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REMARKS ON THE YABLONSKII-VOROB'EV POLYNOMIALS

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Abstract. We study the Yablonskii-Vorob'ev polynomial associated with the second Painlevé equation. To study other special polynomials (Okamoto polynomials, Umemura polynomials) associated with the Painlevé equations, our purely algebraic approach is useful.

Introduction

For a non-negative integer n, let P_n be the rational functions of a variable t determined by the following recurrence relation

$$P_{n+1} = \frac{tP_n^2 - 4(P_n P_n'' - {P_n'}^2)}{P_{n-1}} \tag{1}$$

with initial conditions $P_0 = 1$, $P_1 = t$. Vorob'ev proved the following

PROPOSITION 1. *For every non-negative integer n, P_n is a polynomial.*

The $\{P_n\}$ are called the Yablonskii-Vorob'ev polynomials. In Section 1, we give a proof of Proposition 1 close to the one given by Fukutani, Okamoto and Umemura. (See Fukutani, Okamoto and Umemura [2], Proposition 9.) In the proof of Proposition 1, we show together the following lemmas.

LEMMA 1. *For a non-negative integer n, roots of the algebraic equation $P_n = 0$ are simple.* (See Fukutani, Okamoto and Umemura [2], Proposition 9.)

LEMMA 2. *For a positive integer n, $P_n = 0$, $P_{n-1} = 0$ do not have a common root.* (See Fukutani, Okamoto and Umemura [2], Proposition 9.)

Moreover we prove the following

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Proposition 2. *P_n is divisible by t if and only if $n \equiv 1 \pmod 3$. P_n is a polynomial of t^3 if $n \not\equiv 1 \pmod 3$ and P_n/t is a polynomial of t^3 if $n \equiv 1 \pmod 3$.*

We know that the $\{P_n\}$ satisfy the two Hirota bilinear relations. (See Fukutani, Okamoto and Umemura [2], Definition 3). In Section 2, using one of the Hirota bilinear relations, we prove the following

Theorem 1. *If $n \equiv 1 \pmod 3$, the coefficients of t^4 of the polynomial P_n is equal to 0.*

Kajiwara and Ohta [3] proved the following

Theorem 2.

$$P_n = \left(-\frac{4}{3}\right)^{n(n+1)/6} \left\{\prod_{k=1}^{n}(2k-1)!!\right\} \times \chi_{(n,n-1,\dots,1)}\left(\left(-\frac{3}{4}\right)^{1/3} t, 0, 1, 0, 0, \dots\right), \tag{2}$$

where χ_λ is the Schur polynomial for a partition λ.

In Section 4, we give another proof of Theorem 2 as well as by Noumi and Yamada [5]. Namely we check that the right hand side satisfies the recurrence relation (18). Moreover we show that the Hirota bilinear relation (23) follows from a Plücker relation.

§1. The second Painlevé equation

In this section we review how the Yablonskii-Vorob'ev Polynomials arise from the second Painlevé equation. For detail see Okamoto [6]. By the second Painlevé equation, we mean the differential equation

$$y'' = 2y^3 + ty + \alpha, \tag{3}$$

where t is the independent variable and α is a parameter. The second Painlevé equation is equivalent to the Hamiltonian system

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{\partial H}{\partial z} = z - y^2 - \dfrac{t}{2}, \\[2mm] \dfrac{dz}{dt} = -\dfrac{\partial H}{\partial y} = 2yz + \alpha + \dfrac{1}{2}, \end{cases} \tag{4}$$

where the Hamiltonian H is given by

$$H(\alpha, y, z) = \frac{1}{2}z^2 - \left(y^2 + \frac{1}{2}t\right) z - \left(\alpha + \frac{1}{2}\right) y. \tag{5}$$

For a solution $(y(t), z(t))$ of the Hamiltonian system (4), we have

$$\frac{d}{dt}H(\alpha, y(t), z(t)) = \left.\frac{\partial H(\alpha, y, z)}{\partial t}\right|_{y=y(t), z=z(t)} = -\frac{1}{2}z(t), \tag{6}$$

which we denote by $H'(\alpha, y, z)$.

We denote the set of solutions of the Hamiltonian system (4) for a parameter α by $\Sigma(\alpha)$.

We define a transformation $I^\alpha : \Sigma(\alpha) \longrightarrow \Sigma(-\alpha - 1))$ by

$$I^\alpha(y, z) = \begin{cases} \left(y + \dfrac{\alpha + 1/2}{z},\ z\right), & \text{if } \alpha \neq -\dfrac{1}{2}, \\ (y, z), & \text{if } \alpha = -\dfrac{1}{2}. \end{cases} \tag{7}$$

If $z = 0$ then $\alpha = -\frac{1}{2}$. So, the denominator in (7) is not equal to 0. Similarly, we note that the denominators in the following definitions is not equal to 0. We define a transformation $T_-^\alpha : \Sigma(\alpha) \longrightarrow \Sigma(\alpha - 1)$ by

$$T_-^\alpha(y, z) = \begin{cases} \left(-y - \dfrac{\alpha - 1/2}{2y^2 - z + t},\ 2y^2 - z + t\right), & \text{if } \alpha \neq \dfrac{1}{2}, \\ (-y,\ 2y^2 - z + t), & \text{if } \alpha = \dfrac{1}{2} \end{cases} \tag{8}$$

and a transformation $T_+^\alpha : \Sigma(\alpha) \longrightarrow \Sigma(\alpha + 1)$ by

$$T_+^\alpha(y, z) = \begin{cases} \left(-y - \dfrac{\alpha + 1/2}{z},\ 2\left(y + \dfrac{\alpha + 1/2}{z}\right)^2 - z + t\right), & \text{if } \alpha \neq -\dfrac{1}{2}, \\ \left(-y,\ 2y^2 - z + t\right), & \text{if } \alpha = -\dfrac{1}{2}. \end{cases} \tag{9}$$

We note that $T_+^{\alpha-1} \circ T_-^\alpha = \mathrm{id}_{\Sigma(\alpha)}$ and $T_-^{\alpha+1} \circ T_+^\alpha = \mathrm{id}_{\Sigma(\alpha)}$.

Now, for $\gamma \in \mathbf{C}$ and an integer $n \geq 0$, we define $(y_{\gamma-n}, z_{\gamma-n})$ by the recurrence relation

$$(y_{\gamma-n}, z_{\gamma-n}) = T_-^{\gamma-(n-1)}(y_{\gamma-(n-1)}, z_{\gamma-(n-1)}). \tag{10}$$

For $\beta \in \gamma + \mathbf{Z}$, we set $h(\beta) = H(\beta, y_\beta, z_\beta)$. If $\gamma \notin 1/2 + \mathbf{Z}$, then we have by definition

$$\begin{aligned} h(\beta - 1) &= H(\beta - 1, y_{\beta-1}, z_{\beta-1}) \\ &= \frac{1}{2} z_\beta^2 - \left(y_\beta^2 + \frac{1}{2} t \right) z_\beta - \left(\beta + \frac{1}{2} \right) y_\beta + y_\beta \\ &= h(\beta) + y_\beta. \end{aligned}$$

Namely, we have

$$y_\beta = h(\beta - 1) - h(\beta). \tag{11}$$

By the Hamiltonian system (4) and (11)

$$\begin{aligned} \frac{z'_\beta}{z_\beta} &= 2y_\beta + \frac{\beta + 1/2}{z_\beta} \\ &= y_\beta - y_{\beta+1} \\ &= h(\beta - 1) - 2h(\beta) + h(\beta + 1). \end{aligned} \tag{12}$$

We here introduce the so-called τ function by

$$\frac{d}{dt} \log \tau(\beta) = h(\beta) \tag{13}$$

so that

$$-2 \frac{d^2}{dt^2} \log \tau(\beta) = c \frac{\tau(\beta - 1)\tau(\beta + 1)}{\tau^2(\beta)} \tag{14}$$

by (6) (12) and (13), where c is a constant. The relation (14) is called the Toda equation. The second Painlevé equation has a rational solution if and only if α is an integer. For detail see Umemura and Watanabe [9]. It is easy to see that for $\alpha = 0$ the Hamiltonian system (4) has a unique rational solution $(0, t/2)$. Hence, if we put $\gamma = 0$ and choose (y_0, z_0) as $(0, t/2)$, then we immediately obtain

$$h(0) = -\frac{1}{8} t^2, \tag{15}$$

$$\tau(0) = A_0 \exp\left(-\frac{1}{24} t^3 \right), \tag{16}$$

$$\tau(-1) = A_{-1} \exp\left(-\frac{1}{24} t^3 \right), \tag{17}$$

A_0, A_{-1} being constants. We define a function $P_n(t)$ by

$$\tau(-n-1) = A_{-n-1}P_n(t)\exp\left(-\frac{1}{24}t^3\right), \tag{18}$$

for a non-negative integer n, where A_n is a constant. So we have $P_{-1}(t) = P_0(t) = 1$. Substituting (18) into the Toda equation (14), we find that

$$-\frac{2}{c}\left(-\frac{t}{4} + \frac{P_nP_n'' - P_n'^2}{P_n^2}\right) = \frac{A_{-n}A_{-n-2}}{A_{-n-1}^2}\frac{P_{n-1}P_{n+1}}{P_n^2}. \tag{19}$$

Setting $A_n = 1$ and $c = 1/2$ in the above formula (19), we have

$$P_{n+1} = \frac{tP_n^2 - 4(P_nP_n'' - {P_n'}^2)}{P_{n-1}}.$$

This is just the recurrence relation (1) satisfied by the Yablonskii-Vorob'ev polynomials.

We know that the τ function is an entire function. For detail see Okamoto [6]. Admitting this fact, we easily see from (18) that P_n is a polynomial. We here make a remark that a rational solution (y_{-n-1}, z_{-n-1}) of the Hamiltonian system (4) is represented by the following formulas

$$y_{-n-1} = h(-n-2) - h(-n-1) = \frac{d}{dt}\log\frac{P_{n+1}}{P_n} \tag{20}$$

by (11), (13) and (18), and

$$z_{-n-1} = \frac{\tau(-n-2)\tau(-n)}{2\tau^2(-n-1)} = \frac{P_{n-1}P_{n+1}}{2P_n^2} \tag{21}$$

by (6), (13), (14) and (18).

Now we review the Hirota bilinear relation. For detail see Fukutani, Okamoto and Umemura [2], Definition 3.

From the Hamiltonian system (4), we have

(22)

$$\begin{aligned} z_{-n-1} &= y'_{-n-1} + y^2_{-n-1} + \frac{1}{2} \\ &= \frac{P_{n+1}(P_n^2 - 4P_nP_n'' + 4P_n'^2) + 2P_n(P_{n+1}P_n'' + P_{n+1}''P_n - 2P_{n+1}'P_n')}{2P_{n+1}P_n^2}. \end{aligned}$$

Combining (22) with (21), we have

$$P_{n+1}P_n'' + P_{n+1}''P_n - 2P_{n+1}'P_n' = 0. \tag{23}$$

This equation is one of the Hirota bilinear relation satisfied by P_{n+1} and P_n. (See Fukutani, Okamoto and Umemura [2], Proposition 9.)

Substituting the equation (20) and (21) into the Hamiltonian system (4) written by the following

$$\frac{d}{dt}z_{-n-1} = 2\ y_{-n-1}\ z_{-n-1} - n - \frac{1}{2},$$

we have

$$\frac{P_{n+1}'P_{n-1}}{2P_n^2} + \frac{P_{n+1}P_{n-1}'}{2P_n^2} - \frac{P_{n+1}P_{n-1}P_n'}{P_n^3}$$
$$= \frac{P_{n+1}'P_{n-1}}{P_n^2} - \frac{P_{n+1}P_{n-1}P_n'}{P_n^3} - n - \frac{1}{2}.$$

Hence we obtain

$$P_{n+1}'P_{n-1} - P_{n+1}P_{n-1}' = (2n+1)P_n^2. \tag{24}$$

§2. Proofs of Proposition 1 and Proposition 2

We define the operator l_t by

$$l_t(f) = f\ \frac{df^2}{dt^2} - \left(\frac{df}{dt}\right)^2, \tag{25}$$

$$l_t(f,g) = f\ \left(\frac{d^2g}{dt^2}\right) - \left(\frac{df}{dt}\right)\left(\frac{dg}{dt}\right) + \left(\frac{d^2g}{dt^2}\right)\ g, \tag{26}$$

for the functions f, g of a variable t. We then have the following formulas

$$l_t(cf) = c^2 f, \tag{27}$$

$$l_t(fg) = f^2\ l_t(g) + g^2\ l_t(f), \tag{28}$$

$$l_t(f+g) = l_t(f) + l_t(f,g) + l_t(g), \tag{29}$$

$$l_t(t) = -1, \tag{30}$$

$$l_t(t^3 + c) = -3t(t^3 - 2c), \tag{31}$$

for a constant c. We here note that the recurrence relation (1) is written as

$$P_{n+1} = \frac{tP_n^2 - 4\, l_t(P_n)}{P_{n-1}}. \tag{32}$$

We shall prove Proposition 1, Lemma 1 and Lemma 2 together by mathematical induction on n. As we have $P_0 = 1$, $P_1 = t$, $P_2 = t^3 + 4$ and $P_3 = t^6 + 20t^3 - 80$. Proposition 1 and Lemma 1 hold for $0 \le n \le 3$ and Lemma 2 holds for $1 \le n \le 3$. We now make the following

Assumption 1. *If $3 \le n \le N$, then P_n is a polynomial, roots of an algebraic equation $P_n = 0$ are simple and $P_n = 0$ and $P_{n-1} = 0$ have not a common root.*

We have to show Proposition 1, Lemma 1 and Lemma 2 for $n = N+1$. Let f be an arbitrary polynomial and let $' = \frac{d}{dt}$. Setting $h = tf^2 - 4\, l_t(f) = tf^2 - 4(ff'' - f'^2)$, we have

$$h = 4f'^2 + f \times (\text{a polynomial}), \tag{33}$$

$$\begin{aligned} h' &= f^2 + 2tff' - 4\left(ff''' - f'f''\right) \\ &= 4f'f'' + f \times (\text{a polynomial}), \\ h'' &= 4ff' + 2tf'^2 + 2tff'' - 4\left(ff'''' - f''^2\right) \\ &= 2tf'^2 + 4f''^2 + f \times (\text{a polynomial}). \end{aligned} \tag{34}$$

Then we can see

$$\begin{aligned} l_t(h) &= hh'' - h'^2 \\ &= 8tf'^4 + 16f'^2f''^2 - 16f'^2f''^2 + f \times (\text{a polynomial}) \\ &= 8tf'^4 + f \times (\text{a polynomial}), \end{aligned} \tag{35}$$

$$\begin{aligned} 2th^2 - 4\, l_t(h) &= 32tf'^4 - 32tf'^4 + f \times (\text{a polynomial}) \\ &= f \times (\text{a polynomial}). \end{aligned} \tag{36}$$

Hence we have

$$f \;\mid\; 2th^2 - 4\, l_t(h). \tag{37}$$

Here the symbol | means that the right hand side is divisible by the left hand side. Now, replacing f by P_{N-1}, we have $h = P_{N-2}P_N$ and

$$P_{N-1} \ \mid \ 2tP_{N-2}^2P_N^2 - 4\, l_t(P_{N-2}P_N). \tag{38}$$

By (28), we obtain that

$$\begin{aligned} &2tP_{N-2}^2P_N^2 - 4\, l_t(P_{N-2}P_N) \\ &= P_{N-2}^2\left\{tP_N^2 - 4\, l_t(P_N)\right\} + \ P_N^2\left\{tP_{N-2}^2 - 4\, l_t(P_{N-2})\right\} \\ &= P_{N-2}^2\left\{tP_N^2 - 4\, l_t(P_N)\right\} + P_N^2P_{N-3}P_{N-1}. \end{aligned} \tag{39}$$

Hence, we see

$$P_{N-1} \ \mid \ tP_N^2 - 4\, l_t(P_N). \tag{40}$$

Combining this result with (1), we can conclude that P_{N+1} is a polynomial. If $P_N = 0$ and $P_{N+1} = 0$ have a common root r, then $P_N'(r) = 0$ by (1). This contradicts Assumption 1. So, $P_N = 0$ and $P_{N+1} = 0$ have not a common root. If a root r of $P_{N+1} = 0$ is not simple, then $P_N(r) = 0$ by (24), a contradiction! We hence verified that roots of $P_{N+1} = 0$ are simple. Consequently, we have completed mathematical induction and hence proved Proposition 1, Lemma 1 and Lemma 2.

Now, we prove Proposition 2. The following simple proof was proposed by H. Kawamuko during a discussion about our original proof. Let ω be a primitive cube root of 1. In order to prove Proposition 2 we show by mathematical induction on n the following

$$P_n(\omega t) = \begin{cases} P_n(t), & \text{if } n \not\equiv 1 \pmod 3, \\ \omega P_n(t), & \text{if } n \equiv 1 \pmod 3, \end{cases} \tag{41}$$

for a non-negative integer n. As we have $P_0 = 1$ and $P_1 = t$. The equation (41) hold for $n = 0, 1$. Suppose that the equation (41) is proved for all $n \le N$, $N \ge 1$. Then we have to show the equation (41) for $n = N + 1$. Assume first that $N \equiv 1 \pmod 3$. By induction hypothesis, then, we see $P_{N-1}(\omega t) = P_{N-1}(t)$ and $P_N(\omega t) = \omega P_N(t)$. So we have $P_N'(\omega t) = P_N'(t)$ and $P_N''(\omega t) = \frac{1}{\omega}P_N''(t) = \omega^2 P_N''(t)$. Then, replacing t by ωt in the recurrence relation (18), we have

$$P_{N+1}(\omega t) = \frac{\omega t P_N^2(\omega t) - 4(P_N(\omega t)P_N''(\omega t) - P_N'(\omega t)^2)}{P_{N-1}(\omega t)}$$
$$= P_{N+1}(t).$$

Hence we have verified the equation (41) for $n = N+1$, $N \equiv 1 \pmod 3$.

Next, if $N \equiv 2 \pmod 3$ then we have $P_{N-1}(\omega t) = \omega P_{N-1}(t)$, and $P_N(\omega t) = P_N(t)$ by induction hypothesis. So we can see $P_N'(\omega t) = \omega^2 P_N'(t)$ and $P_N''(\omega t) = \omega P_N''(t)$. Hence, replacing t by ωt in the recurrence relation (18), we have $P_{N+1}(\omega t) = P_{N+1}(t)$, which proved the equation (41) for $n = N+1$, $N \equiv 2 \pmod 3$.

Next, if $N \equiv 0 \pmod 3$ then we have $P_{N-1}(\omega t) = P_{N-1}(t)$, $P_N(\omega t) = P_N(t)$, $P_N'(\omega t) = \omega^2 P_N'(t)$ and $P_N''(\omega t) = \omega P_N''(t)$ by induction hypothesis. Hence, replacing t by ωt in the recurrence relation (18), we have $P_{N+1}(\omega t) = \omega P_{N+1}(t)$, which proved the equation (41) for $n = N+1$, $N \equiv 0 \pmod 3$.

With these result, we have verified the equation (41) for $n = N+1$ and hence obtained the equation (41) by mathematical induction. Therefore, combining the equation (41) with Proposition 1, we have Proposition 2.

§3. Proof of Theorem 1

To illustrate Theorem 1, we have

$$P_4 = t^{10} + 60t^7 + 11200t,$$

$$P_7 = t^{28} + 504t^{25} + 75600t^{22} + 5174400t^{19} + 62092800t^{16} + 13039488000t^{13} - 828731904000t^{10} - 49723914240000t^7 - 3093932441600000t.$$

In order to prove Theorem 1, for a non-negative integer n, we define the rational function ${}^aP_n(t)$ by

$$(42) \qquad {}^aP_n(t) = \begin{cases} P_n(t), & \text{if } n \not\equiv 1 \pmod 3, \\ P_n(t)/t, & \text{if } n \equiv 1 \pmod 3. \end{cases}$$

and the rational function ${}^bP_n(v)$ of variable v by

$$(43) \qquad {}^bP_n(v) = {}^aP_n(t), v = t^3.$$

From Proposition 2, we have that ${}^aP_n(t)$ and ${}^bP_n(v)$ are polynomials for a non-negative integer n.

We shall prove Theorem 1 by mathematical induction. As we have $P_1(t) = t$ and see that Theorem 1 holds for $n = 1$. Let N be $N \equiv 1 \pmod{3}$. Suppose that Theorem 1 is proved for $n = N-3$. We have to show Theorem 1 for $n = N$. From (24), we have

$$(44)\quad {}^bP_N{}^bP_{N-2} + 3v({}^bP'_N{}^bP_{N-2} - {}^bP_N{}^bP'_{N-2}) = (2N-1){}^bP^2_{N-1},$$

$$(45)\quad -{}^bP_{N-1}{}^bP_{N-3} + 3v({}^bP'_{N-1}{}^bP_{N-3} - {}^bP_{N-1}{}^bP'_{N-3}) = (2N-3){}^bP^2_{N-2},$$

$$(46)\quad 3({}^bP'_{N-2}{}^bP_{N-4} - {}^bP_{N-2}{}^bP'_{N-4}) = (2N-5){}^bP^2_{N-3}.$$

Substituting $v = 0$ into (44), (45) and these derivation, we have

$$(47)\qquad {}^bP_N(0){}^bP_{N-2}(0) = (2N-1){}^bP^2_{N-1}(0),$$

$$(48)\qquad -{}^bP_{N-1}(0){}^bP_{N-3}(0) = (2N-3){}^bP^2_{N-2}(0),$$

$$(49)\qquad 2{}^bP'_N(0){}^bP_{N-2}(0) - {}^bP_N(0){}^bP'_{N-2}(0) = (2N-1){}^bP_{N-1}(0){}^bP'_{N-1}(0),$$

$$(50)\qquad {}^bP'_{N-1}(0){}^bP_{N-3}(0) - 2{}^bP_{N-1}(0){}^bP'_{N-3}(0) = (2N-3){}^bP_{N-2}(0){}^bP'_{N-2}.$$

Combining (47) with (49), we have

$$(51)\qquad {}^bP_N(0){}^bP'_{N-1}(0){}^bP_{N-2}(0) - 2{}^bP'_N(0){}^bP_{N-1}(0){}^bP_{N-2}(0) + {}^bP_N(0){}^bP_{N-1}(0){}^bP'_{N-2}(0) = 0.$$

Combining (48) with (50), we have

$$(52)\qquad -{}^bP_{N-1}(0){}^bP'_{N-2}(0){}^bP_{N-3}(0) - {}^bP'_{N-1}(0){}^bP_{N-2}(0){}^bP_{N-3}(0) + 2{}^bP_{N-1}(0){}^bP_{N-2}(0){}^bP'_{N-3}(0) = 0.$$

We have ${}^bP'_{N-3}(0) = 0$ by induction hypothesis and ${}^bP_{N-3}(0) \neq 0$ by Proposition 2. By (52), we hence see

$$(53)\qquad {}^bP_{N-1}(0){}^bP'_{N-2}(0) + {}^bP'_{N-1}(0){}^bP_{N-2}(0) = 0.$$

Combining (53) with (51), we have

$$ {}^bP'_N(0)\,{}^bP_{N-1}(0)\,{}^bP_{N-2}(0) = 0. \tag{54} $$

We see ${}^bP_{N-1}(0) \neq 0$ and ${}^bP_{N-2}(0) \neq 0$ since $P_N(0) = P_{N-3}(0) = 0$ and Lemma 2. Consequently we obtain

$$ {}^bP'_N(0) = 0. \tag{55} $$

We have completed mathematical induction and hence verified Theorem 1.

On the other hand, another proof of Theorem 1 can be carried out as follows: From (24), we have

$$ \frac{P'_{n+1}}{P_{n+1}} - \frac{P'_{n-1}}{P_{n-1}} = \frac{(2n+1)P_n^2}{P_{n+1}P_{n-1}}. \tag{56} $$

From the differential of (1), we have

$$ \frac{P'_{n+1}}{P_{n+1}} + \frac{P'_{n-1}}{P_{n-1}} = \frac{P_n^2 + 2tP_nP'_n - 4(P_nP'''_n - P'_nP''_n)}{P_{n+1}P_{n-1}}. \tag{57} $$

Combining (57) with (1) and (56), we have

$$ \frac{P'_{n+1}}{P_{n+1}} = \frac{(n+1)P_n^2 + tP_nP'_n - 2(P_nP'''_n - P'_nP''_n)}{t\,P_n^2 - 4(P_nP''_n - P'^2_n)}. \tag{58} $$

Substituting (58) into (20), we have

$$ y_{-n-1} = \frac{(n+1)P_n^2 + tP_nP'_n - 2(P_nP'''_n - P'_nP''_n)}{t\,P_n^2 - 4(P_nP''_n - P'^2_n)} - \frac{P'_n}{P_n}. \tag{59} $$

Substituting (59) into the second Painlevé equation (3), we find a differential equation satisfied by P_n. Similarly, we can make several differential equations satisfied by P_n. From the reduction of these differential equations, we conclude that P_n satisfies the following differential equations

$$ \begin{aligned} (60)\quad & 4y^{(1)2}\left(ty^{(1)2} - 4y^{(1)}y^{(3)} + 3y^{(2)2}\right) \\ & + 4y\left(-2ty^{(1)2}y^{(2)} - 2y^{(1)3} + 2y^{(1)2}y^{(4)} + 2y^{(1)}y^{(2)}y^{(3)} - 2y^{(2)3}\right) \\ & + 2y^2\left(t^2y^{(1)2} - 4ty^{(1)}y^{(3)} + 5y^{(2)2} + 3y^{(1)}y^{(2)} - 4y^{(2)}y^{(4)} + 2y^{(3)3}\right) \\ & + 2y^3\left(-2t^2y^{(2)} + ty^{(4)} - y^{(3)}\right) - n(n+1)y^4 = 0 \end{aligned} $$

and

$$2y^{(1)}\left(ty^{(1)2}-4y^{(1)}y^{(3)}+3y^{(2)2}\right) \tag{61}$$
$$+y\left(-3ty^{(1)}y^{(2)}-2y^{(1)2}+5y^{(1)}y^{(4)}-2y^{(2)}y^{(3)}\right)$$
$$+y^2\left(ty^{(3)}+2y^{(2)}-y^{(5)}\right)=0,$$

where $y^{(n)m}$ is defined by $y^{(n)m}=(d^ny/dt^n)^m$. From (60) and (61), we can verify that

$$P_n|t(P_n')^2-4P_n'P_n'''+3(P_n'')^2, \tag{62}$$

$$P_n|t(P_n')^2P_n''+(P_n')^2P_n''''-6P_n'P_n''P_n'''+4(P_n'')^3. \tag{63}$$

From the last formula, we obtain $P_n''''(0)=0$ if $P_n(0)=0$, which completes the proof of Theorem 1.

§4. Proof of Theorem 2

We review the Plücker relation and present useful relation. Let A be a commutative ring. We consider the free A-module

$$V_\infty=\{{}^t(v_1,v_2,v_3,\ldots)|v_i\in A \text{ for } i=1,2,\ldots\}. \tag{64}$$

We define

$$e_i={}^t(0,\ldots,0,\underset{i\text{-th place}}{1},0,\ldots). \tag{65}$$

For $v_j={}^t(v_{1j},v_{2j},v_{3j},\ldots)\in V_\infty$ $(j=1,2,3,\ldots,n)$, we define

$$|v_1\wedge v_2\wedge v_3\wedge\cdots\wedge v_n|=\det(v_{ij})_{i,j=1,2,3,\ldots,n} \tag{66}$$
$$=\det\begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1n}\\ v_{21} & v_{22} & \cdots & v_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ v_{n1} & v_{n2} & \cdots & v_{nn}\end{pmatrix}.$$

By the Plücker relation, we mean

$$\sum_{j=1}^{n+1}(-1)^j\left\{\left|v_1\wedge v_2\cdots\wedge v_{n-1}\wedge v_j'\right|\right. \tag{67}$$
$$\left.\times\left|v_1'\wedge\cdots\wedge v_{j-1}'\wedge v_{j+1}'\wedge\cdots\wedge v_{n+1}'\right|\right\}=0$$

where $v_1, v_2, \dots v_{n-1}, v'_1, \dots, v'_{n+1} \in V_\infty$. (For detail see Date, Jimbo and Miwa [1], p.70.) Now, for $v = {}^t(v_1, v_2, v_3, \dots) \in V_\infty$, we set $v^+ = {}^t(v_2, v_3, v_4, \dots)$. For positive integers n, $m \le n$, we define

$$(68) \qquad \Sigma_m^n = \{(\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbf{Z}^m | 1 \le \lambda_1 < \lambda_2 < \cdots < \lambda_m \le n\}.$$

For positive integers n, m, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \in \Sigma_m^n$ and $v_i \in V_\infty$ $(i = 1, 2, \dots, n)$, we give $D_\lambda(v_1, v_2, \dots, v_n)$ by

$$(69) \qquad D_\lambda(v_1, v_2, \dots, v_n) = \left| v_1 \wedge v_2 \wedge v_3 \wedge \cdots \wedge v_{\lambda_1}^+ \wedge \cdots \wedge v_{\lambda_2}^+ \wedge \cdots \wedge v_{\lambda_m}^+ \wedge \cdots \wedge v_n \right|.$$

We then have

$$(70) \qquad \sum_{\lambda \in \Sigma_m^n} D_\lambda(v_1, v_2, \dots, v_n) = (-1)^m \left| v_1 \wedge v_2 \wedge v_3 \wedge \cdots \wedge v_n \wedge e_{n+1-m} \right|.$$

We here give our original proof. H. Kawamuko tought us another simple proof of (70). See the appendix in this paper for detail.

First, we prove the equation (70) for $m = 1$ by mathematical induction on n. As we can see $|v_1^+| = (-1)|v_1 \wedge e_1|$. The equation (70) holds for $m = n = 1$. Suppose that the equation (70) is verified for $m = 1$ and $n \le N - 1$. We shall prove the equation (70) for $m = 1$, $n = N$. From the Laplace expansion of N-th row vector, for a positive integer $j = 1, 2, \dots, N$, we have

$$(71) \qquad \begin{aligned} D_{(j)}(v_1, v_2, \dots, v_N) &= \sum_{k=1}^{j-1} (-1)^{N+k}\, v_{Nk}\, D_{(j-1)}(v_1, v_2, \dots, v_{k-1}, v_{k+1}, \dots, v_N) \\ &\quad + (-1)^{N+j} v_{n+1\ j}\ |v_1 \wedge v_2 \wedge \cdots \wedge v_{j-1} \wedge v_{j+1} \wedge \cdots \wedge v_N| \\ &\quad + \sum_{k=j+1}^{N} (-1)^{N+k}\, v_{Nk}\, D_{(j)}(v_1, v_2, \dots, v_{k-1}, v_{k+1}, \dots, v_N). \end{aligned}$$

So, we have

$$
\begin{aligned}
(72)\quad & \sum_{j=1}^{N} D_{(j)}(v_1, v_2, \dots, v_N) \\
& = \sum_{k=1}^{N} (-1)^{N+k} v_{Nk} \left\{ \sum_{j=1}^{N-1} D_{(j)}(v_1, v_2, \dots, v_{k-1}, v_{k+1}, \dots, v_N) \right\} \\
& \quad + \sum_{k=1}^{N} (-1)^{N+k} v_{N+1\ k} \left| v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1} \wedge v_{k+1} \wedge \cdots \wedge v_N \right| .
\end{aligned}
$$

By induction hypotheses, we have

$$
\begin{aligned}
(73)\quad & \sum_{k=1}^{N} (-1)^{N+k} v_{Nk} \left\{ \sum_{j=1}^{N-1} D_{(j)}(v_1, v_2, \dots, v_{k-1}, v_{k+1}, \dots, v_N) \right\} \\
& = \sum_{k=1}^{N} (-1)^{N+k+1} v_{Nk} \left| v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1} \wedge v_{k+1} \wedge \cdots \wedge v_N \wedge e_{N-1} \right| \\
& = 0.
\end{aligned}
$$

On the other hand, we can see

$$
\begin{aligned}
(74)\quad & \sum_{k=1}^{N} (-1)^{N+k} v_{N+1\ k} \left| v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1} \wedge v_{k+1} \wedge \cdots \wedge v_N \right| \\
& \qquad = (-1) \left| v_1 \wedge v_2 \wedge v_3 \wedge \cdots \wedge v_N \wedge e_N \right| .
\end{aligned}
$$

Combining (73), (74) with (72), we have

$$
(75)\quad \sum_{j=1}^{N} D_{(j)}(v_1, v_2, \dots, v_N) = (-1) \left| v_1 \wedge v_2 \wedge v_3 \wedge \cdots \wedge v_N \wedge e_N \right| .
$$

We have verified the equation (70) for $m = 1$, $n = N$ and hence obtained the equation (70) for $m = 1$ by mathematical induction on n.

We shall prove the equation (70) for a positive integer $m \leq n$. Suppose that the equation (70) is verified for $n \leq N - 1$ and for $m \leq M - 1$, $n = N$. We have to prove the equation (70) for $m = M$, $n = N$. By the Laplace expansion of N-th row vector and induction hypotheses, we have

$$\begin{aligned}(76)\quad &\sum_{\lambda\in\Sigma_M^N} D_\lambda(v_1,v_2,\dots,v_N)\\ &=\sum_{k=1}^{N}(-1)^{N+k}\ v_{Nk}\ \left\{\sum_{\lambda\in\Sigma_M^{N-1}} D_\lambda(v_1,v_2,\dots,v_{k-1},v_{k+1},\dots,v_N)\right\}\\ &\quad+\sum_{k=1}^{N}(-1)^{N+k}v_{N+1\ k}\left\{\sum_{\lambda\in\Sigma_{M-1}^{N-1}} D_\lambda(v_1,v_2,\dots,v_{k-1},v_{k+1},\dots,v_N)\right\}\\ &=\sum_{k=1}^{N}(-1)^{N+k+M}v_{Nk}\left|v_1\wedge v_2\wedge\cdots\wedge v_{k-1}\wedge v_{k+1}\wedge\cdots\wedge v_N\wedge e_{N-M}\right|\\ &\quad+\sum_{k=1}^{N}(-1)^{N+k+M-1}v_{N+1\ k}\\ &\qquad\times\left|v_1\wedge v_2\wedge\cdots\wedge v_{k-1}\wedge v_{k+1}\wedge\cdots\wedge v_N\wedge e_{N+1-M}\right|\\ &=(-1)^M\left|v_1\wedge v_2\wedge v_3\wedge\cdots\wedge v_N\wedge e_{N+1-M}\right|.\end{aligned}$$

We have verified the equation (70) for $m=M$, $n=N$ and hence obtained the equation (70) by mathematical induction.

By the elementary Schur polynomial, we mean, for a non-negative integer n,

$$S_n=\sum_{\substack{l_1\ge 0,l_2\ge 0,\dots,l_n\ge 0\\ l_1+2l_2+\cdots+nl_n=n}}\frac{t_1^{l_1}t_2^{l_2}\cdots t_n^{l_n}}{(l_1!)(l_2!)\cdots(l_n!)}\tag{77}$$

so that

$$\exp\left(\sum_{i=1}^{\infty}t_ix^i\right)=\sum_{n=0}^{\infty}S_nx^n.\tag{78}$$

For a negative integer n, we define $S_n=0$. Now, for a partition $\lambda=(\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_k)$, we define the Schur polynomial χ_λ by

$$\chi_\lambda=\det(S_{j-i+\lambda_{k-j+1}})_{i,j=1,2,\dots,k}.\tag{79}$$

Let $\overline{S}_n={}^t(S_n,S_{n-1},S_{n-2},\dots,)$. Here we note

$$\chi_\lambda=\left|\overline{S}_{\lambda_k}\wedge\overline{S}_{1+\lambda_{k-1}}\wedge\cdots\wedge\overline{S}_{k-1+\lambda_1}\right|.\tag{80}$$

For example, we have

$$\chi_{(4,3,3,1,1)} = \left|\overline{S}_1 \wedge \overline{S}_2 \wedge \overline{S}_5 \wedge \overline{S}_6 \wedge \overline{S}_8\right| = \det \begin{pmatrix} S_1 & S_2 & S_5 & S_6 & S_8 \\ 1 & S_1 & S_4 & S_5 & S_7 \\ 0 & 1 & S_3 & S_4 & S_6 \\ 0 & 0 & S_2 & S_3 & S_5 \\ 0 & 0 & S_1 & S_2 & S_4 \end{pmatrix}.$$

For $T_n = S_n(t, 0, 1, 0, 0, \ldots)$, we note from (77) and the differential of the variable x of (78)

$$\frac{d}{dt} T_n(t) = T_{n-1}(t), \tag{81}$$

$$nT_n(t) = tT_{n-1}(t) + 3T_{n-3}(t). \tag{82}$$

For a positive integer n, we define χ_n by

$$\chi_n = \chi_{(n,n-1,\ldots,1)}(t, 0, 1, 0, 0, \ldots) = \det \begin{pmatrix} T_1 & T_3 & T_5 & \cdots & T_{2n-1} \\ 1 & T_2 & T_4 & \cdots & T_{2n-2} \\ 0 & T_1 & T_3 & \cdots & T_{2n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & T_n \end{pmatrix}$$

and define $\chi_0 = 1$. Setting $\overline{T}_n = {}^t(T_n, T_{n-1}, T_{n-2}, \ldots,)$, we have for a positive integer n

$$\chi_n = \left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1}\right|. \tag{83}$$

Here we prove the following

Proposition 3. *For a positive integer n, $\chi_n(t)$ satisfy the following relations*

$$(2n+1)\chi_{n-1}\chi_{n+1} = t\left(\chi_n\right)^2 + 3\left\{\chi_n \left(\frac{d^2}{dt^2}\chi_n\right) - \left(\frac{d}{dt}\chi_n\right)^2\right\}, \tag{84}$$

$$\left(\frac{d^2}{dt^2}\chi_{n+1}\right)\chi_n - 2\left(\frac{d}{dt}\chi_{n+1}\right)\left(\frac{d}{dt}\chi_n\right) + \chi_{n+1}\left(\frac{d^2}{dt^2}\chi_n\right) = 0, \tag{85}$$

$$\left(\frac{d}{dt}\chi_{n+1}\right)\chi_{n-1} - \chi_{n+1}\left(\frac{d}{dt}\chi_{n-1}\right) = \left(\chi_n\right)^2. \tag{86}$$

Proof. As we have $\chi_0 = 1$, $\chi_1 = t$ and $\chi_2 = \frac{1}{3}t^3 - 1$. So Proposition 3 holds for $n = 1$. Hence we have to prove Proposition 3 for $n \geq 2$.

First, we shall prove the equation (84) for $n \geq 2$. Using the Plücker relation for $\overline{T}_1, \overline{T}_3, \dots, \overline{T}_{2n-3}, e_n$ and $\overline{T}_1, \overline{T}_3, \dots, \overline{T}_{2n+1}, e_{n+1}$, we have

(87)
$$\begin{aligned}
&\chi_{n-1}(t)\chi_{n+1}(t) \\
&= \left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-3} \wedge e_n \wedge e_{n+1}\right| \left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge \overline{T}_{2n+1}\right| \\
&= -\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-3} \wedge e_n \wedge \overline{T}_{2n-1}\right| \\
&\qquad\qquad \times \left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-3} \wedge \overline{T}_{2n+1} \wedge e_{n+1}\right| \\
&+ \left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-3} \wedge e_n \wedge \overline{T}_{2n+1}\right| \left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge e_{n+1}\right| \\
&= \left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-3} \wedge \overline{T}_{2n-1} \wedge e_n\right| \\
&\qquad\qquad \times \left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-3} \wedge \overline{T}_{2n+1} \wedge e_{n+1}\right| \\
&- \left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-3} \wedge \overline{T}_{2n+1} \wedge e_n\right| \left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge e_{n+1}\right|.
\end{aligned}$$

Here we set

(88) $$\psi_n(t) = \left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-3} \wedge \overline{T}_{2n+1}\right|.$$

By (81), we note

(89) $$\frac{d}{dt}\overline{T}_n = (\overline{T}_n)^+,$$

for an integer n. By (89) and (70), we have

(90) $$\begin{aligned}
\frac{d}{dt}\chi_n(t) &= \sum_{i=1}^{n} \left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \left(\overline{T}_{2i-1}\right)^+ \wedge \cdots \wedge \overline{T}_{2n-1}\right| \\
&= -\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge e_n\right|,
\end{aligned}$$

(91) $$\begin{aligned}
\frac{d}{dt}\psi_n(t) &= \sum_{i=1}^{n-1} \left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \left(\overline{T}_{2i-1}\right)^+ \wedge \cdots \wedge \overline{T}_{2n-3} \wedge \overline{T}_{2n+1}\right| \\
&\quad + \left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-3} \wedge \left(\overline{T}_{2n+1}\right)^+\right| \\
&= -\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-3} \wedge \overline{T}_{2n+1} \wedge e_n\right|.
\end{aligned}$$

By (70), (89), (90), we note

$$\frac{d^2}{dt^2}\chi_n(t) = -\sum_{i=1}^{n} \left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \left(\overline{T}_{2i-1}\right)^+ \wedge \cdots \wedge \overline{T}_{2n-1} \wedge e_n\right| \tag{92}$$
$$= \left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge e_n \wedge e_{n+1}\right| + \left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge e_{n-1}\right|.$$

On the other hand, we have

$$\frac{d^2}{dt^2}\chi(t) = 2\sum_{\lambda\in\Sigma_2^n} D_\lambda(\overline{T}_1, \overline{T}_3, \ldots, \overline{T}_{2n-1}) \tag{93}$$
$$= 2\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge e_{n-1}\right|.$$

Hence, by (92) and (93), we can see

$$\frac{d^2}{dt^2}\chi(t) = 2\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge e_n \wedge e_{n+1}\right|. \tag{94}$$

Now, combining (87) with (90) and (91), we have

$$\chi_{n-1}(t)\chi_{n+1}(t) = \chi_n(t)\left(\frac{d}{dt}\psi_n(t)\right) - \left(\frac{d}{dt}\chi_n(t)\right)\psi_n(t). \tag{95}$$

Hence

(96)

$$(2n+1)\chi_{n-1}(t)\chi_{n+1}(t) - t\left(\chi_n(t)\right)^2$$
$$= \chi_n(t)\left\{(2n+1)\left(\frac{d}{dt}\psi_n(t)\right) - t\chi_n(t)\right\} - \left(\frac{d}{dt}\chi_n(t)\right)\{(2n+1)\psi_n(t)\}$$
$$= \chi_n(t)\frac{d}{dt}\left\{(2n+1)\psi_n(t) - \frac{t^2}{2}\chi_n(t)\right\}$$
$$- \left(\frac{d}{dt}\chi_n(t)\right)\left\{(2n+1)\psi_n(t) - \frac{t^2}{2}\chi_n(t)\right\}.$$

In order to prove the recurrence relation (84), we show the following

LEMMA 3. *For an integer $n \geq 2$ and an integer $i = 1, 2, \ldots, n+1$, we have*

(97)

$$-\frac{t^2}{2}\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge e_i\right| + t\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge e_{i+1}\right|$$

$$+(i+1)\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge e_{i+2}\right| - 3\frac{d}{dt}\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge e_i\right|$$

$$+(2n+1)\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-3} \wedge \overline{T}_{2n+1} \wedge e_i\right| = 0.$$

In particular, for $i = n+1$, we have

$$-\frac{t^2}{2}\chi_n(t) + (2n+1)\psi_n(t) = 3\frac{d}{dt}\chi_n(t). \tag{98}$$

(Further see Noumi and Yamada [5], p.65, Lemma 3.)

Proof. Let $y_n(i)$ be the left-hand side of the equation (97). Now, for $n \geq 2$, $i = 1, 2, 3, \ldots, n$, noting

$$\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge {}^t(tT_{2i}, T_{2i}, 0, 0, \ldots)\right| = 0 \tag{99}$$

by $\overline{T}_1 = {}^t(t, 1, 0, 0, \ldots)$, we have

(100)

$$(y_n(1), y_n(2), \ldots, y_n(n+1))\ {}^t(T_{2i-1}, T_{2i-2}, \ldots, T_{2i-n-1})$$

$$= -\frac{t^2}{2}\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge \overline{T}_{2i-1}\right| + t\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge \overline{T}_{2i}\right|$$

$$+\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge \left[\mathrm{diag}(0, 1, 2, \ldots)\overline{T}_{2i+1}\right]\right|$$

$$-3\frac{d}{dt}\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge \overline{T}_{2i-1}\right| + 3\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge \overline{T}_{2i-2}\right|$$

$$+(2n+1)\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-3} \wedge \overline{T}_{2n+1} \wedge \overline{T}_{2i-1}\right|$$

$$= \left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge \left[\mathrm{diag}(2i+1, 2i, \ldots)\overline{T}_{2i+1}\right]\right|$$

$$-3\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge \overline{T}_{2i-2}\right|$$

$$+\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge \left[\mathrm{diag}(0, 1, \ldots)\overline{T}_{2i+1}\right]\right|$$

$$+3\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge \overline{T}_{2i-2}\right|$$

$$
\begin{aligned}
&+(2n+1)\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-3} \wedge \overline{T}_{2n+1} \wedge \overline{T}_{2i-1}\right| \\
&= (2i+1)\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-3} \wedge \overline{T}_{2n-1} \wedge \overline{T}_{2i+1}\right| \\
&\quad +(2n+1)\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-3} \wedge \overline{T}_{2n+1} \wedge \overline{T}_{2i-1}\right|,
\end{aligned}
$$

where $\operatorname{diag}(i_1, i_2, \ldots)$ is a diagonal matrix defined by

$$
\operatorname{diag}(i_1, i_2, \ldots) = \begin{pmatrix} i_1 & 0 & \cdots \\ 0 & i_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.
$$

We prove Lemma 3 by induction on n.

As we have by the equation (100) $(y_2(1), y_2(2), y_2(3))\ {}^t(T_1, T_0, 0) = 0$ and $(y_2(1), y_2(2), y_2(3))\ {}^t(T_3, T_2, T_1) = 0$. Moreover we can show

$$
y_2(3) = -\frac{1}{2}t^2\left|\overline{T}_1 \wedge \overline{T}_3\right| + 5\left|\overline{T}_1 \wedge \overline{T}_5\right| - 3\frac{d}{dt}\left|\overline{T}_1 \wedge \overline{T}_3\right| = 0. \tag{101}
$$

Hence we have $y_2(1) = y_2(2) = y_2(3) = 0$ and verified that Lemma 3 holds for $n = 2$. Suppose that Lemma 3 is proved for $2 \leq n \leq N-1$. We shall prove Lemma 3 for $n = N$. Using the Laplace expansion of the N-th column vector of $y_N(N+1)$ and the equation (82), we have

$$
\begin{aligned}
&y_N(N+1) \\
&= (y_{N-1}(1), y_{N-1}(2), \ldots, y_{N-1}(N))\ {}^t(T_{2N-1}, T_{2N-2}, \ldots, T_N).
\end{aligned} \tag{102}
$$

We hence obtain $y_N(N+1) = 0$ by the induction hypothesis. Moreover, by the equation (100), for $i = 1, 2, \ldots, N$, we have

$$
(y_N(1), y_N(2), \ldots, y_N(N+1))\ {}^t(T_{2i-1}, T_{2i-2}, \ldots, T_{2i-N-1}) = 0. \tag{103}
$$

Consequently, we have $y_N(i) = 0$ for $i = 1, 2, \ldots, N+1$, which proved Lemma 3 for $n = N$. We hence obtain Lemma 3 by induction on n.

Combining Lemma 3 with (96), we have the recurrence relation (84).

Next, we shall prove the equation (86) for $n \geq 2$. Using the Plücker relation and the equation (70), we have

(104)

$$
\begin{aligned}
&\left(\frac{d}{dt}\chi_{n+1}(t)\right)\chi_{n-1}(t)\\
&= -\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n+1} \wedge e_{n+1}\right| \left|e_1 \wedge \overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge e_{n+2}\right|\\
&= -\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n+1} \wedge e_{n+2}\right| \left|e_1 \wedge \overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge e_{n+1}\right|\\
&\quad +(-1)^{n+1}\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n+1} \wedge e_1\right|\\
&\qquad\qquad \times \left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge e_{n+1} \wedge e_{n+2}\right|\\
&= \chi_{n+1}(t)\left(\frac{d}{dt}\chi_{n-1}(t)\right) + \chi_n(t)^2.
\end{aligned}
$$

Next, we shall prove the equation (85) for $n \geq 2$. Using the Plücker relation and (90), (94), we have

(105)

$$
\begin{aligned}
&\left(\frac{d^2}{dt^2}\chi_{n+1}(t)\right)\chi_n(t)\\
&= 2\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n+1} \wedge e_n\right| \left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge e_{n+1} \wedge e_{n+2}\right|\\
&= 2\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n+1} \wedge e_{n+1}\right| \left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge e_n \wedge e_{n+2}\right|\\
&\quad -2\left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n+1} \wedge e_{n+2}\right| \left|\overline{T}_1 \wedge \overline{T}_3 \wedge \cdots \wedge \overline{T}_{2n-1} \wedge e_n \wedge e_{n+1}\right|\\
&= 2\left(\frac{d}{dt}\chi_{n+1}(t)\right)\left(\frac{d}{dt}\chi_n(t)\right) - \chi_{n+1}(t)\left(\frac{d^2}{dt^2}\chi_n(t)\right).
\end{aligned}
$$

Hence we have completed a proof of Proposition 3.

Using Proposition 3, we shall verify Theorem 2, the Hirota bilinear relation and the relation (24). Let $a = (-3/4)^{1/3}$. For a non-negative integer n, we set

$$b_n = a^{-\frac{n(n+1)}{2}}, \tag{106}$$

$$c_n = \prod_{k=1}^{n}(2k-1)!! \tag{107}$$

$$Q_n(u) = b_n c_n \chi_n(u), \quad u = at. \tag{108}$$

Then we see

$$b_{n+1}b_{n-1} = \frac{1}{a}b_n^2, \tag{109}$$

$$c_{n+1}c_{n-1} = (2n+1)c_n^2, \tag{110}$$

for a positive integer n. We have $Q_0(u) = P_0(t) = 1$ and $Q_1(u) = P_1(t) = t$. Now, by (84), (109) and (110), for a positive integer n, we have

(111)
$$\begin{aligned}
&Q_{n+1}(u)Q_{n-1}(u)\\
&= b_{n+1}b_{n-1}c_{n+1}c_{n-1}\chi_{n+1}(u)\chi_{n-1}(u)\\
&= \frac{1}{a}b_n^2c_n^2\left[u(\chi_n(u))^2 + 3\left\{\chi_n(u)\left(\frac{d^2}{du^2}\chi_n(u)\right) - \left(\frac{d}{du}\chi_n(u)\right)^2\right\}\right]\\
&= b_n^2c_n^2\left[t(\chi_n(u))^2 + \frac{3}{a^3}\left\{\chi_n(u)\left(\frac{d^2}{dt^2}\chi_n(u)\right) - \left(\frac{d}{dt}\chi_n(u)\right)^2\right\}\right]\\
&= tQ_n(u) - 4\left\{Q_n(u)\left(\frac{d^2}{dt^2}Q_n(u)\right) - \left(\frac{d}{dt}Q_n(u)\right)^2\right\},
\end{aligned}$$

which is just equal to the recurrence relation (1). From the uniqueness of recurrence relation, we hence conclude $P_n(t) = Q_n(u)$ for a non-negative integer n which is Theorem 2.

By (85), for a positive integer n, we have

$$\begin{aligned}
&\left(\frac{d^2}{dt^2}Q_{n+1}(u)\right)Q_n(u) - 2\left(\frac{d}{dt}Q_{n+1}(u)\right)\left(\frac{d}{dt}Q_n(u)\right)\\
&+ Q_{n+1}(u)\left(\frac{d^2}{dt^2}Q_n(u)\right)\\
&= a^2b_{n+1}b_nc_{n+1}c_n\left\{\left(\frac{d^2}{du^2}\chi_{n+1}(u)\right)\chi_n(u)\right.\\
&\quad\left.-2\left(\frac{d}{du}\chi_{n+1}(u)\right)\left(\frac{d}{du}\chi_n(u)\right) + \chi_{n+1}(u)\left(\frac{d^2}{du^2}\chi_n(u)\right)\right\}\\
&= 0.
\end{aligned} \tag{112}$$

By (86), (109) and (110), for a positive integer n, we have

$$\begin{aligned}
(113)\qquad & \left(\frac{d}{dt}Q_{n+1}(u)\right)Q_{n-1}(u) - Q_{n+1}(u)\left(\frac{d}{dt}Q_{n-1}(u)\right) \\
&= b_{b+1}b_{n-1}c_{n+1}c_{n-1}\left\{\left(\frac{d}{dt}\chi_{n+1}(u)\right)\chi_{n-1}(u)\right. \\
&\qquad \left. -\chi_{n+1}(u)\left(\frac{d}{dt}\chi_{n-1}(u)\right)\right\} \\
&= (2n+1)b_n^2c_n^2\left\{\left(\frac{d}{du}\chi_{n+1}(u)\right)\chi_{n-1}(u)\right. \\
&\qquad \left. -\chi_{n+1}(u)\left(\frac{d}{du}\chi_{n-1}(u)\right)\right\} \\
&= (2n+1)\left(Q_n(u)\right)^2.
\end{aligned}$$

We hence have verified (23) and (24).

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Appendix by Hiroyuki Kawamuko

We give a simple proof of equality (70). Let n be a positive integer and x be a variable. We set $X_n = {}^t(x^n, x^{n-1}, x^{n-2}, \ldots)$. For a $v_i = {}^t(v_{1i}, v_{2i}, v_{3i}, \ldots) \in V_\infty$ $(i = 1, 2, \ldots, n)$, we consider

$$(114)\qquad K(x) = |v_1 \wedge v_2 \wedge v_3 \wedge \cdots \wedge v_n \wedge X_n|$$

$$= \begin{vmatrix} v_{11} & v_{12} & v_{13} & \cdots & v_{1n} & x^n \\ v_{21} & v_{22} & v_{23} & \cdots & v_{2n} & x^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{n1} & v_{n2} & v_{n3} & \cdots & v_{nn} & x \\ v_{n+1\ 1} & v_{n+1\ 2} & v_{n+1\ 3} & \cdots & v_{n+1\ n} & 1 \end{vmatrix}.$$

We have

$$(115)\qquad K(x) = \sum_{m=0}^{n} |v_1 \wedge v_2 \wedge v_3 \wedge \cdots \wedge v_n \wedge e_{n+1-m}|\, x^m.$$

On the other hand, we have

(116)

$$K(x) = \begin{vmatrix} v_{11} - xv_{21} & v_{12} - xv_{22} & \dots & v_{1n} - xv_{2n} & 0 \\ v_{21} - xv_{31} & v_{22} - xv_{32} & \dots & v_{2n} - xv_{3n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{n1} - xv_{n+1\ 1} & v_{n2} - xv_{n+1\ 2} & \dots & v_{nn} - xv_{n+1\ n} & 0 \\ v_{n+1\ 1} & v_{n+1\ 2} & \dots & v_{n+1\ n} & 1 \end{vmatrix}$$

$$= (-1)^n \begin{vmatrix} xv_{21} - v_{11} & xv_{22} - v_{12} & \dots & xv_{2n} - v_{1n} \\ xv_{31} - v_{21} & xv_{32} - v_{22} & \dots & xv_{3n} - v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ xv_{n+1\ 1} - v_{n1} & xv_{n+1\ 2} - v_{n2} & \dots & xv_{n+1\ n} - v_{nn} \end{vmatrix}$$

$$= (-1)^n \left| (xv_1^+ - v_1) \wedge (xv_2^+ - v_2) \wedge \cdots \wedge (xv_n^+ - v_n) \right|,$$

by multilinearity of determinant

$$= \sum_{\lambda \in \Sigma_m^n} (-1)^m D_\lambda(v_1, v_2, \dots, v_n) x^m.$$

Comparing the coefficients of x^m of (115), (116), we get (70).

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