})\cosh z - 3z^{-1}\sinh z),\\
&K_{1/2}(z) = \sqrt{\pi/2z}e^{-z}, \qquad K_{3/2}(z) = \sqrt{\pi/2z}(1+z^{-1})e^{-z},\\
&K_{5/2}(z) = \sqrt{\pi/2z}(1+3z^{-1}+3z^{-2})e^{-z}.
\end{aligned}$$

(5) Wronskians. $W(f,g) \equiv fg' - f'g$.

$$\begin{aligned}
&W(K_\nu(z), I_\nu(z)) = I_\nu(z)K_{\nu+1}(z) - I_{\nu+1}(z)K_\nu(z) = 1/z,\\
&W(I_\nu(z), I_{-\nu}(z)) = I_\nu(z)I_{-(\nu+1)}(z) - I_{\nu+1}(z)I_{-\nu}(z) = -2/\Gamma(\nu)\Gamma(1-2\nu)z.
\end{aligned}$$

(6) Two integral representations.

$$\begin{aligned}
K_\nu(z) &= \frac{1}{2}\Big(\frac{z}{2}\Big)^\nu \int_0^\infty e^{-t-z^2/4t}t^{-\nu-1}dt\\
&= \frac{2^\nu\Gamma(\nu+1/2)}{z^\nu\sqrt{\pi}}\int_0^\infty \frac{\cos zt}{(1+t^2)^{\nu+1/2}}dt \qquad (\operatorname{Re}\nu > -1/2).
\end{aligned}$$

(7) Asymptotics as $z \downarrow 0$.

$$\begin{aligned}
&I_\nu(z) = \frac{z^\nu}{2^\nu\Gamma(1+\nu)}(1+o(1)),\\
&K_\nu(z) = \frac{2^{\nu-1}\Gamma(\nu)}{z^\nu}(1+o(1)),\ \nu > 0, \qquad K_0(z) = \Big(\log\frac{2}{z}\Big)(1+o(1)).
\end{aligned}$$

(8) Asymptotics as $z \uparrow \infty$.

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}}(1+O(z^{-1})), \qquad K_\nu(z) = \Big(\frac{z}{2\pi}\Big)^{1/2}e^{-z}(1+O(z^{-1})).$$

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Hiroyuki Matsumoto
School of Informatics and Sciences
Nagoya University
Chikusa-ku, Nagoya 464-8601
Japan
FAX: 052-789-4800
matsu@info.human.nagoya-u.ac.jp

Marc Yor
Laboratoire de Probabilités
Université Pierre et Marie Curie
4, Place Jussieu, Tour 56, F-75252 Paris Cedex 05
France
FAX: +33 1 44 27 72 23