A. K. Dutta
Nagoya Math. J.
Vol. 159 (2000), 45–51

ON SEPARABLE A^1-FORMS

AMARTYA KUMAR DUTTA

Abstract. We show that for any field k, separable A^1-forms over commutative k-algebras are trivial.

Introduction

Suppose that k is a field and A a commutative k-algebra. An A-algebra B is said to be an $\boldsymbol{A}^1$-form over A (with respect to k) if $B \otimes_k \overline{k}$ is a polynomial ring in one variable over $A \otimes_k \overline{k}$, where $\overline{k}$ denotes the algebraic closure of k. It is well-known that if k is a perfect field then any $\boldsymbol{A}^1$-form over k is a polynomial ring in one variable over k (see Lemma 5 below). This need not be true if k is not perfect. In [BD, 3.7], it was shown that if A is a noetherian normal k-domain over a perfect field k and B is an $\boldsymbol{A}^1$-form over A (with respect to k) then B is A-isomorphic to the symmetric algebra of an invertible ideal of A. In this paper we show that this result can be extended to any commutative k-algebra A. More precisely, we prove the following:

Theorem. *Let k be a field, A a commutative k-algebra and L a separable field extension of k. Let B be an A-algebra such that $B \otimes_k L$ is isomorphic to the symmetric algebra of a finitely generated rank one projective module over $A \otimes_k L$. Then B is isomorphic to the symmetric algebra of a finitely generated rank one projective module over A.*

The author sincerely thanks S.M. Bhatwadekar and Amit Roy for their valuable suggestions and help during the preparation of this paper. He is also grateful to Mohan Kumar for discussion during the work [BD] which led to the investigation of the above problem.

Received February 18, 1994.
Revised May 17, 1999.
1991 Mathematics Subject Classification: 13B25, 12F10.

Main Theorem

We first fix some notations and terminology. Throughout this paper all rings will be assumed to be commutative with unity. For a ring R, $R^{[n]}$ denotes a polynomial ring in n variables over R and R^* the group of units of R.

A finitely generated flat A-algebra B is said to be an $\boldsymbol{A}^n$*-fibration* if $B_P/PB_P = (A_P/PA_P)^{[n]}$ for all $P \in \operatorname{Spec} A$. An A-algebra B is said to be *locally* $\boldsymbol{A}^n$ if $B_P = {A_P}^{[n]}$ for each maximal ideal P of A. By a result of Bass-Connell-Wright [BCW, 4.4], a finitely presented locally $\boldsymbol{A}^n$-algebra is isomorphic to the symmetric algebra of a finitely generated projective A-module of rank n.

If k is a field with algebraic closure $\overline{k}$, then a k-algebra L is said to be *separable* if $L \otimes_k \overline{k}$ is a reduced ring. If a field extension L over k has a separating transcendence basis then it is a separable k-algebra. For further details on separability, see [M].

We first prove a few technical lemmas.

Lemma 1. *Let k be a field, A a k-algebra and L a field extension of k. If B is an A-algebra such that $B \otimes_k L$ is finitely presented over $A \otimes_k L$, then B is finitely presented over A.*

Proof. It is enough to assume that $A \hookrightarrow B$ (and hence $A \otimes_k L \hookrightarrow B \otimes_k L$). Let $B \otimes_k L = (A \otimes_k L)[x_1, \ldots, x_m]$. Let $x_i = \sum_j (b_{ij} \otimes \alpha_{ij})$, where $b_{ij} \in B$ and $\alpha_{ij} \in L$ for all i, j. Then $B_1 = A[\{b_{ij}\}_{i,j}] \hookrightarrow B$ and the induced map $B_1 \otimes_k L \hookrightarrow B \otimes_k L$ is clearly an isomorphism. As L is faithfully flat over k, it follows that $B = B_1$. Thus B is finitely generated over A, say, generated by r elements.

Let ϕ be a surjection $A[X_1, \ldots, X_r] \to B$ and $I = \operatorname{Ker} \phi$. Then $I \otimes_k L$ is the kernel of the induced surjection

$$\phi_L : (A \otimes_k L)[X_1, \ldots, X_r] \to B \otimes_k L,$$

and hence is finitely generated. Let $I \otimes_k L = (f_1, \ldots, f_m)$ where $f_i = \sum_j (a_{ij} \otimes \beta_{ij})$, where $a_{ij} \in I$ and $\beta_{ij} \in L$ for all i, j. Let J be the ideal in $A[X_1, \ldots, X_r]$ generated by the a_{ij}'s. Then $J \subseteq I$ and $J \otimes_k L = I \otimes_k L$. Therefore, L being faithfully flat over k, we have $J = I$ showing that I is finitely generated. Thus B is finitely presented over A. $\square$

LEMMA 2. *Let k be a field, A a k-algebra, L a field extension of k and M a finitely generated projective module over $A \otimes_k L$. Then there exists a subfield K of L which is finitely generated over k and a finitely generated projective module N over $A \otimes_k K$ such that $N \otimes_K L \cong M$ as $(A \otimes_k L)$-modules.*

Proof. Let $A_L = A \otimes_k L$. Let M be a direct summand of a free A_L-module F of rank m with basis $\{e_1, \ldots, e_m\}$. The projection map $\phi : F \to M$ defines an idempotent A_L-endomorphism of F. Let

$$\phi(e_i) = \sum_{1 \leq j \leq m} \left(\sum_r (a_{ijr} \otimes \alpha_{ijr}) \right) e_j,$$

where $a_{ijr} \in A$ and $\alpha_{ijr} \in L$. Let $K = k(\{\alpha_{ijr}\}_{i,j,r})$ and $A_K = A \otimes_k K$ (identified with its image in A_L). Let E be the free A_K-module with basis $\{e_1, \ldots, e_m\}$, considered as a subgroup of F. Let $\psi = \phi \mid_E$ and $N = \psi(E)$. Then ψ is an idempotent A_K-endomorphism of E so that N is a finitely generated projective A_K-module. Since $E \otimes_K L = F$, it follows that $N \otimes_K L = M$. ∎

LEMMA 3. *Let k be a field, A a k-algebra and L a field extension of k. If C and D are finitely generated A-algebras such that $C \otimes_k L \cong D \otimes_k L$ as $(A \otimes_k L)$-algebras, then there exists a subfield K of L such that K is finitely generated over k and $C \otimes_k K \cong D \otimes_k K$ as $(A \otimes_k K)$-algebras.*

Proof. Let $C = A[c_1, \ldots, c_m]$ and $D = A[d_1, \ldots, d_n]$. Suppose that $\phi : C \otimes_k L \to D \otimes_k L$ is an $(A \otimes_k L)$-isomorphism. Let $\phi(c_i \otimes 1) = \sum_j (b_{ij} \otimes \beta_{ij})$, where $b_{ij} \in D$, $\beta_{ij} \in L$ and let $\phi^{-1}(d_i \otimes 1) = \sum_j (a_{ij} \otimes \alpha_{ij})$, where $a_{ij} \in C$, $\alpha_{ij} \in L$. Let K be the subfield of L generated by k, $\{\alpha_{ij}\}_{i,j}$ and $\{\beta_{ij}\}_{i,j}$. Identify $C \otimes_k K$ and $D \otimes_k K$ with their images in $C \otimes_k L$ and $D \otimes_k L$ respectively. Now it is easy to see that the restriction of ϕ to $C \otimes_k K$ induces an $(A \otimes_k K)$-isomorphism $C \otimes_k K \to D \otimes_k K$. ∎

LEMMA 4. *Let C be a ring, D a finitely generated C-algebra and S a multiplicatively closed set in C whose elements are non-zero divisors in C. Suppose that there exists a finitely generated projective $S^{-1}C$-module P such that $S^{-1}D \cong \operatorname{Sym} P$ as $S^{-1}C$-algebras. Then there exists an element $f \in S$ and a finitely generated projective C_f-module Q such that $D_f \cong \operatorname{Sym} Q$ as C_f-algebras.*

Proof. Let P be a direct summand of a free $S^{-1}C$-module F of rank m with basis $\{e_1, \dots, e_m\}$ and $\phi : F \to P$ be the projection map. Now it is easy to see that there exists $g \in S$ such that $\phi(e_i) = (\sum_{1 \le j \le n} a_j e_j)/g$ for some $a_1, \dots, a_n \in C$. Let E be the free C_g-module generated by $\{e_1, \dots, e_m\}$ (considered as a subgroup of F). Let $\psi = \phi \mid_E$ and $N = \psi(E)$. As in the proof of Lemma 2, N is a finitely generated projective C_g-module and $S^{-1}N = P$. Thus

$$S^{-1}D_g \cong \operatorname{Sym}_{S^{-1}C_g} S^{-1}N = S^{-1}(\operatorname{Sym}_{C_g} N).$$

Now as D is finitely generated over C, it is easy to see that there exists $h \in S$ such that $D_{gh} \cong (\operatorname{Sym}_{C_g} N)_h$. Let $f = gh$ and $Q = N_h$. Then Q is a finitely generated projective C_f-module such that $D_f \cong \operatorname{Sym}_{C_f} Q$. ∎

We shall now prove the main theorem. For the convenience of the reader, we first give a simple proof of the following well-known result.

Lemma 5. *Let k be a field and let L be a finite separable extension of k. Suppose that B is an overdomain of k such that $B \otimes_k L = L^{[1]}$. Then $B = k^{[1]}$.*

Proof. Let $B \otimes_k L = L[T]$. We identify B with its image in $B \otimes_k L$ under the map $b \to b \otimes 1$. Replacing L by its splitting field, we may assume L to be finite Galois over k with Galois group G, say. Any $\sigma \in G$ can be extended to a B-automorphism of $B \otimes_k L (= L[T])$ by defining $\sigma(b \otimes \alpha) = b \otimes \sigma(\alpha)$ for $b \in B, \alpha \in L$. Let

$$T = 1 \otimes \alpha_0 + e_1 \otimes \alpha_1 + \cdots + e_r \otimes \alpha_r,$$

where $1, e_1, \dots, e_r$ form part of a k-basis of B and $\alpha_i \in L$. Since the bilinear map $L \times L \to k$ given by $(x, y) \to \operatorname{Tr}(xy)$ is non-degenerate, replacing T by $\alpha T (\alpha \in L^*)$ if necessary, we assume that $\operatorname{Tr}(\alpha_i) \neq 0$ for some $i \ge 1$. Thus

$$W = \sum_{\sigma \in G} \sigma(T) = 1 \otimes \operatorname{Tr}(\alpha_0) + e_1 \otimes \operatorname{Tr}(\alpha_1) + \cdots + e_r \otimes \operatorname{Tr}(\alpha_r)$$

is an element of $B \backslash k$. Since $L[T] = \sigma(L[T]) = L[\sigma(T)]$, clearly $\sigma(T)$ is linear in T for each σ and hence $\deg_T W \le 1$. But as $B \cap L = k$, it follows that $W \notin L$ so that $\deg_T W = 1$. Hence, $k[W] \otimes_k L = L[W] = L[T] = B \otimes_k L$. Therefore, L being faithfully flat over k, we obtain $B = k[W] (= k^{[1]})$. ∎

We now generalise Lemma 5 as follows.

PROPOSITION 6. *Let k be a field, L a finite separable extension of k, A a k-algebra and B an A-algebra such that $B \otimes_k L$ is isomorphic to the symmetric algebra of a finitely generated rank one projective module over $A \otimes_k L$. Then B is isomorphic to the symmetric algebra of a finitely generated rank one projective module over A.*

Proof. By Lemma 1, B is finitely presented over A. Hence, by [BCW, 4.4], it is enough to assume that A is local. Now $A \otimes_k L$, being a finite extension of A, is semilocal. Hence $B \otimes_k L = (A \otimes_k L)^{[1]}$, say, $B \otimes_k L = (A \otimes_k L)[Y]$. Let $B = A[b_1, \ldots, b_r]$ and let $b_i \otimes 1 = \sum_j a_{ij} \otimes f_j(Y)$, where $a_{ij} \in A, f_j \in L^{[1]}$. Let $A_1 = k[\{a_{ij}\}_{i,j}](\hookrightarrow A)$ and $B_1 = A_1[b_1, \ldots, b_r](\hookrightarrow B)$. Then clearly $B_1 \otimes_k L = (A_1 \otimes_k L)[Y]$ (identifying them by their isomorphic images in $B \otimes_k L$). Now the canonical map $B_1 \otimes_{A_1} A \to B$ is clearly surjective and as

$$\begin{aligned}(B_1 \otimes_{A_1} A) \otimes_k L &= (B_1 \otimes_k L) \otimes_{A_1} A \\ &= (A_1 \otimes_k L)[Y] \otimes_{A_1} A \\ &= (A \otimes_k L)[Y] \\ &= B \otimes_k L,\end{aligned}$$

the map $B_1 \otimes_{A_1} A \to B$ is actually an isomorphism.

Thus, replacing A by A_1 and B by B_1 if necessary, we assume that A is an affine k-algebra; in particular, A is noetherian. By [BCW, 4.4], we can assume that A is a k-spot. Since A is now noetherian, to prove that $B = A^{[1]}$, it is enough to prove that $B/(\operatorname{nil} A)B = (A/\operatorname{nil} A)^{[1]}$, so that we may further assume A to be a reduced ring. Replacing L by its splitting field, we may assume L to be a Galois extension of k. We are thus reduced to proving the following statement:

(*) *Let L be a finite Galois extension of k with Galois group G, A a reduced k-spot and B a finitely generated A-algebra such that $B \otimes_k L = (A \otimes_k L)^{[1]}$. Then $B = A^{[1]}$.*

We now use Itoh's result in [I] on weak normality to deduce (*). (The author thanks the referee for drawing his attention to the paper of Itoh which has simplified the proof of (*) and Amit Roy for his help in the following deduction.) Note that any $\sigma \in G$ can be extended to an A-automorphism of $A \otimes_k L$ by defining $\sigma(a \otimes \alpha) = a \otimes \sigma(\alpha)$ for $a \in A, \alpha \in L$. Now let $x \in A \otimes_k L$ be such that $x^2, x^3 \in A$. Then $\sigma(x^2) = x^2$ and $\sigma(x^3) = x^3$ for all $\sigma \in G$ and hence it follows that $(\sigma(x) - x)^3 = 0$ for all $\sigma \in G$. Therefore, as $A \otimes_k L$ is reduced, $\sigma(x) = x$ for all $\sigma \in G$, showing that $x \in A$. Thus A is seminormal in $A \otimes_k L$. Now, let $z \in A \otimes_k L$ be such that $z^p, pz \in A$ for some prime p. If $p \neq \operatorname{ch} k$, then already $z \in A$. If $p = \operatorname{ch} k$, then for any $\sigma \in G$, $(\sigma(z) - z)^p = \sigma(z)^p - z^p = 0$, showing that $\sigma(z) = z$ for all $\sigma \in G$ and hence $z \in A$. Therefore, by [I, Prop. 1], A is weakly normal in $A \otimes_k L$ and hence (*) follows from the main theorem in [I]. ∎

We now prove the main theorem.

THEOREM 7. *Let k be a field, L a separable field extension of k, A a k-algebra and B an A-algebra such that $B \otimes_k L$ is isomorphic to the symmetric algebra of a finitely generated rank one projective module over $A \otimes_k L$. Then B is isomorphic to the symmetric algebra of a finitely generated rank one projective module over A.*

Proof. Using Lemmas 2 and 3 successively, we see that there exists a subfield L_1 of L which is finitely generated over k and a finitely generated projective $(A \otimes_k L_1)$-module P of rank one such that $B \otimes_k L_1 \cong \operatorname{Sym}_{A \otimes_k L_1} P$. Now L_1 is a finite separable extension of a rational function field $K = k(X_1, \ldots, X_n)$. Hence, by Proposition 6, $B \otimes_k K$ is the symmetric algebra of a finitely generated rank one projective $A \otimes_k K$-module Q. By Lemma 4, there exists an element $f \in k^{[n]}$ such that $B[X_1, \ldots, X_n, 1/f(X_1, \ldots, X_n)]$ is the symmetric algebra of a finitely generated rank one projective $A[X_1, \ldots, X_n, 1/f(X_1, \ldots, X_n)]$-module.

Let $D = k[X_1, \ldots, X_n, 1/f(X_1, \ldots, X_n)]$. If k is a finite field, then for any maximal ideal N of D, $F = D/N$ is a finite extension of k (by Nullstellensatz) and separable over k. Now $B \otimes_k F$ is the symmetric algebra of a rank one projective $A \otimes_k F$-module. Hence, by Proposition 6, B is the symmetric algebra of a finitely generated rank one projective A-module.

If k is infinite, then we can choose $z_1, \ldots, z_n \in k$ such that $f(z_1, \ldots, z_n) \neq 0$. In D, consider the maximal ideal

$$N = (X_1 - z_1, \ldots, X_n - z_n, 1/f(X_1, \ldots, X_n) - 1/f(z_1, \ldots, z_n)).$$

Then $D/N = k$ so that B is the symmetric algebra of a finitely generated rank one projective A-module. This completes the proof. ∎

Note that, in general, a separable $\boldsymbol{A}^1$-form need not be $\boldsymbol{A}^1$ as the following well-known example illustrates : Let $A = \boldsymbol{R}[X,Y]/(X^2+Y^2-1)$ where $\boldsymbol{R}$ denotes the field of real numbers, let I be an invertible ideal of A which is not principal and let $B = \mathrm{Sym}_A(I)$. Then $B \neq A^{[1]}$ but $B \otimes_{\boldsymbol{R}} \boldsymbol{C} = (A \otimes_{\boldsymbol{R}} \boldsymbol{C})^{[1]}$.

In [K], Kambayashi has proved that separable $\boldsymbol{A}^2$-forms over a field are trivial. One may ask a similar question over rings:

Question. *Let K be a field of characteristic zero, A a noetherian K-algebra (say, A is regular) and B an A-algebra such that $B \otimes_K \overline{K} = (A \otimes_K \overline{K})^{[2]}$ (where $\overline{K}$ denotes the algebraic closure of K). Then, is B isomorphic to the symmetric algebra of a projective A-module of rank two?*

Remark 8. If A is a Dedekind domain, then the above question has an affirmative answer.

Proof. Using Kambayashi's result [K] and the arguments in Proposition 6 and Theorem 7, it would follow that B is an $\boldsymbol{A}^2$-fibration over A. Therefore, B is locally $\boldsymbol{A}^2$ over A by Sathaye's result [S, Theorem 1], and hence a symmetric algebra by [BCW, 4.4] □

References

[BCW] H. Bass, E. H. Connell and D. L. Wright, *Locally polynomial algebras are symmetric algebras.*, Invent. Math., **38** (1977), 279–299.

[BD] S. M. Bhatwadekar, A. K. Dutta, *On A^1-fibrations of subalgebras of polynomial algebras*, Comp. Math., **95(3)** (1995), 263–285.

[I] S. Itoh, *On Weak Normality and Symmetric Algebras*, J. Algebra, **85(1)** (1983), 40–50.

[M] H. Matsumura, Commutative Algebra, 2nd Edn., Benjamin, 1980.

[K] T. Kambayashi, *On the absence of nontrivial separable forms of the affine plane*, J. Algebra, **35** (1975), 449–456.

[S] A. Sathaye, *Polynomial Ring in Two Variables over a D.V.R.:A Criterion*, Invent. Math., **74** (1983), 159–168.

Stat-Math Unit
Indian Statistical Institute
203, B.T. Road
Calcutta 700 035
India
amartya@isical.ac.in