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A NOTE ON WEIGHTED BERGMAN SPACES AND THE CESÁRO OPERATOR

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Dedicated to Professor John Benedetto

Abstract. Let B denote the unit ball in $\mathbb{C}^n$, and $dV(z)$ normalized Lebesgue measure on B. For $\alpha > -1$, define $dV^\alpha(z) = (1-|z|^2)^\alpha dV(z)$. Let $\mathcal{H}(B)$ denote the space of holomorhic functions on B, and for $0 < p < \infty$, let $\mathcal{A}^p(dV_\alpha)$ denote $L^p(dV_\alpha) \cap \mathcal{H}(B)$. In this note we characterize $\mathcal{A}^p(dV_\alpha)$ as those functions in $\mathcal{H}(B)$ whose images under the action of a certain set of differential operators lie in $L^p(dV_\alpha)$. This is valid for $1 \le p < \infty$. We also show that the Cesàro operator is bounded on $\mathcal{A}^p(dV_\alpha)$ for $0 < p < \infty$. Analogous results are given for the polydisc.

§0. Introduction

Let $B = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z| < 1\}$ be the unit ball in $\mathbb{C}^n$, let $\mathcal{H}(B)$ be the class of all holomorphic functions defined on B, and let $dV^\alpha(z) = (1-|z|^2)^\alpha dV(z)$ where $dV(z)$ is Lebesgue measure on B normalized to make the volume of the unit ball equal 1, *i.e,*

$$\int_B dV(z) = \frac{n\Gamma(n)}{\pi^n}\int_0^1 \int_{\partial B} r^{2n-1} dr d\sigma = 1,$$

(see [K, page 58]). We are interested in the holomorhpic functions which lie in $L^p(dV_\alpha)$ for various $0 < p < \infty$ and $\alpha > -1$. The case $p = 2$ and $\alpha = 0$ involves the classical Bergman projection operator – one of the most important operators in the theory of functions of several complex variables. It has been used to characterize biholomorphic mappings of finite type

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pseudoconvex domains (see Fefferman [F], Bell and Ligocka [BL], Catlin [Ca]).

It is worth noting that $dV_\alpha(z)$ is a natural measure on B since the projection operator from $\mathbb{C}^n$ onto $\mathbb{C}^m$, $1 \le m < n$, restricted to the sphere in $\mathbb{C}^n$ is naturally expressed as integration with respect to $dV_\alpha(z)$ over the unit ball B_m in $\mathbb{C}^m$ with $\alpha = n - m - 1$. More specifically, Forelli [Fo] has shown that

$$\int_{\partial B} (f \circ \mathbf{P}) d\sigma = \frac{(n-1)!}{m!(n-m-1)!} \int_{B_m} \left(1 - \sum_{k=1}^{m} |z_k|^2\right)^{n-m-1} f(z_1, \ldots, z_m) dV_m(z).$$

where $d\sigma$ is the area measure on ∂B and dV_m is the Lebesgue measure on $\mathbb{C}^m$.

We define $\mathcal{A}^p(dV_\alpha)$ to be the intersection of the spaces $L^p(dV_\alpha)$ and $\mathcal{H}(B)$, and call this the weighted Bergman space. It turns out that $\mathcal{A}^p(dV_\alpha)$ is a closed subspace of $L^p(dV_\alpha)$ and so it is natural to consider the projection

$$\mathbf{B}_\alpha : L^p(dV_\alpha) \to \mathcal{A}^p(dV_\alpha).$$

This projection, known as the weighted Bergman projection, is given an integral over B and is known as the weighted Bergman integral on B. The object of this paper is to give another characterization of $\mathcal{A}^p(dV_\alpha)$, namely, that $\mathcal{A}^p(dV_\alpha)$ consists of those holomorphic functions whose images under a certain set of differential operators lie in $L^p(dV_\alpha)$. This is the content of Theorem 1.7. We also observe in Theorem 1.8 that a similar characterization exists when the ball is replaced by the polydisc. A second objective of the paper is to study the Cesàro operator on $\mathcal{A}^p(dV_\alpha)$, for both the ball and the polydisc. Here we prove that in both of these cases the Cesàro operator is a bounded operator. This is the content of Theorems 2.4 and 2.5.

The paper is organized as follows. In section 1 we record the relevant definitions and lemmas, omitting proofs when they are available elsewhere in the literature. The main Theorems 1.7 and 1.8, characterizing $\mathcal{A}^p(dV_\alpha)$ are proven. In section 2 we define the Cesàro operator for the polydisc and the "slice" Cesàro operator for the ball. In Theorem 2.4 we prove that the Cesàro operator is a bounded operator on $\mathcal{A}^p(dV_\alpha)$ for the polydisc. We state the corresponding result for the slice Cesàro operator on the ball. The proof is omitted since it is similar to the case of the polydisc.

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§1. Weighted Bergman space

The space $L^p(dV_\alpha)$ consists of all Lebesgue measurable functions f defined on B satisfying

$$\|f\|^p_{L^p(dV_\alpha)} = \int_B |f(z)|^p (1-|z|^2)^\alpha dV(z) < \infty$$

where $dV(z)$ is the volume measure on B normalized so that $V(B) = 1$. It is easy to see that for $1 \le p < \infty, L^p(dV_\alpha)$ is a Banach space with norm $\|\cdot\|_{L^p(dV_\alpha)}$ and for $0 < p < 1, L^p(dV_\alpha)$ is an F-space under the metric

$$d(f,g) = \|f-g\|^p_{L^p(dV_\alpha)}.$$

Definition. For $0 < p < \infty$, and $\alpha > -1$

$$\mathcal{A}^p(dV_\alpha) = L^p(dV_\alpha) \cap \mathcal{H}(B).$$

The following lemma is proved, for example, in Djrbashian and Shamoian [DS], page 14, Corollary 1 to Theorem 1.1 and pages 128–136, §6.1.

Lemma 1.1. *For $1 \le p < \infty$, $\mathcal{A}^p(dV_\alpha)$ is a closed subspace of $L^p(dV_\alpha)$.*

For $p = 2$ we define the weighted Bergman projection

$$\mathbf{B}_\alpha : L^2(dV_\alpha) \to \mathcal{A}^2(dV_\alpha), \qquad \alpha > -1$$

as follows:

$$\begin{aligned} (1.1) \qquad & \mathbf{B}_\alpha(f)(z) \\ &= \frac{(\alpha+1)\cdots(\alpha+n)}{n!} \int_B \frac{f(w)}{(1-z\cdot\bar{w})^{n+1+\alpha}} (1-|w|^2)^\alpha dV(w) \\ &= \int_B f(w) K_\alpha(z,w)(1-|w|^2)^\alpha dV(w). \end{aligned}$$

The following result allows the extention to other values of p.

PROPOSITION 1.2. *Let $-1 < \alpha < \infty$, $-1 < \beta < \infty$ and $1 < p < \infty$. Then the projection operator $\mathbf{B}_\beta$ can be extended as a bounded operator from $L^p(dV_\alpha)$ onto $\mathcal{A}^p(dV_\alpha)$ if and only if $(1+\beta)p > 1+\alpha$. Moreover, in this case we have the reproducing formula $\mathbf{B}_\beta(f) = f$ for all $f \in \mathcal{A}^p(dV_\alpha)$.*

For the proof, see [DS, pages 33–36], § 2.1 and pages 128–136. We also refer the readers to [BG], [BCG], [CL] and references in there for further discussions.

Remark. Consider the case $\beta = \alpha$ in the hypothesis of Proposition 1.2, it is obvious that the operator $\mathbf{B}_\alpha$ is not bounded on $L^1(dV_\alpha)$ (see also [CNS], [CL]). However, for $\alpha > -1$, we may consider the following projection operator:

$$\tilde{\mathbf{B}}_\alpha(f)(z) = \frac{(\alpha+2)\cdots(\alpha+n+1)}{n!} \times \\ \times \int_B \frac{f(w)}{(1 - z\cdot\bar{w})^{n+2+\alpha}}(1-|w|^2)^{\alpha+1} dV(w). \tag{1.2}$$

Then it can be shown that $\tilde{\mathbf{B}}_\alpha : L^1(dV_\alpha) \to \mathcal{A}^1(dV_\alpha)$ boundedly. Moreover, $\tilde{\mathbf{B}}_\alpha(f) = f$ for all $f \in \mathcal{A}^1(dV_\alpha)$ (see *e.g.*, [Z, Chapter 4]). Proofs for the results in this section for the operator $\tilde{\mathbf{B}}_\alpha$ are identical to the proofs of the corresponding results for the operator $\mathbf{B}_\alpha$. Therefore, we will not repeat them.

Let $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$. A multi-index $\mathbf{k} = (k_1, \ldots, k_n)$ is an element in $(\mathbb{Z}_+)^n$. If $\mathbf{k}$ is a multi-index, let $|\mathbf{k}| = k_1 + \cdots + k_n$ and define the operator $\mathcal{Q}_\mathbf{k}$ as follows:

$$\mathcal{Q}_\mathbf{k}(f)(z) = (1-|z|^2)^{|\mathbf{k}|} \int_B \frac{\bar{w}_1^{k_1}\cdots\bar{w}_n^{k_n} f(w)}{(1-z\cdot\bar{w})^{n+1+\alpha+|\mathbf{k}|}}(1-|w|^2)^\alpha dV(w).$$

LEMMA 1.3. *Let $1 < p < \infty$ and $\alpha > -1$. Then the operator $\mathcal{Q}_\mathbf{k}$ can be extended as a bounded operator from $L^p(dV_\alpha)$ into $L^p(dV_\alpha)$.*

Proof. Let q be the conjugate exponent of p, *i.e.*, $\frac{1}{p} + \frac{1}{q} = 1$. Since $w \in B$, $|\bar{w}_1^{k_1}\cdots\bar{w}_n^{k_n}| \le 1$, we know that

$$|\mathcal{Q}_\mathbf{k}(f)(z)|^p \\ \le \left\{(1-|z|^2)^{|\mathbf{k}|}\int_B \frac{|f(w)|(1-|w|^2)^{-\frac{\alpha+1}{2pq}}(1-|w|^2)^{\frac{\alpha+1}{2pq}}}{|1-z\cdot\bar{w}|^{n+1+\alpha+|\mathbf{k}|}}(1-|w|^2)^\alpha dV(w)\right\}^p.$$

Then by Hölder's inequality, we have

$$|\mathcal{Q}_{\mathbf{k}}(f)(z)|^p \leq (1-|z|^2)^{p|\mathbf{k}|}\left\{\int_B \frac{(1-|w|^2)^{-\frac{\alpha+1}{2p}}}{|1-z\cdot\bar{w}|^{n+1+\alpha}}(1-|w|^2)^{\alpha}dV(w)\right\}^{\frac{p}{q}}$$
$$\times\left\{\int_B \frac{|f(w)|^p(1-|w|^2)^{\frac{\alpha+1}{2q}}}{|1-z\cdot\bar{w}|^{n+1+\alpha+p|\mathbf{k}|}}(1-|w|^2)^{\alpha}dV(w)\right\}$$
$$\leq C\cdot(1-|z|^2)^{p|\mathbf{k}|-\frac{\alpha+1}{2q}}\left\{\int_B \frac{|f(w)|^p(1-|w|^2)^{\alpha+\frac{\alpha+1}{2q}}}{|1-z\cdot\bar{w}|^{n+1+\alpha+p|\mathbf{k}|}}dV(w)\right\}.$$

Since $p|\mathbf{k}|+\alpha-\frac{\alpha+1}{2q}>-1$, by [R, Proposition 1.4.10], we obtain

$$\int_B |\mathcal{Q}_{\mathbf{k}}(f)(z)|^p(1-|z|^2)^{\alpha}dV(z)$$
$$\leq C\int_B (1-|w|^2)^{\alpha+\frac{\alpha+1}{2q}}|f(w)|^p\left\{\int_B \frac{(1-|z|^2)^{p|\mathbf{k}|+\alpha-\frac{\alpha+1}{2q}}}{|1-z\cdot\bar{w}|^{n+1+\alpha+p|\mathbf{k}|}}dV(z)\right\}dV(w)$$
$$\leq C\int_B |f(w)|^p(1-|w|^2)^{\alpha+\frac{\alpha+1}{2q}-\frac{\alpha+1}{2q}}dV(w)\leq C\cdot\|f\|^p_{L^p(dV_\alpha)}.$$

This concludes the proof of Lemma 1.3. ∎

LEMMA 1.4. *Let* $1<p<\infty$, $\alpha>-1$ *and* $\mathbf{k}\in(\mathbb{Z}_+)^n$. *Then* $(1-|z|^2)^{|\mathbf{k}|}\frac{\partial^{|\mathbf{k}|}f}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}(z)\in L^p(dV_\alpha)$ *for all* $f\in\mathcal{A}^p(dV_\alpha)$.

Proof. Since $f\in\mathcal{A}^p(dV_\alpha)$, we know that from (1.2),

$$f(z)=\frac{(\alpha+1)\cdots(\alpha+n)}{n!}\int_B \frac{f(w)}{(1-z\cdot\bar{w})^{n+1+\alpha}}(1-|w|^2)^{\alpha}dV(w).$$

It follows that

$$(1.3)\qquad (1-|z|^2)^{|\mathbf{k}|}\frac{\partial^{|\mathbf{k}|}f}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}(z)$$
$$=\frac{(\alpha+1)\cdots(\alpha+n+|\mathbf{k}|)}{n!}(1-|z|^2)^{|\mathbf{k}|}\times$$
$$\times\int_B \frac{\bar{w}_1^{k_1}\cdots\bar{w}_n^{k_n}f(w)}{(1-z\cdot\bar{w})^{n+1+\alpha+|\mathbf{k}|}}(1-|w|^2)^{\alpha}dV(w).$$

From Lemma 1.3, there is a constant C depending on $\mathbf{k}$, n, α and p such that

$$\left\|(1-|z|^2)^{|\mathbf{k}|}\frac{\partial^{|\mathbf{k}|}f}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}\right\|_{L^p(dV_\alpha)}\le C\|f\|_{\mathcal{A}^p(dV_\alpha)}.$$

Now the result follows immediately. □

Similarly, we have the following result for the case $p=1$ (cf. [Z, Lemma 4.2.7]).

LEMMA 1.5. *Let $\alpha>-1$ and $\mathbf{k}\in(\mathbb{Z}_+)^n$. Then*

$$(1-|z|^2)^{|\mathbf{k}|}\frac{\partial^{|\mathbf{k}|}f}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}(z)\in L^1(dV_\alpha)$$

for all $f\in\mathcal{A}^1(dV_\alpha)$.

COROLLARY 1.6. *Let $f\in\mathcal{A}^p(dV_\alpha)$. Then for all $\mathbf{k}\in(\mathbb{Z}_+)^n$*

$$\frac{\partial^{|\mathbf{k}|}f}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}(0)$$

$$=\frac{(\alpha+1)\cdots(\alpha+n+|\mathbf{k}|)}{n!}\int_B\bar{w}_1^{k_1}\cdots\bar{w}_n^{k_n}f(w)(1-|w|^2)^\alpha dV(w).$$

Furthermore,

$$\left|\frac{\partial^{|\mathbf{k}|}f}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}(0)\right|\le C\cdot\|f\|_{\mathcal{A}^p(dV_\alpha)}.$$

Proof. From the equation (1.3), we have the first assertion of the Corollary immediately (cf. [DS, Theorem 6.1]). Now by Hölder's inequality, we obtain

$$\left|\frac{\partial^{|\mathbf{k}|}f(0)}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}\right|$$

$$=\left|\frac{(\alpha+1)\cdots(\alpha+n+|\mathbf{k}|)}{n!}\int_B\bar{w}_1^{k_1}\cdots\bar{w}_n^{k_n}f(w)(1-|w|^2)^\alpha dV(w)\right|$$

$$\le\frac{(\alpha+1)\cdots(\alpha+n+|\mathbf{k}|)}{n!}\int_B|f(w)|(1-|w|^2)^\alpha dV(w)$$

$$\leq \frac{(\alpha+1)\cdots(\alpha+n+|\mathbf{k}|)}{n!} \cdot \|f\|_{\mathcal{A}^p(dV_\alpha)} \cdot \left[\int_B (1-|w|^2)^\alpha dV(w)\right]^{\frac{1}{q}}$$

$$\leq C(\alpha, |\mathbf{k}|, p) \cdot \|f\|_{\mathcal{A}^p(dV_\alpha)},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. ∎

Now we are in a position to prove the main theorem of this section.

THEOREM 1.7. *Let $1 \leq p < \infty$, $\alpha > -1$, N be a fixed positive integer and $\mathbf{k} = (k_1, \ldots, k_n) \in (\mathbb{Z}_+)^n$. Let f be a holomorphic function defined on the unit ball B in $\mathbb{C}^n$. Then $f \in \mathcal{A}^p(dV_\alpha)$ if and only if*

$$(1-|z|^2)^N \frac{\partial^N f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}(z) \in L^p(dV_\alpha), \quad \textit{for all} \quad |\mathbf{k}| = N.$$

Moreover,

$$(1.4) \qquad \|f\|_{L^p(dV_\alpha)}$$

$$\approx \left(\sum_{|\mathbf{k}|=0}^{N-1} \left| \frac{\partial^{|\mathbf{k}|} f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}(0) \right| + \sum_{|\mathbf{k}|=N} \left\| (1-|z|^2)^N \frac{\partial^N f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} \right\|_{L^p(dV_\alpha)} \right).$$

Proof. One direction has been proved in Lemmas 1.4 and 1.5. Now let us turn to the other direction. Assume that

$$\sum_{|\mathbf{k}|=N} (1-|z|^2)^N \left| \frac{\partial^N f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}(z) \right| \in L^p(dV_\alpha).$$

Without loss of generality, we may assume $\frac{\partial^{|\mathbf{k}|} f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}(0) = 0$ for $|\mathbf{k}| \leq 2N$. Fix $\mathbf{k} \in (\mathbb{Z}_+)^n$ with $|\mathbf{k}| = N$. Now let us consider the function

$$g(z) = \frac{(1-|z|^2)^N}{\bar{z}_1^{k_1} \cdots \bar{z}_n^{k_n}} \frac{\partial^N f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}(z).$$

Then it holds that $g(z) \in L^p(dV_\alpha)$. Therefore,

$$G(z) = \frac{(\alpha+1)\cdots(\alpha+n)}{n!} \times$$

$$\times \int_B \frac{(1-|w|^2)^N (1-|w|^2)^\alpha}{\bar{w}_1^{k_1} \cdots \bar{w}_n^{k_n} (1-z\cdot\bar{w})^{n+1+\alpha}} \frac{\partial^N f}{\partial w_1^{k_1} \cdots \partial w_n^{k_n}}(w) dV(w)$$

is a function in $\mathcal{A}^p(dV_\alpha)$. It follows that,

$$\frac{\partial^N G}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}(z) = \frac{(\alpha+1)\cdots(\alpha+n+N)}{n!}\times$$
$$\times\int_B \frac{(1-|w|^2)^N(1-|w|^2)^\alpha}{(1-z\cdot\bar{w})^{n+1+\alpha+N}}\frac{\partial^N f}{\partial w_1^{k_1}\cdots\partial w_n^{k_n}}(w)dV(w).$$

Now by Proposition 1.2, we know that

$$\frac{\partial^N G}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}(z) = \frac{\partial^N f}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}(z)$$

for all $z \in B$. For $0 \le |\mathbf{j}| \le N-1$,

$$\frac{\partial^{|\mathbf{j}|} G}{\partial z_1^{j_1}\cdots\partial z_n^{j_n}}(0) = \frac{(\alpha+1)\cdots(\alpha+n+|\mathbf{j}|)}{n}\times$$
$$\times\int_B \frac{(1-|w|^2)^N(1-|w|^2)^\alpha}{\bar{w}_1^{k_1-j_1}\cdots\bar{w}_n^{k_n-j_n}}\frac{\partial^N f}{\partial w_1^{k_1}\cdots\partial w_n^{k_n}}(w)dV(w) = 0.$$

Thus we have $f(z) = G(z) = \mathbf{B}_\alpha(g)(z)$ for all $z \in B$. Since $\mathbf{B}_\alpha$ is bounded from $L^p(dV_\alpha)$ onto $\mathcal{A}^p(dV_\alpha)$ for $1 < p < \infty$, this gives us $f \in \mathcal{A}^p(dV_\alpha)$ for $1 < p < \infty$. (For the case $p = 1$, we use the operator $\tilde{\mathbf{B}}_\alpha$.)

Fix p, $1 \le p < \infty$. Define

$$\mathcal{B}_N = \left\{ f \in \mathcal{H}(B) \text{ with } \frac{\partial^N f}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}} \in \mathcal{A}^p(dV_{\alpha+pN}) \text{ for all } |\mathbf{k}| = N \right\}.$$

It is easy to see that $\mathcal{B}_N$ is a Banach space under the norm

$$\|f\|_{\mathcal{B}_N} = \sum_{|\mathbf{k}|=0}^{N-1}\left|\frac{\partial^{|\mathbf{k}|} f}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}(0)\right| + \sum_{|\mathbf{k}|=N}\left\|\frac{\partial^N f}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}\right\|_{L^p(dV_{\alpha+pN})}.$$

If a sequence $\{f_k\}$ converges to f_0 in $\mathcal{B}_N$, then we know that $f_k \to f_0$ uniformly on compact subsets of B. Now let us prove the estimate (1.4). By (1.5) and Corollary 1.6, we know that $\|f\|_{\mathcal{B}_N} \le C\|f\|_{\mathcal{A}^p(dV_\alpha)}$. Next, let $\mathbf{I} : \mathcal{B}_N \to \mathcal{A}^p(dV_\alpha)$ be the identity operator. If $\|\mathbf{I}(f_k) - F\|_{L^p(dV_\alpha)} \to 0$ and $\|f_k - f_0\|_{\mathcal{B}_N} \to 0$ as $k \to \infty$, then $f_k \to F$ uniformly on compact subsets of B. Henc, $\mathbf{I}(f_0) = F$. By the closed graph theorem, we know that there exists a constant C such that $\|f\|_{\mathcal{A}^p(dV_\alpha)} \le C\|f\|_{\mathcal{B}_N}$. The proof of the theorem is therefore complete. ∎

We finish this section by considering the case of the polydisc. Let

$$\mathbb{D}^n = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| < 1,\, j = 1, \ldots, n\}$$

be the polydisc in $\mathbb{C}^n$ and let $\mathcal{H}(\mathbb{D}^n)$ be the class of all holomorphic functions f defined on $\mathbb{D}^n$. Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_j > -1$ for $j = 1, \ldots, n$. The space $L^p(dV_{\vec{\alpha}})$ consists of all Lebesgue measurable functions defined on $\mathbb{D}^n$ satisfying

$$\|f\|^p_{L^p(dV_{\vec{\alpha}})} = \int_{\mathbb{D}^n} |f(z)|^p \prod_{j=1}^{n} (1 - |z_j|^2)^{\alpha_j} dV(z_j) < \infty.$$

Here $dV(z_j)$ is the normalized volume measure on the unit disc $\mathbb{D}$, *i.e.*,

$$\int_{\mathbb{D}} dV(z_j) = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} r dr d\theta_j = 1.$$

Now the weighted Bergman space $\mathcal{A}^p(dV_{\vec{\alpha}})$ is the intersection of $L^p(dV_{\vec{\alpha}})$ and $\mathcal{H}(\mathbb{D}^n)$.

From computations in this section, it is easy to see that the kernel $B_{\vec{\alpha}}(z, w)$ for the weighted Bergman projection $\mathbf{B}_{\vec{\alpha}} : L^2(dV_{\vec{\alpha}}) \to \mathcal{A}^2(dV_{\vec{\alpha}})$ is

$$B_{\vec{\alpha}}(z, w) = \prod_{j=1}^{n} \frac{(\alpha_j + 1)}{(1 - z_j \bar{w}_j)^{\alpha_j + 2}}.$$

It can be shown that the operator $\mathbf{B}_{\vec{\alpha}}$ can be extended as a bounded operator from $L^p(dV_{\vec{\alpha}})$ onto $\mathcal{A}^p(dV_{\vec{\alpha}})$ giving the following theorem:

THEOREM 1.8. *Let N be a fixed positive integer and let $\mathbf{k} = (k_1, \ldots, k_n) \in (\mathbb{Z}_+)^n$. Let f be a holomorphic function defined on the polydisc $\mathbb{D}^n$ in $\mathbb{C}^n$. Then for $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $f \in \mathcal{A}^p(dV_{\vec{\alpha}})$ if and only if*

$$\left[\prod_{j=1}^{n} (1 - |z_j|^2)^{k_j}\right] \frac{\partial^{|\mathbf{k}|} f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}(z) \in L^p(dV_{\vec{\alpha}})$$

for $1 \le p < \infty$, $\alpha_j > -1$, $j = 1, \ldots, n$. Moreover,

$$\|f\|_{\mathcal{A}^p(dV_{\vec{\alpha}})} \approx \left(\sum_{|\mathbf{k}|=0}^{N-1} \left| \frac{\partial^N f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}(0) \right| + \sum_{|\mathbf{k}|=N} \left\| \left[\prod_{j=1}^{n} (1 - |z_j|^2)^{k_j}\right] \frac{\partial^N f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} \right\|_{L^p(dV_{\vec{\alpha}})} \right). \tag{1.5}$$

§2. The Cesàro operator

In this section we study the Cesàro operator for the polydisc and ball. We start with the polydisc. Let f be a holomorphic function defined on the polydisc $\mathbb{D}^n$. It follows that

$$f = \sum_{|\mathbf{k}|=0}^{\infty} a_{\mathbf{k}} z^{\mathbf{k}} = \sum_{k_1+\cdots+k_n=0}^{\infty} a_{k_1 k_2 \cdots k_n} z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$$

where $\mathbf{k} = (k_1, \ldots, k_n) \in (\mathbb{Z}_+)^n$. Let $\mathbf{m} = (m_1, \ldots, m_n) \in (\mathbb{Z}_+)^n$ be another n-tuple. We say that $\mathbf{m} \leq \mathbf{k}$ if and only if $m_j \leq k_j$ for $1 \leq j \leq n$. The Cesàro operator $\mathcal{C}$ is defined by

$$\mathcal{C}(f)(z) = \sum_{|\mathbf{k}|=0}^{\infty} \left(\frac{1}{(k_1+1)\cdots(k_n+1)} \sum_{\mathbf{m} \leq \mathbf{k}} a_{\mathbf{m}} \right) z^{\mathbf{k}}.$$

It is easy to see that

$$\begin{aligned} \mathcal{C}(f)(z) &= \int_0^1 \cdots \int_0^1 \frac{f(t_1 z_1, \ldots, t_n z_n)}{(1-t_1 z_1)\ldots(1-t_n z_n)} dt_1 \cdots dt_n \\ &= \int_Q \frac{f(t \cdot z)}{\prod_{j=1}^n (1-t_j z_j)} dt, \end{aligned}$$

where $Q = [0,1]^n$ and $dt = dt_1 \cdots dt_n$.

In preparation for the proof of Theorem 2.4, we record some preliminary lemmas. The proof of the following lemma is an easy consequence of the plurisubharmonicity of the function $|f|^p$.

Lemma 2.1. *Let* $1 \leq p < \infty$ *and* $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$ *with* $\alpha_j > -1$ *for* $j = 1, \ldots, n$. *Then for each* $\ell \in \{1, \cdots, n\}$, *there exists a universal constant* C_ℓ *such that*

$$\|f\|_{\mathcal{A}^p(dV_{\vec{\alpha}})} \leq C_\ell \|z_\ell f\|_{\mathcal{A}^p(dV_{\vec{\alpha}})}$$

for every $f \in \mathcal{A}^p(dV_{\vec{\alpha}})$.

The next lemma is just an n-fold version of a result of Duren [D, page 65].

LEMMA 2.2. *If $s_j > 1$ and $0 \le r_j < 1$ for $j = 1, \dots, n$, then there is a constant γ depending only on s_j, $j = 1, \dots, n$, such that*

$$\int_{[-\pi,\pi]^n} \prod_{j=1}^{n} \left|1 - r_j e^{i\theta_j}\right|^{-s_j} d\theta \le \gamma \cdot \prod_{j=1}^{n} (1 - r_j)^{-s_j+1}$$

where $[-\pi,\pi]^n = [-\pi,\pi] \times \cdots \times [-\pi,\pi]$ and $d\theta = \prod_{j=1}^{n} d\theta_j$.

The following lemma was first proved by Hardy-Littlewood [HL, pp. 412 and 414] in the case $n = 1$. It is not difficult to generalize their result to higher dimensional cases by taking the limit of the sequence of partial sums of the power series expansion of the holomorphic function f.

LEMMA 2.3. *Let $0 < p < 1$, $1 < q < \infty$ and $0 < r_j < 1$ for $j = 1, \dots, n$. Then there exists two universal constants C_1 and C_2 such that*

$$\int_{[-\pi,\pi]^n} \sup_{0 \le t_j < 1, 1 \le j \le n} |f(t_1 r_1 e^{i\theta_1}, \dots, t_n r_n e^{i\theta_n})|^p d\theta \\ \le C_1 \int_{[-\pi,\pi]^n} |f(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})|^p d\theta,$$

and

$$\int_Q \left\{ \int_{[-\pi,\pi]^n} |f(t_1 r_1 e^{i\theta_1}, \dots, t_n r_n e^{i\theta_n})|^{pq} d\theta \right\}^{\frac{1}{q}} \prod_{j=1}^{n} (1 - t_j)^{-\frac{1}{q}} dt \\ \le C_2 \int_{[-\pi,\pi]^n} |f(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})|^p d\theta,$$

for all holomorphic functions f defined on $\mathbb{D}^n$.

Now we are in a position to prove the first of our two theorems on the Cesàro operator.

THEOREM 2.4. *Let $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$ with $\alpha_j > -1$ for $j = 1, \dots, n$. Then the Cesàro operator $\mathcal{C}$ is bounded on $\mathcal{A}^p(dV_{\vec{\alpha}})$ for $0 < p < \infty$.*

Proof. We have to split the proof of this theorem into two cases.

Case 1. $1 \leq p < \infty$. Suppose that $f \in \mathcal{A}^p(dV_{\vec{\alpha}})$ and let $F = \mathcal{C}(f)$. By direct computation, we obtain

$$z_1 \cdots z_n \frac{\partial^n F(z)}{\partial z_1 \cdots \partial z_n} = \frac{f(z_1, \ldots, z_n)}{(1-z_1)\cdots(1-z_n)} \tag{2.1}$$

$$+ \sum_{q=1}^{n-1} (-1)^q \times$$

$$\sum_{1 \leq j_1 < \cdots < j_q \leq n} \int_{[0,1]^q} \frac{f(z_1, \ldots, t_{j_1} z_{j_1}, \ldots, t_{j_q} z_{j_q}, \ldots, z_n) dt_{j_1} \cdots dt_{j_q}}{(1-z_1)\cdots(1-t_{j_1}z_{j_1})\cdots(1-t_{j_q}z_{j_q})\cdots(1-z_n)}$$

$$+ (-1)^n \int_{[0,1]^n} \frac{f(t_1 z_1, \ldots, t_n z_n)}{(1-t_1 z_1)\cdots(1-t_n z_n)} dt_1 \cdots dt_n.$$

It is easy to see that the first term on the right hand side of (2.1) satisfies the following estimate.

$$\int_{\mathbb{D}^n} \left| \prod_{j=1}^n (1-|z_j|^2) \frac{f(z_1, \ldots, z_n)}{(1-z_j)} \right|^p \prod_{j=1}^n (1-|z_j|^2)^{\alpha_j} dV(z)$$

$$\leq \int_{\mathbb{D}^n} |f(z_1, \ldots, z_n)|^p \prod_{j=1}^n (1+|z_j|)^p (1-|z_j|^2)^{\alpha_j} dV(z)$$

$$\leq 2^{np} \int_{\mathbb{D}^n} |f(z)|^p \prod_{j=1}^n (1-|z_j|^2)^{\alpha_j} dV(z).$$

For the last term on the right hand side of (2.1), we have

$$\left\| \prod_{j=1}^n (1-|z_j|^2) \int_Q \frac{f(t \cdot z)}{(1-t_1 z_1)\cdots(1-t_n z_n)} dt \right\|_{L^p(dV_{\vec{\alpha}})}$$

$$\leq \int_Q \left\{ \int_{\mathbb{D}^n} \left[\prod_{j=1}^n \frac{(1-|z_j|^2)}{|1-t_j z_j|} |f(t \cdot z)| \right]^p \prod_{j=1}^n (1-|z_j|^2)^{\alpha_j} dV(z) \right\}^{\frac{1}{p}} dt$$

$$\leq 2^n \int_Q \left\{ \int_{\mathbb{D}^n} |f(t_1 z_1, \ldots, t_n z_n)|^p \prod_{j=1}^n (1-|z_j|^2)^{\alpha_j} dV(z) \right\}^{\frac{1}{p}} dt$$

$$\leq 2^n \int_Q \left\{ \int_{\mathbb{D}^n} |f(z_1, \ldots, z_n)|^p \prod_{j=1}^n (1-|z_j|^2)^{\alpha_j} dV(z) \right\}^{\frac{1}{p}} dt$$

$$= 2^n \|f\|_{\mathcal{A}^p(dV_{\vec{\alpha}})}.$$

Let $J = (j_1, \dots, j_q)$ with $1 \le j_1 < \cdots < j_q \le n$. For terms in between on the right hand side of (2.1), we have

$$\Bigg\| \prod_{j=1}^{n} (1 - |z_j|^2) \times$$

$$\times \int_{[0,1]^q} \frac{f(z_1, \dots, t_{j_1} z_{j_1} \dots, t_{j_q} z_{j_q}, \dots, z_n)}{(1 - z_1) \cdots (1 - t_{j_1} z_{j_n}) \cdots (1 - t_{j_q} z_{j_q}), \cdots (1 - z_n)} dt \Bigg\|_{L^p(dV_{\vec{\alpha}})}$$

$$\le \int_{[0,1]^q} \Bigg\{ \int_{\mathbb{D}^n} \left[\prod_{j \notin J} \frac{(1 - |z_j|^2)}{|1 - z_j|} \prod_{j \in J} \frac{(1 - |z_j|^2)}{|1 - t_j z_j|} |f(t \cdot z)| \right]^p \times$$

$$\times \prod_{j=1}^{n} (1 - |z_j|^2)^{\alpha_j} dV(z_j) \Bigg\}^{\frac{1}{p}} dt$$

$$\le 2^n \int_{[0,1]^q} \Bigg\{ \int_{\mathbb{D}^n} |f(z_1, \dots, t_{j_1} z_{j_1}, \dots, t_{j_q} z_{j_q}, \dots, z_n)|^p \times$$

$$\times \prod_{j=1}^{n} (1 - |z_j|^2)^{\alpha_j} dV(z_j) \Bigg\}^{\frac{1}{p}} dt$$

$$\le 2^n \int_{[0,1]^q} \left\{ \int_{\mathbb{D}^n} |f(z_1, \dots, z_n)|^p \prod_{j=1}^{n} (1 - |z_j|^2)^{\alpha_j} dV(z_j) \right\}^{\frac{1}{p}} dt$$

$$= 2^n \|f\|_{\mathcal{A}^p(dV_{\vec{\alpha}})}.$$

Here we use the Minkowski integral inequality and the monotonicity of the function $U(t_1, \dots, t_n) = \int_0^{2\pi} \cdots \int_0^{2\pi} |f(t_1 r_1 e^{i\theta_1}, \dots, t_n r_n e^{i\theta_n})|^p d\theta_1 \cdots d\theta_n$. Combining the above computations, we obtain

$$\left\| \prod_{j=1}^{n} (1 - |z_j|^2) \left[z_1 \cdots z_n \frac{\partial^n F(z)}{\partial z_1 \cdots \partial z_n} \right] \right\|_{L^p(dV_{\vec{\alpha}})} \le 2^{2n} \|f\|_{\mathcal{A}^p(dV_{\vec{\alpha}})}.$$

But the left hand side of the above inequality is equivalent to

$$z_1 \cdots z_n \frac{\partial^n F(z)}{\partial z_1 \cdots \partial z_n} \in \mathcal{A}^p(dV_{\vec{\alpha} + \mathbf{p}})$$

with $\vec{\alpha}+\mathbf{p}=(\alpha_1+p,\dots,\alpha_n+p)$. By Lemma 2.1, there exists a universal constant C such that

$$\left\|\prod_{j=1}^{n}(1-|z_j|^2)\frac{\partial^n F(z)}{\partial z_1\cdots\partial z_n}\right\|_{L^p(dV_{\vec{\alpha}})}$$

$$=\left\|\frac{\partial^n F(z)}{\partial z_1\cdots\partial z_n}\right\|_{\mathcal{A}^p(dV_{\vec{\alpha}+\mathbf{p}})}\le C'\left\|z_1\cdots z_n\frac{\partial^n F(z)}{\partial z_1\cdots\partial z_n}\right\|_{\mathcal{A}^p(dV_{\vec{\alpha}+\mathbf{p}})}$$

$$=C'\left\|\prod_{j=1}^{n}(1-|z_j|^2)z_1\cdots z_n\frac{\partial^n F(z)}{\partial z_1\cdots\partial z_n}\right\|_{L^p(dV_{\vec{\alpha}})}.$$

It follows that

$$\left\|\prod_{j=1}^{n}(1-|z_j|^2)\frac{\partial^n F(z)}{\partial z_1\cdots\partial z_n}\right\|_{L^p(dV_{\vec{\alpha}})}\le c\|f\|_{\mathcal{A}^p(dV_{\vec{\alpha}})}.$$

Since $F(0)=f(0)$, by Theorem 1.8, we have

$$\begin{aligned}\|F\|_{\mathcal{A}^p(dV_{\vec{\alpha}})}&=\|\mathcal{C}(f)\|_{\mathcal{A}^p(dV_{\vec{\alpha}})}\\&\le c\left(|f(0)|+\left\|\prod_{j=1}^{n}(1-|z_j|^2)\frac{\partial^n F(z)}{\partial z_1\cdots\partial z_n}\right\|_{L^p(dV_{\vec{\alpha}})}\right)\\&\le c'\|f\|_{\mathcal{A}^p(dV_{\vec{\alpha}})}.\end{aligned}$$

Here c and c' are universal constants depending on p and α only.

Case 2. $0<p<1$. Without loss of generality, we may just assume $n=2$. Let $f\in\mathcal{A}^p(dV_{\vec{\alpha}})$ and $F=\mathcal{C}(f)$. Suppose that $1<q<\frac{1}{1-p}$ and q' is the conjugate exponent of q, *i.e.*, $\frac{1}{q}+\frac{1}{q'}=1$. Then by Lemma 2.2 and Hölder's inequality, we have

$$\begin{aligned}&\int_{[-\pi,\pi]^2}\left|\frac{f(t_1r_1e^{i\theta_1},t_2r_2e^{i\theta_2})}{(1-t_1r_1e^{i\theta_1})(1-t_2r_2e^{i\theta_2})}\right|^p d\theta_1 d\theta_2 \qquad (2.2)\\&\le\left[\int_{[-\pi,\pi]^2}\left|\frac{1}{(1-t_1r_1e^{i\theta_1})(1-t_2r_2e^{i\theta_2})}\right|^{pq'}d\theta_1 d\theta_2\right]^{\frac{1}{q'}}\times\end{aligned}$$

$$\times \left[\int_{[-\pi,\pi]^2} \left| f(t_1 r_1 e^{i\theta_1}, t_2 r_2 e^{i\theta_2}) \right|^{pq} d\theta_1 d\theta_2 \right]^{\frac{1}{q}}$$

$$\leq C_\gamma \prod_{j=1}^{2} (1-t_j)^{\frac{1-pq'}{q'}} \left[\int_{[-\pi,\pi]^2} \left| f(t_1 r_1 e^{i\theta_1}, t_2 r_2 e^{i\theta_2}) \right|^{pq} d\theta_1 d\theta_2 \right]^{\frac{1}{q}}.$$

Now let us consider a partition on the unit interval $[0,1]$ with $\lambda_j = 1 - 2^{-j}$ and $\lambda_k = 1 - 2^{-k}$ for $j, k \in \mathbb{Z}_+$. Then we obtain

$$(2.3) \quad \int_{[-\pi,\pi]^2} |F(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})|^p d\theta_1 d\theta_2$$

$$\leq \int_{[-\pi,\pi]^2} \left\{ \int_{[0,1]^2} \left| \frac{f(t_1 r_1 e^{i\theta_1}, t_2 r_2 e^{i\theta_2})}{(1 - t_1 r_1 e^{i\theta_1})(1 - t_2 r_2 e^{i\theta_2})} \right| dt_1 dt_2 \right\}^p d\theta_1 d\theta_2$$

$$\leq \sum_{j,k=1}^{\infty} \int_{[-\pi,\pi]^2} \left\{ \int_{\lambda_{k-1}}^{\lambda_k} \int_{\lambda_{j-1}}^{\lambda_j} \left| \frac{f(t_1 r_1 e^{i\theta_1}, t_2 r_2 e^{i\theta_2})}{(1 - t_1 r_1 e^{i\theta_1})(1 - t_2 r_2 e^{i\theta_2})} \right| dt_1 dt_2 \right\}^p d\theta_1 d\theta_2$$

$$\leq \sum_{j,k=1}^{\infty} \frac{1}{2^{(j+k)p}} \times$$

$$\times \int_{[-\pi,\pi]^2} \left\{ \sup_{0 \leq t_1 \leq \lambda_j, 0 \leq t_2 \leq \lambda_k} \left| \frac{f(t_1 r_1 e^{i\theta_1}, t_2 r_2 e^{i\theta_2})}{(1 - t_1 r_1 e^{i\theta_1})(1 - t_2 r_2 e^{i\theta_2})} \right|^p \right\} d\theta_1 d\theta_2.$$

Next let us analyse the last line of (2.3). By the Hardy-Littlewood inequality, we know that

$$(2.4) \quad \int_{[-\pi,\pi]^2} \left\{ \sup_{0 \leq t_1 \leq \lambda_j, 0 \leq t_2 \leq \lambda_k} \left| \frac{f(t_1 r_1 e^{i\theta_1}, t_2 r_2 e^{i\theta_2})}{(1 - t_1 r_1 e^{i\theta_1})(1 - t_2 r_2 e^{i\theta_2})} \right|^p \right\} d\theta_1 d\theta_2$$

$$\leq \int_{[-\pi,\pi]^2} \left\{ \sup_{0 \leq t_1 < \lambda_j, 0 \leq t_2 < \lambda_k} \left| \frac{f(t_1 r_1 e^{i\theta_1}, t_2 r_2 e^{i\theta_2})}{(1 - t_1 r_1 e^{i\theta_1})(1 - t_2 r_2 e^{i\theta_2})} \right|^p + \right.$$

$$\left. + \left| \frac{f(\lambda_j r_1 e^{i\theta_1}, \lambda_k r_2 e^{i\theta_2})}{(1 - \lambda_j r_1 e^{i\theta_1})(1 - \lambda_k r_2 e^{i\theta_2})} \right|^p \right\} d\theta_1 d\theta_2$$

$$\leq (C_1 + 1) \int_{[-\pi,\pi]^2} \left| \frac{f(\lambda_j r_1 e^{i\theta_1}, \lambda_k r_2 e^{i\theta_2})}{(1 - \lambda_j r_1 e^{i\theta_1})(1 - \lambda_k r_2 e^{i\theta_2})} \right|^p d\theta_1 d\theta_2.$$

For $\lambda_{j-1} \le t_1 < 1$ and $\lambda_{k-1} \le t_2 < 1$, we use a similar trick to obtain

$$\begin{aligned}(2.5)\quad &\int_{[-\pi,\pi]^2}\left|\frac{f(\lambda_{j-1}r_1e^{i\theta_1},\lambda_{k-1}r_2e^{i\theta_2})}{(1-\lambda_{j-1}r_1e^{i\theta_1})(1-\lambda_{k-1}r_2e^{i\theta_2})}\right|^p d\theta_1 d\theta_2\\ &\le \int_{[-\pi,\pi]^2}\left\{\sup_{0\le\rho_1<t_1,0\le\rho_2<t_2}\left|\frac{f(\rho_1r_1e^{i\theta_1},\rho_2r_2e^{i\theta_2})}{(1-\rho_1r_1e^{i\theta_1})(1-\rho_2r_2e^{i\theta_2})}\right|^p\right\} d\theta_1 d\theta_2\\ &\le (C_1+1)\int_{[-\pi,\pi]^2}\left|\frac{f(t_1r_1e^{i\theta_1},t_2r_2e^{i\theta_2})}{(1-t_1r_1e^{i\theta_1})(1-t_2r_2e^{i\theta_2})}\right|^p d\theta_1 d\theta_2.\end{aligned}$$

Combining (2.4) and (2.5), we get the following

$$\begin{aligned}(2.6)\quad &\sum_{j,k=1}^{\infty}\frac{1}{2^{(j+k)p}}\times\\ &\times\int_{[-\pi,\pi]^2}\left\{\sup_{0\le t_1\le\lambda_j,0\le t_2\le\lambda_k}\left|\frac{f(t_1r_1e^{i\theta_1},t_2r_2e^{i\theta_2})}{(1-t_1r_1e^{i\theta_1})(1-t_2r_2e^{i\theta_2})}\right|^p\right\} d\theta_1 d\theta_2\\ &\le (C_1+1)\sum_{j,k=1}^{\infty}\frac{1}{2^{(j+k)p}}\int_{[-\pi,\pi]^2}\left|\frac{f(\lambda_jr_1e^{i\theta_1},\lambda_kr_2e^{i\theta_2})}{(1-\lambda_jr_1e^{i\theta_1})(1-\lambda_kr_2e^{i\theta_2})}\right|^p d\theta_1 d\theta_2.\end{aligned}$$

In fact, (2.6) can be estimates as follows:

$$\begin{aligned}(2.6) \le\ & 2^2(C_1+1)\sum_{j,k=1}^{\infty}(1-\lambda_j)^{p-1}(1-\lambda_k)^{p-1}\times\\ &\times\left\{\int_{[-\pi,\pi]^2}\left|\frac{f(\lambda_jr_1e^{i\theta_1},\lambda_kr_2e^{i\theta_2})}{(1-\lambda_jr_1e^{i\theta_1})(1-\lambda_kr_2e^{i\theta_2})}\right|^p d\theta_1 d\theta_2\right\}\times\\ &\qquad\times(\lambda_{j+1}-\lambda_j)(\lambda_{k+1}-\lambda_k)\\ \le\ & 2^4(C_1+1)\sum_{j,k=1}^{\infty}(1-\lambda_j)^{p-1}(1-\lambda_k)^{p-1}\times\\ &\times\left\{\int_{[-\pi,\pi]^2}\left|\frac{f(\lambda_jr_1e^{i\theta_1},\lambda_kr_2e^{i\theta_2})}{(1-\lambda_jr_1e^{i\theta_1})(1-\lambda_kr_2e^{i\theta_2})}\right|^p d\theta_1 d\theta_2\right\}\times\\ &\qquad\times(\lambda_j-\lambda_{j-1})(\lambda_k-\lambda_{k-1})\end{aligned}$$

$$\leq 2^4(C_1+1)^2 \sum_{j,k=1}^{\infty} \int_{\lambda_{j-1}}^{\lambda_j} \int_{\lambda_{k-1}}^{\lambda_k} \prod_{\ell=1}^{2} (1-t_\ell)^{p-1} \times$$

$$\times \left\{ \int_{[-\pi,\pi]^2} \left| \frac{f(t_1 r_1 e^{i\theta_1}, t_2 r_2 e^{i\theta_2})}{(1-t_1 r_1 e^{i\theta_1})(1-t_2 r_2 e^{i\theta_2})} \right|^p d\theta_1 d\theta_2 \right\} dt_1 dt_2.$$

This is actually bounded by the following:

$$C_\gamma 2^4 (C_1+1)^2 \int_{[0,1]^2} \prod_{\ell=1}^{2} (1-t_\ell)^{p-1} (1-t_\ell)^{\frac{1-pq'}{q'}} \times$$

$$\times \left[\int_{[-\pi,\pi]^2} |f(t_1 r_1 e^{i\theta_1}, t_2 r_2 e^{i\theta_2})|^{pq} d\theta_1 d\theta_2 \right]^{\frac{1}{q}} dt_1 dt_2$$

$$\leq C_3 \int_{[0,1]^2} \prod_{\ell=1}^{2} (1-t_\ell)^{-\frac{1}{q}} \left[\int_{[-\pi,\pi]^2} |f(t_1 r_1 e^{i\theta_1}, t_2 r_2 e^{i\theta_2})|^{pq} d\theta_1 d\theta_2 \right]^{\frac{1}{q}} dt_1 dt_2$$

$$\leq C_4 \int_{[-\pi,\pi]^2} |f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})|^p d\theta_1 d\theta_2.$$

Here we use (2.2) and Lemma 2.3. It follows that

$$\|\mathcal{C}(f)\|_{\mathcal{A}^p(dV_{\vec{\alpha}})} \leq C \cdot \|f\|_{\mathcal{A}^p(dV_{\vec{\alpha}})}.$$

The proof of this theorem is therefore complete. □

We next consider a Cesàro operator on $\mathcal{A}^p(dV_\alpha)$, which we define below. Let f be a holomorphic function defined on the unit ball B. Assume that $f(z) = \sum_{|\mathbf{k}|=0}^{\infty} a_{\mathbf{k}} z^{\mathbf{k}}$. For $\ell \in \mathbb{Z}_+$, let $F_\ell(z) = \sum_{|\mathbf{k}|=\ell} a_{\mathbf{k}} z^{\mathbf{k}}$. It follows that F_ℓ is homogeneous of degree ℓ, and the power series can be rewritten as the *homogeneous expansion* as follows:

$$f(z) = \sum_{\ell=0}^{\infty} F_\ell(z).$$

Now fix a point $\zeta \in \partial B$, then

$$f(z) = f(\zeta \cdot \xi) = f_\zeta(\xi) = \sum_{\ell=0}^{\infty} F_\ell(\zeta) \xi^\ell$$

for $\xi \in \mathbb{D}$. It has been shown that the infinite series $\sum_{\ell=0}^{\infty} F_\ell(\zeta)\xi^\ell$ converges uniformly to $f(z)$ on every compact subset in $\mathbb{D}$ (see Rudin [R, pages 19–22]). It is obvious that $|\xi| = |\zeta \cdot \xi| = |z| = r$. We define the *"slice Cesàro operator"* as follows:

$$\mathcal{C}_s(f)(z) = \mathcal{C}_s(f_\zeta)(\xi) = \sum_{k=0}^{\infty} \left[\frac{1}{k+1} \sum_{\ell=0}^{k} F_\ell(\zeta) \right] \xi^k.$$

It is easy to see that

$$\mathcal{C}_s(f)(z) = \int_0^1 \frac{f_\zeta(t\xi)}{(1-t\xi)} dt.$$

An argument similar to the one above can be used to prove the following theorem:

THEOREM 2.5. *Cesàro operator $\mathcal{C}_s$ is bounded on $\mathcal{A}^p(dV_\alpha)$ for $0 < p < \infty$.*

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