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A REMARK ON THE THEOREM OF OHSAWA-TAKEGOSHI

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§1. Introduction and main result

If $D \subset \mathbb{C}^n$ is a pseudoconvex domain and $X \subset D$ a closed analytic subset, the famous theorem B of Cartan-Serre asserts, that the restriction operator $r : \mathcal{O}(D) \longrightarrow \mathcal{O}(X)$ mapping each function F to its restriction $F|X$ is surjective. A very important question of modern complex analysis is to ask what happens to this result if certain growth conditions for the holomorphic functions on D and on X are added. If the L^2-norm with respect to the Lebesgue-measure and a plurisubharmonic weight function is taken as growth condition, then the Cartan-Serre extension has the following analogue:

THEOREM 1.1. (Ohsawa-Takegoshi [3]) *Let $D \subset\subset \mathbb{C}^n$ be a bounded pseudoconvex domain, $H \subset \mathbb{C}^n$ a complex affine hyperplane with $D' := D \cap H \neq \emptyset$ and $\varphi : D \longrightarrow \mathbb{R} \cup \{-\infty\}$ a plurisubharmonic function. Then there is a constant $C > 0$, depending only on the diameter of D, such that for each function f holomorphic on D' satisfying the growth condition*

$$\int_{D'} |f|^2 e^{-\varphi}\, dV_{n-1} < \infty,$$

where dV_{n-1} denotes the Lebesgue-measure on $X \cong \mathbb{R}^{2n-2}$ there is a holomorphic function F on D such that $r(F) = F|D' = f$ and

$$\int_D |F|^2 e^{-\varphi}\, dV_n \leq C \int_{D'} |f|^2 e^{-\varphi}\, dV_{n-1}.$$

This theorem has, meanwhile, found a lot of applications in complex analysis and in algebraic geometry. Therefore, it is important to ask what kind of generalizations are possible. We mention the work of L. Manivel [2]

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who treats the extension problem for certain sections of suitable holomorphic vector bundles. More directly, it can be observed, that the Theorem carries over to the case, where H is replaced by a closed complex analytic subvariety $X = \{z \in U : h(z) = 0\}$ of an open neighborhood U of $\overline{D}$ given by a holomorphic function h on U, such that on $X \cap U \cap \overline{D}$ there is at most a finite number of points z, all lying in D, with $\partial h(z) = 0$. (Hence X may have singularities, but only inside a compact subset of D.)

In this article we will show, that there is, however, no general Ohsawa-Takegoshi theorem for algebraic complex hypersurfaces of $\mathbb{C}^n$, even not if they are algebraic principal divisors intersecting ∂D transversally and if D is strictly pseudoconvex.

In order to formulate our result more precisely, we denote by B_n the unit ball in $\mathbb{C}^n$, by $\mathcal{O}(B_n)$ the algebra of holomorphic functions on B_n and by $H^2(B_n) := \mathcal{O}(B_n) \cap L^2(B_n)$ the Hilbert space of square-integrable (with respect to the Lebesgue measure) holomorphic functions on B_n. For a complex hypersurface $X \subset \mathbb{C}^n$ with $X \cap B_n \neq \emptyset$ we mean by $H^\infty(B_n)$ the space of bounded holomorphic functions on B_n. We will show

Theorem 1.2. *There is an irreducible algebraic complex hypersurface X in $\mathbb{C}^3$ (with singularities) of the form $X = \{z \in \mathbb{C}^3 : h(z) = 0\}$ for a polynomial h such that $X \cap B_3 \neq \emptyset$ and $\dim_{\mathbb{C}}(T_z^{\mathbb{C}}(\partial B_3) \cap T_z X) = 1$ (here $T_z X$ denotes the tangent cone to X at z) and a function $f \in H^\infty(X \cap B_3)$, such that f has no holomorphic extension F to B_3 belonging to $H^2(B_3)$.*

Before we come to the proof of this theorem we remark that the techniques for constructing the desired counterexample are very close to those used in [1] for constructing smooth hypersurfaces in pseudoconvex domains with very astonishing behavior with respect to the extension of holomorphic functions.

§2. Some auxiliary facts

For the convenience of the reader we give here at first the proof of a classical lemma needed later. For this we denote

$$S_n(\varepsilon) := \left\{ z \in \mathbb{C}^n : |z_1| = \varepsilon^{1/2}, z_2 = 0, \dots, z_{n-1} = 0, z_n = 1 - \varepsilon \right\}.$$

For $\varepsilon > 0$ sufficiently small, one obviously has $S_n(\varepsilon) \subset B_n$. The following estimate holds:

LEMMA 2.1. *For $F \in H^2(B_n)$ and all $z \in S_n(\varepsilon)$, one has*

$$|F(z)| \leq c\|F\|_{H^2(B_n)}\varepsilon^{-(n+1)/2} \tag{2.1}$$

with a universal constant $c > 0$.

Proof. Notice that there are positive constants $a_1, \ldots, a_n$ such that for $\varepsilon > 0$ sufficiently small the polydisc around any point $z \in S_n(\varepsilon)$ given by

$$P(z) := \Big\{\zeta \in \mathbb{C}^n : |\zeta_1 - z_1| \leq a_1\varepsilon^{1/2}, \ldots, |\zeta_{n-1} - z_{n-1}| \leq a_{n-1}\varepsilon^{1/2}, |\zeta_n - z_n| \leq a_n\varepsilon\Big\}$$

is contained in B_n. Applying the Cauchy estimates to it we obtain immediately

$$\begin{aligned} |F(z)| &\leq \frac{1}{\mathrm{vol}\,(P(z))}\int_{P(z)} |F(\zeta)|\, d\lambda(\zeta) \\ &\leq \frac{1}{\mathrm{vol}\,(P(z))}\left(\int_{P(z)} |F(\zeta)|^2\, d\lambda(\zeta)\right)^{1/2}\left(\int_{P(z)} d\lambda(\zeta)\right)^{1/2} \\ &\leq \left(\frac{1}{\mathrm{vol}\,(P(z))}\right)^{1/2}\|F\|_{H^2(B_n)} \\ &\leq c\|F\|_{H^2(B_n)}\varepsilon^{-(n+1)/2}. \end{aligned}$$

This proves the Lemma. $\square$

In the next proposition we construct the desired hypersurface X in $\mathbb{C}^3$ and the crucial holomorphic function f on $X \cap B_3$.

PROPOSITION 2.2. *We put $X := \{z = (z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^2 + z_2^q = 0\}$ for any fixed uneven integer $q > 3$ and define the holomorphic function f on B_3 by*

$$f(z) := \frac{z_1}{(1 - z_3)^{q/4}}. \tag{2.2}$$

Then f is bounded on $X' := X \cap B_3$ and, if $\theta > 0$ is any given constant and $\varepsilon > 0$ is sufficiently small (independently of the choice of θ), then f does not have a holomorphic extension F from X' to B_3 satisfying the estimate

$$|F(z)| \leq c\varepsilon^{1/2+\theta-q/4}, \tag{2.3}$$

for all $z \in S_3(\varepsilon)$ and any constant $c > 0$.

Proof. For all $z \in B_3$ we have $|1 - z_3| \geq c(|z_1|^2 + |z_2|^2)$. Furthermore, $|z_1| = |z_2|^{q/2}$ on X'. Hence f is bounded on X'.

In order to show the second part of the Proposition, we argue by contradiction. Let us suppose for some holomorphic extension F of $f|X'$ to B_3 there exists a constant $\theta > 0$ such that for all $z \in S_3(\varepsilon)$ with $\varepsilon > 0$ small enough and a suitable constant $c > 0$ inequality (2.3) holds. Since the function $z \longmapsto z_1^2 + z_2^q$ is irreducible, the function F can be written in the form

$$F(z) = \frac{1}{(1 - z_3)^{q/4}} \big(z_1 + \big(z_1^2 + z_2^q\big) g(z)\big), \tag{2.4}$$

with a holomorphic function g on B_3. Then it follows from (2.3) that g verifies for each z_1 with $|z_1| = \varepsilon^{1/2}$ the inequality

$$\left| \frac{1}{z_1} + g(z_1, 0, 1 - \varepsilon) \right| \leq c\varepsilon^{\theta - 1/2}. \tag{2.5}$$

Consequently,

$$\int_{|z_1| = \varepsilon^{1/2}} \left(\frac{1}{z_1} + g(z_1, 0, 1 - \varepsilon) \right) dz_1 = O\Big(\varepsilon^{\theta}\Big). \tag{2.6}$$

However,

$$\int_{|z_1| = \varepsilon^{1/2}} g(z_1, 0, 1 - \varepsilon)\, dz_1 = 0, \tag{2.7}$$

since g is holomorphic on B_3, and, hence,

$$2\pi i = \int_{|z_1| = \varepsilon^{1/2}} \frac{1}{z_1}\, dz_1 = \int_{|z_1| = \varepsilon^{1/2}} \left(\frac{1}{z_1} + g(z_1, 0, 1 - \varepsilon) \right) dz_1. \tag{2.8}$$

The equations (2.6) and (2.8) obviously contradict each other. ∎

§3. Proof of Theorem 1.1

Let X and f be as in Proposition 2.2. We may assume, that the uneven integer $q > 3$ has been chosen such that $1/2 - q/4 < -2$. A simple calculation shows that the transversality condition for X at ∂B_3 required in Theorem 1.1 is satisfied. If now f would have a holomorphic extension $F \in H^2(B_3)$, then, according to Lemma 2.1, we would have for any point $z \in S_3(\varepsilon)$ (with ε sufficiently small) the inequality

$$|F(z)| \leq c \|F\|_{H^2(B_3)} \varepsilon^{-2}.$$

Because of the choice of q such that $1/2 - q/4 < -2.$ and Proposition 2.2 this is, however, impossible. ∎

References

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