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TWO-WEIGHTED INEQUALITIES FOR THE DERIVATIVES OF HOLOMORPHIC FUNCTIONS AND CARLESON MEASURES ON THE UNIT BALL

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Abstract. We characterize those positive measure μ's on the higher dimensional unit ball such that "two-weighted inequalities" hold for holomorphic functions and their derivatives. Characterizations are given in terms of the Carleson measure conditions. The results of this paper also distinguish between the fractional and the tangential derivatives.

§1. Introduction and statements of results

Let U be the open unit disc in $\mathbb{C}$ with the boundary ∂U. In relation to the embedding of Hardy spaces into various Sobolev spaces of holomorphic functions, the following problem has been extensively studied and completely resolved [C], [D], [L1], [L2], [Sh1], [Sh2]: Characterize those positive measures μ on U such that the inequality

$$\left(\int_U |f^{(k)}(z)|^q \, d\mu(z) \right)^{1/q} \lesssim \|f\|_{H^p} \quad \text{for any } f \in H^p \tag{1.1}$$

holds. Such characterizations of the measure μ are given in terms of the Carleson measure type criterion: If either $p = q \geq 2$ or $0 < p < q < \infty$, then the inequality (1.1) holds if and only if

$$\mu(\hat{I}) \lesssim \ell(I)^{(1+kp)q/p} \tag{1.2}$$

for any arc $I \subset \partial U$. Here $\hat{I}$ is the tent over I, i.e.,

$$\hat{I} = \{re^{i\theta} : |1 - l(I)/2| \leq r < 1, \, e^{i\theta} \in I\}$$

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and $l(I)$ denotes the length of I. In the other cases, namely, $0 < p = q < 2$ or $0 < q < p$, the characterization is given in terms of the so-called "balayées" [L2].

This paper is concerned with the problem of characterizing those μ's such that the two-weighted inequality holds, namely,

$$(1.3) \qquad \left(\int_U |f^{(k)}(z)|^q \, d\mu(z) \right)^{1/q} \lesssim \|f\|_{H^p(\omega)} \quad \text{for any } f \in H^p(\omega)$$

where ω is an $\mathcal{A}_p$-weight of Muckenhoupt and $H^p(\omega)$ is the Hardy space weighted by ω. As far as we are aware of, [G] was the first paper where this problem was considered. If $k = 0$ and either $p = q \geq 2$ or $1 < p < q < \infty$, Gu showed that the inequality (1.3) holds if and only if

$$(1.4) \qquad \mu(\hat{I}) \lesssim \omega(I)^{q/p}$$

for any arc $I \subset \partial U$. Here $\omega(I) = \int_I d\omega$. When $k \geq 1$, only partial results are known. Girela, Lorente, and Sarrion found a necessary condition for (1.3) to hold when $p = q \geq 1$. When $p = q \geq 2$, they also found a sufficient condition for (1.3) to hold when $\omega(\theta) = |\theta|^\alpha$ $(-1 < \alpha < p - 1)$ which is an $\mathcal{A}_p$-weight on ∂U [GLS].

In this paper we give a complete characterization of a positive measure μ such that the inequality (1.3) holds when $k \geq 0$ and either $p = q \geq 2$ or $1 < p < q < \infty$. Moreover we consider the problem on the higher dimensional balls. In the higher dimensional balls, the tangential derivative and the normal derivative of holomorphic functions behave differently. In fact, the tangential derivative is half order better than the normal derivative. The results of this paper comply with this phenomenon and distinguish between the normal and the tangential derivatives. Let $\mathcal{R}^\alpha f$ be the fractional derivative of order α and $\nabla_T^k f$ be the tangential derivative of order k. Main results of this paper are as follows.

Theorem A. *Let μ be a positive Borel measure on the unit ball $B^n \subset \mathbb{C}^n$ and $\omega \in \mathcal{A}_p$ on ∂B^n. Assume either $p = q \geq 2$ or $1 < p < q < \infty$ and $\alpha \geq 0$. Then*

$$(1.5) \qquad \left(\int_{B^n} |\mathcal{R}^\alpha f(z)|^q \, d\mu(z) \right)^{1/q} \lesssim \|f\|_{H^p(\omega)} \quad \textit{for all } f \in H^p(\omega)$$

if and only if

$$(1.6) \qquad \mu(\hat{B}) \lesssim \omega(B)^{q/p} \sigma(B)^{q\alpha/n} \quad \textit{for all Koranyi ball } B \subset \partial B^n,$$

where σ denotes the surface measure on ∂B^n.

For the tangential derivative, we have the following theorem:

THEOREM B. *Let μ and ω be as above. Assume either $p = q \geq 2$ or $1 < p < q < \infty$ and k is a nonnegative integer. Then,*

$$\left(\int_{B^n} |\nabla_T^k f(z)|^q \, d\mu(z) \right)^{1/q} \lesssim \|f\|_{H^p(\omega)} \quad \text{for all } f \in H^p(\omega) \tag{1.7}$$

if and only if

$$\mu(\hat{B}) \lesssim \omega(B)^{q/p} \sigma(B)^{qk/2n} \quad \text{for all Koranyi ball } B \subset \partial B^n. \tag{1.8}$$

This paper organizes as follows: In Section 2, we define necessary terminologies and prove some preliminary lemmas. In Section 3, we prove an integral inequality with weights on fractional derivatives. With help of this inequality we are able to reduce Theorem A to the radial derivative case. In Section 4, we give pointwise estimates for the gradient and the tangential derivatives of holomorphic functions which are necessary to prove Theorem B. Proofs of Theorems A and B are given in Section 5. For the convenience of readers, we include a proof of the L^p boundedness with weight of the area integrals in Section 6.

Throughout this paper "$\alpha \lesssim \beta$" implies that there exists a constant C such that $\alpha \leq C\beta$. Also we write "$\alpha \approx \beta$" if $\alpha \lesssim \beta$ and $\beta \lesssim \alpha$. The constant C may depend on some parameters such as p, q and k, but it will be always independent of the particular functions, measures, or points, etc.

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§2. Preliminary lemmas

Let us fix some notations. B^n denotes the unit ball in $\mathbb{C}^n$ and ∂B^n its boundary. For $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_n) \in \overline{B^n}$, let $\langle z, w \rangle := z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$, $|z| := \langle z, z \rangle^{1/2}$, $r(z) := 1 - |z|^2$, and $\pi(z) := z/|z|$ the normal projection from $B^n \setminus \{0\}$ onto ∂B^n. For $\zeta \in \partial B^n$ and $\delta > 0$, the Koranyi ball $B(\zeta, \delta)$ is defined by

$$B(\zeta, \delta) := \{\eta \in \partial B^n : |1 - \langle \zeta, \eta \rangle| < \delta\}.$$

Note that the distance $|1 - \langle \zeta, \eta \rangle|$ satisfies the pseudo-triangular inequality:

$$|1 - \langle \zeta, \eta \rangle| \leq 2\big(|1 - \langle \zeta, \xi \rangle| + |1 - \langle \xi, \eta \rangle|\big) \quad \text{for } \zeta, \eta, \xi \in \partial B^n.$$

The tent space $\hat{B}(\zeta,\delta)$ over $B(\zeta,\delta)$ is defined by

$$\hat{B}(\zeta,\delta) := \{z \in B^n \setminus \{0\} : \pi(z) \in B(\zeta,\delta),\, r(z) < \delta\}.$$

For $z \in B^n \setminus \{0\}$ and $c > 0$, let $B(z) = B(\pi(z), r(z))$ and $cB(z) = B(\pi(z), cr(z))$. For $|z| > 1/2$ and a small positive number ϵ which is fixed once and for all, we define

$$P_\epsilon(z) := \{w \in B^n : \pi(w) \in B(\pi(z), \epsilon r(z)),\, |r(z) - r(w)| \le \epsilon r(z)\}.$$

And for $|z| \le 1/2$, let $P_\epsilon(z) = \{w \in B^n : |z - w| < \epsilon\}$. Then for $|z| > 1/2$, $P_\epsilon(z)$ is a twisted polydisc in B^n centered at z and of size $\epsilon r(z)$ in the normal direction and $\sqrt{\epsilon r(z)}$ in the complex tangential direction. The following lemma collects some relevant facts on the Koranyi ball and polydiscs. They are probably well known. However, we include brief proofs of them for readers' convenience.

Lemma 2.1. *For $z \in \mathbb{C}^n \setminus \{0\}$, let $T(z) := \{\eta \in \mathbb{C}^n : |\eta| = 1,\, \langle z, \eta\rangle = 0\}$, the set of unit vectors normal to z. Let $0 < \delta < 1/2$ and ϵ be a small positive number.*

(i) *For $\zeta \in \partial B^n$,*

$$B(\zeta,\delta) = \{w = (1+t)\zeta + s\eta : |t| < \delta,\, 1 = |1+t|^2 + |s|^2,\, t, s \in \mathbb{C},\, \eta \in T(\zeta)\}. \tag{2.1}$$

(ii) *If $w \in P_\epsilon(z)$ for $|z| > 1/2$, then $B(\pi(w), r(w)) \subset B(\pi(z), 3r(z))$.*

(iii) $\hat{B}(\zeta,\delta) \subset \{w = (1+t)\zeta + s\eta \in B^n : \eta \in T(\zeta),\, |t| < 2\delta,\, |s| < \sqrt{2\delta}\}$.

(iv) *Let $Q_\epsilon(z) := \{w = (1+t)z + s\eta \in B^n : \eta \in T(z),\, |t| < \epsilon r(z),\, |s| < \sqrt{\epsilon r(z)}\}$. There exists a constant $c > 0$ and $C > 0$ such that for all $z \in B^n$*

$$P_\epsilon(z) \subset Q_{c\epsilon}(z) \subset P_{C\epsilon}(z), \tag{2.2}$$

and for all $w \in P_\epsilon(z)$

$$P_\epsilon(z) \subset P_{C\epsilon}(w). \tag{2.3}$$

(v) *For any holomorphic function f in B^n,*

$$|f(z)| \lesssim \frac{1}{|P_\epsilon(z)|} \int_{P_\epsilon(z)} |f(w)|\, dm(w), \tag{2.4}$$

where m is the Lebesgue measure on $\mathbb{C}^n$ and $|P_\epsilon(z)| = m(P_\epsilon(z))$.

Proof. Let $w \in B(\zeta, \delta)$ and write $w = (1+t)\zeta + s\eta$ where $t, s \in \mathbb{C}$ and $\eta \in T(\zeta)$. Then,

$$1 = |w|^2 = |1+t|^2 + |s|^2 \quad \text{and} \quad |t| = |1 - \langle \zeta, w \rangle| < \delta.$$

Hence, we have (2.1). It is easy to see (ii).

To prove (iii), note that in (i)

$$|s|^2 = 1 - |1+t|^2 < 2|t| < 2\delta.$$

If $z \in \hat{B}(\zeta, \delta)$, then $z/|z| \in B(\zeta, \delta)$ and $1 - |z| < \delta$. Hence, by (i) there exists $\eta \in T(\zeta)$ and $t, s \in \mathbb{C}$ with $|t| < \delta$ and $|s| < \sqrt{2\delta}$ such that

$$z = |z|(1+t)\zeta + |z|s\eta = (1 + (|z| - 1 + |z|t))\zeta + |z|s\eta.$$

Since $|(|z| - 1 + |z|t)| < \delta + |t| < 2\delta$ and $|z||s| < \sqrt{2\delta}$, we have (iii).

If $|z| \le 1/2$, then (2.2) is trivial. If $|z| > 1/2$ and $w \in P_\epsilon(z)$, then by the definition of $P_\epsilon(z)$ and (2.1) we have

$$P_\epsilon(z) = \left\{ w = \frac{|w|(1+t)}{|z|} z + s|w|\eta : |t| < \epsilon r(z),\, 1 = |1+t|^2 + |s|^2, \right.$$
$$\left. |r(z) - r(w)| \le \epsilon r(z),\, t, s \in \mathbb{C},\, \eta \in T(z) \right\}$$

Now let $w \in P_\epsilon(z)$, $a = \left(-1 + \frac{|w|}{|z|} + \frac{t|w|}{|z|}\right)$, and $b = s|w|$, then

$$w = \frac{|w|(1+t)}{|z|} z + s|w|\eta = (1+a)z + b\eta.$$

Since $|r(w) - r(z)| \le \epsilon r(z)$, we have

$$\left| 1 - \frac{|w|}{|z|} \right| = \frac{|r(z) - r(w)|}{|z|(|z| + |w|)} \le \frac{\epsilon r(z)}{|z|(|z| + |w|)} < 4\epsilon r(z).$$

Hence, there exists a constant $c > 0$ such that $|a| < c\epsilon r(z)$ and $|b| < \sqrt{c\epsilon r(z)}$. Thus we have

$$P_\epsilon(z) \subset Q_{c\epsilon}(z).$$

Now let $w = (1+a)z + b\eta \in Q_{c\epsilon}(z)$ with $\langle z, \eta \rangle = 0$. Define t and s by $a = \left(-1 + \frac{|w|}{|z|} + \frac{t|w|}{|z|}\right)$ and $b = s|w|$. Then for some constant $C > 0$, $|t| < C\epsilon r(z)$ and $|r(z) - r(w)| \le C\epsilon r(z)$. Since $w = \frac{|w|}{|z|}(1+t)z + s|w|\eta$, it follows that $w \in P_{C\epsilon}(z)$. Hence $Q_{c\epsilon}(z) \subset P_{C\epsilon}(z)$.

(2.3) for $|z| \le 1/2$ is also trivial. If $|z| > 1/2$, it follows from the pseudo-triangular inequality of the distance $|1 - \langle \zeta, \eta \rangle|$.

For the submean value property (v), we may assume that $z = (\sqrt{1-\delta}, 0, \dots, 0)$ after a unitary change of coordinates if necessary. Choose a small constant c_1 so that $Q_{c_1\epsilon}(z) \subset P_\epsilon(z)$. It then follows from (2.2) and the submean value property of holomorphic functions over balls that

$$\begin{aligned} |f(z)| &\lesssim \frac{1}{(\epsilon\delta)^2} \int_{|w_1 - z_1| < c_1\epsilon\delta} |f(w_1, 0, \dots, 0)|\, dm(w_1) \\ &\lesssim \frac{1}{(\epsilon\delta)^{n+1}} \int_{Q_{c_1\epsilon}(z)} |f(w)|\, dm(w) \\ &\lesssim \frac{1}{|P_\epsilon(z)|} \int_{P_\epsilon(z)} |f(w)|\, dm(w). \end{aligned}$$

This completes the proof. □

Throughout this paper, ω is an $\mathcal{A}_p$-weight of Muckenhoupt. Note that an $\mathcal{A}_p$-weight ω has the doubling property, i.e., there exists a constant C_ω such that

$$\omega(B(\zeta, 2\delta)) \le C_\omega \omega(B(\zeta, \delta))$$

for all Koranyi ball $B(\zeta, \delta)$ where $\omega(B(\zeta, \delta)) = \int_{B(\zeta,\delta)} \omega(\eta)\, d\sigma(\eta)$.

Lemma 2.2. *For $1 \le A$ and $-n-1 < B$ let $d\nu(z) = \omega(B(z))^A r(z)^B\, dm(z)$, then*

$$\nu(\hat{B}(\zeta, \delta)) \lesssim \omega(B(\zeta, \delta))^A \delta^{B+n+1}.$$

Proof. Fix $\zeta \in \partial B^n$ and $\delta > 0$. For $\eta_{k,m} \in \partial B^n$ let $B_m^k := \{w \in B^n \setminus \{0\} : \delta/2^{k+1} \le r(w) < \delta/2^k,\ \pi(w) \in B(\eta_{k,m}, \delta/2^k)\}$. One can find points $\eta_{k,m} \in \overline{B(\zeta, \delta)}$ such that

$$\hat{B}(\zeta, \delta) \subset \bigcup_{k=0}^{\infty} \bigcup_{m=1}^{N_k} B_m^k$$

and

$$\sum_{m=1}^{N_k} \omega\Big(B\Big(\eta_{k,m}, \frac{\delta}{2^k}\Big)\Big) \lesssim \omega(B(\zeta, \delta)), \quad k = 0, 1, 2, \dots$$

where $\{N_k\}$ are some positive integers. Then

$$\begin{aligned}\int_{\hat{B}(\zeta,\delta)} d\nu &\leq \sum_{k=0}^{\infty}\sum_{m=1}^{N_k}\int_{B_m^k} \omega(B(z))^A r(z)^B\, dm(z)\\ &\lesssim \sum_{k=0}^{\infty}\sum_{m=1}^{N_k}\int_{B_m^k} \omega\Big(B\Big(\eta_{k,m},\frac{\delta}{2^{k-2}}\Big)\Big)^A r(z)^B\, dm(z)\\ &\lesssim \sum_{k=0}^{\infty}\sum_{m=1}^{N_k} \omega\Big(B\Big(\eta_{k,m},\frac{\delta}{2^{k}}\Big)\Big)^A (\delta/2^k)^{B+1}\sigma(\pi(B_m^k))\\ &\lesssim \sum_{k=0}^{\infty}(\delta/2^k)^{B+1+n}\left(\sum_{m=1}^{N_k}\omega(\pi(B_m^k))^A\right).\end{aligned}$$

From the fact $1 \leq A$ and the doubling property of ω we have

$$\sum_{m=1}^{N_k}\left(\frac{\omega(\pi(B_m^k))}{\omega(B(\zeta,\delta))}\right)^A \lesssim \sum_{m=1}^{N_k}\frac{\omega(\pi(B_m^k))}{\omega(B(\zeta,\delta))} \lesssim 1.$$

This completes the proof. $\square$

Lemma 2.3. *Let $1 \leq A$ and $-n-1 < B$. If a constant q satisfies $C_\omega^A < 2^{q-B-n-1}$, then we have, for $z \in B^n \setminus \{0\}$*

$$\int_{B^n}\frac{\omega(B(w))^A r(w)^B}{|1-\langle z,w\rangle|^q}\, dm(w) \lesssim r(z)^{-q+B+n+1}\omega(B(z))^A.$$

Proof. Fix $z \in B^n \setminus \{0\}$. Note that for $w \in \widehat{2^k B(z)} \setminus \widehat{2^{k-1}B}(z)$, $|1-\langle z,w\rangle| \approx 2^k r(z)$ for $k = 1, 2, \dots$ and for $w \in \widehat{B(z)}$, $|1-\langle z,w\rangle| \approx r(z)$. Hence, we get

$$\begin{aligned}&\int_{B^n}\frac{\omega(B(w))^A r(w)^B}{|1-\langle z,w\rangle|^q}\, dm(w)\\ &\quad\lesssim \int_{\widehat{B(z)}} r(z)^{-q}\omega(B(w))^A r(w)^B\, dm(w)\\ &\qquad + \sum_{k=1}^{\infty}\int_{\widehat{2^k B(z)}\setminus\widehat{2^{k-1}B}(z)} (2^k r(z))^{-q}\omega(B(w))^A r(w)^B\, dm(w)\\ &\quad\lesssim \sum_{k=0}^{\infty}(2^k r(z))^{-q}\int_{\widehat{2^k B(z)}} \omega(B(w))^A r(w)^B\, dm(w).\end{aligned}$$

Hence, from Lemma 2.2 we have

$$\begin{aligned}\int_{B^n} \frac{\omega(B(w))^A r(w)^B}{|1-\langle z,w\rangle|^q}\, dm(w) &\lesssim \sum_{k=0}^{\infty} (2^k r(z))^{-q} \omega(2^k B(z))^A (2^k r(z))^{B+n+1} \\ &\lesssim r(z)^{-q+B+n+1} \omega(B(z))^A \sum_{k=0}^{\infty} \frac{C_\omega^{Ak}}{2^{k(q-B-n-1)}} \\ &\lesssim r(z)^{-q+B+n+1} \omega(B(z))^A.\end{aligned}$$

This completes the proof. ∎

LEMMA 2.4. *For* $1 < A$, *let* $d\nu_A(z) = \omega(B(z))^A r(z)^{-n-1}\, dm(z)$, *then*

$$\nu_A(\hat{B}(\zeta,\delta)) \lesssim \omega(B(\zeta,\delta))^A.$$

Proof. Note that from the doubling property of ω, we have

$$\frac{\omega(B(z))}{\omega(B(\zeta,\delta))} \lesssim 1$$

for all $z \in \hat{B}(\zeta,\delta)$. Thus if $A \geq \alpha$, we have

$$\begin{aligned}\omega(B(\zeta,\delta))^{-A} \nu_A(\widehat{B(\zeta,\delta)}) &= \int_{\widehat{B(\zeta,\delta)}} \frac{\omega(B(z))^A}{\omega(B(\zeta,\delta))^A} r(z)^{-n-1}\, dm(z) \\ &\lesssim \omega(B(\zeta,\delta))^{-\alpha} \nu_\alpha(\widehat{B(\zeta,\delta)}).\end{aligned}$$

Thus if Lemma 2.4 holds for some α, then it holds for all $A \geq \alpha$. Since $\omega \in \mathcal{A}_p$, there is a constant $\beta > 1$ such that the following reverse Hölder inequality holds for all $1 \leq \alpha \leq \beta$;

$$|B(\zeta,\delta)|^{\alpha-1} \int_{B(\zeta,\delta)} \omega(\eta)^\alpha\, d\sigma(\eta) \lesssim \left(\int_{B(\zeta,\delta)} \omega(\eta)\, d\sigma(\eta) \right)^\alpha.$$

Here $|B(\zeta,\delta)| = \sigma(B(\zeta,\delta))$. Note that $|B(\zeta,\delta)| \approx \delta^n$. Then if $1 < \alpha < \beta$, we have from Hölder and reverse Hölder inequalities

$$\begin{aligned}&\int_{\widehat{B(\zeta,\delta)}} d\nu_\alpha(z) \\ &\quad\lesssim \int_0^\delta \int_{B(\zeta,\delta)} \left(|B(\eta,r)|^{\alpha-1} \int_{B(\eta,r)} \omega(x)^\alpha\, d\sigma(x) \right) r^{-n-1}\, d\sigma(\eta) dr\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\delta \left(\int_{B(\zeta,\delta)} \int_{B(\zeta,2\delta)} |B(\eta,r)|^{\alpha-1} \omega(x)^\alpha \mathcal{X}_{B(\eta,r)}(x)\, d\sigma(\eta) d\sigma(x) \right) \frac{dr}{r^{n+1}} \\
&\lesssim \int_0^\delta \left(\int_{B(\zeta,2\delta)} \omega(x)^\alpha \, d\sigma(x) \right) r^{n\alpha-n-1}\, dr \\
&\lesssim |B(\zeta,\delta)|^{\alpha-1} \int_{B(\zeta,2\delta)} \omega(x)^\alpha \, d\sigma(x) \\
&\lesssim \left(\int_{B(\zeta,\delta)} \omega(x)\, d\sigma(x) \right)^\alpha .
\end{aligned}
$$

This completes the proof. ∎

§3. Fractional derivatives

Let α be a real number. For a holomorphic function f on B^n with the homogeneous expansion $f(z) = \sum_{k=0}^\infty f_k(z)$, we define

$$\mathcal{R}^\alpha f(z) = \sum_{k=0}^\infty (1+k)^\alpha f_k(z).$$

When $\alpha = 1$, $\mathcal{R} = \mathcal{R}^1 = I + \sum_{j=1}^n z_j(\partial/\partial z_j)$. Notice that $\sum_{j=1}^n z_j(\partial/\partial z_j)$ is the normal differentiation operator. When $\alpha \geq 0$, $\mathcal{R}^\alpha f$ is called the radial fractional derivative of order α of the function f. From Lemma 2.3 we have the following weighted inequality for fractional derivatives. Its proof uses the ideas of that of the unweighted case in [BB].

LEMMA 3.1. *Let $\alpha > 0$, $A \geq 1$ and $p \geq 1$, then for a holomorphic function f on B^n*

$$
\begin{aligned}
&\int_{B^n} |\mathcal{R}^\alpha f(z)|^p \omega(B(z))^A r(z)^{p\alpha-n-1}\, dm(z) \\
&\qquad \lesssim \int_{B^n} |\mathcal{R} f(z)|^p \omega(B(z))^A r(z)^{p-n-1}\, dm(z).
\end{aligned}
$$

Proof. We quote some facts from [BB]. For real numbers a, b and $z \in U$, let

$$G_{a,b}(z) = \sum_{m=0}^\infty \frac{1}{m!} \frac{a(a+1)\cdots(a+m-1)}{(m+1)^b} z^m. \tag{3.1}$$

If $a > b$, then

$$|G_{a,b}(z)| \lesssim |1-z|^{-(a-b)}. \tag{3.2}$$

For $q > 0$, let $dV_q(w) = \frac{1}{\pi^n}\frac{\Gamma(n+q)}{\Gamma(q)}(1-|w|^2)^{q-1}\,dm(w)$. Then for a real number s and a homomorphic function f on B^n, we have

$$f(z) = \int_{B^n} \mathcal{R}^s f(w) G_{n+q,s}(\overline{\langle w, z\rangle})\,dV_q(w) \tag{3.3}$$

if $\int_{B^n} |\mathcal{R}^s f(w)|^2\,dV_{n+q}(w) < \infty$. If $s > n+q$ and $q \geq 0$, then

$$\int_{B^n} |1 - \langle z, w\rangle|^{-s}\,dV_q(w) \approx (1-|z|^2)^{n+q-s}. \tag{3.4}$$

Proofs of (3.2), (3.3), and (3.4) can be found in [BB].

Let $z \in B^n \setminus \{0\}$ and k be an integer so that $2^{-k} \leq r(z) < 2^{-k+1}$. Then

$$\omega(B(z)) \geq \frac{1}{C_\omega^k}\omega(2^k B(z)) \gtrsim \frac{1}{C_\omega^{k+4}}\omega(\partial B^n) \gtrsim \frac{1}{C_\omega^k}.$$

Hence, if $q_1 > 0$ is sufficiently large so that $2^{q_1} > C_\omega$, then for $z \in B^n \setminus \{0\}$

$$\omega(B(z)) \gtrsim r(z)^{q_1}$$

and thus

$$\begin{aligned}\int_{B^n} &|\mathcal{R}f(z)|^p r(z)^{q_1 A + p - n - 1}\,dm(z)\\ &\lesssim \int_{B^n} |\mathcal{R}f(z)|^p \omega(B(z))^A r(z)^{p-n-1}\,dm(z).\end{aligned}$$

Note that for $w \in P_\epsilon(z)$, $r(z) \approx r(w)$. Hence we have

$$\begin{aligned}|\mathcal{R}f(z)|^p &\lesssim \frac{1}{|P_\epsilon(z)|}\int_{P_\epsilon(z)} |\mathcal{R}f(w)|^p\,dm(w)\\ &\lesssim \frac{1}{r(z)^{q_1 A + p}}\int_{B^n} |\mathcal{R}f(w)|^p r(w)^{q_1 A + p - n - 1}\,dm(w)\\ &\lesssim \frac{1}{r(z)^{q_1 A + p}}\int_{B^n} |\mathcal{R}f(w)|^p \omega(B(w))^A r(w)^{p-n-1}\,dm(w).\end{aligned}$$

Suppose that $M := \int_{B^n} |\mathcal{R}f(z)|^p \omega(B(z))^A r(z)^{p-n-1}\,dm(z) < \infty$. If $q > 2(q_1 A + p)/p$, then

$$\int_{B^n} |\mathcal{R}f(z)|^2 r(z)^q\,dm(z) \lesssim M^{2/p}\int_{B^n} r(z)^{q-2(q_1 A+p)/p}\,dm(z) < \infty.$$

Thus for $s = 1 - \alpha$ it follows from (3.3) that

$$\begin{aligned}\mathcal{R}^\alpha f(w) &= \int_{B^n} \mathcal{R}^s(\mathcal{R}^\alpha f)(z) G_{n+q,1-\alpha}(\overline{\langle z, w\rangle})\, dV_q(z)\\ &= \int_{B^n} \mathcal{R} f(z) G_{n+q,1-\alpha}(\overline{\langle z, w\rangle})\, dV_q(z).\end{aligned}$$

For the remainder of the proof we will choose q sufficiently large if necessary. Let p' be the conjugate exponent to p. Then it follows from Hölder's inequality that

$$\begin{aligned}&\int_{B^n} |\mathcal{R}^\alpha f(w)|^p \frac{\omega(B(w))^A}{r(w)^{-\alpha p+n+1}}\, dm(w)\\ &\quad= \int_{B^n} \left| \int_{B^n} \mathcal{R} f(z) G_{n+q,1-\alpha}(\overline{\langle z, w\rangle})\, dV_q(z)\right|^p \frac{\omega(B(w))^A}{r(w)^{-\alpha p+n+1}}\, dm(w)\\ &\quad\lesssim \int_{B^n} \left(\int_{B^n} \left|\mathcal{R} f(z)^p G_{n+q,1-\alpha}(\overline{\langle z, w\rangle})\right| r(z)^{\delta(p-1)}\, dV_q(z)\right)\\ &\qquad\quad \times \left(\int_{B^n} \left|G_{n+q,1-\alpha}(\overline{\langle z, w\rangle})\right| r(z)^{-\delta}\, dV_q(z)\right)^{p/p'} \frac{\omega(B(w))^A}{r(w)^{-\alpha p+n+1}}\, dm(w)\end{aligned}$$

where δ is a number to be chosen. Note that we can not use Hölder's inequality if $p = 1$. However, when $p = 1$, we may use Fubini's theorem instead, then Lemma 3.1 follows from the inequality (3.5) below with $B = \alpha - n - 1$. Thus we assume $p > 1$. Choose δ so that $1 - \alpha < \delta < 1$. Since $n + q - \delta < n + q - 1 + \alpha$, it follows from (3.2) and (3.4) that

$$\begin{aligned}&\int_{B^n} \left|G_{n+q,1-\alpha}(\overline{\langle z, w\rangle})\right| r(z)^{-\delta}\, dV_q(z)\\ &\quad\lesssim \int_{B^n} |1 - \langle z, w\rangle|^{-(n+q-1+\alpha)}\, dV_{q-\delta}(z)\\ &\quad\lesssim r(w)^{1-\alpha-\delta}.\end{aligned}$$

Set $B = \alpha p - n - 1 + (1 - \alpha - \delta)p/p'$, then $B > -n - 1$ since $\delta < 1$. It then follows that

$$\begin{aligned}&\int_{B^n} |\mathcal{R}^\alpha f(w)|^p \frac{\omega(B(w))^A}{r(w)^{-\alpha p+n+1}}\, dm(w)\\ &\quad\lesssim \int_{B^n} \left(\int_{B^n} \left|\mathcal{R} f(z)^p G_{n+q,1-\alpha}(\overline{\langle z, w\rangle})\right| r(z)^{\delta(p-1)}\, dV_q(z)\right)\end{aligned}$$

$$
\begin{aligned}
&\times\omega(B(w))^A r(w)^B\,dm(w)\\
&= \int_{B^n}\left(\int_{B^n}\left|G_{n+q,1-\alpha}(\overline{\langle z,w\rangle})\right|\omega(B(w))^A r(w)^B\,dm(w)\right)\\
&\qquad\times|\mathcal{R}f(z)|^p r(z)^{\delta(p-1)}\,dV_q(z).
\end{aligned}
$$

Now, choose q sufficiently large, then from (3.2) and Lemma 2.3 we have

$$
\begin{aligned}
(3.5)\qquad &\int_{B^n}\left|G_{n+q,1-\alpha}(\overline{\langle z,w\rangle})\right|\omega(B(w))^A r(w)^B\,dm(w)\\
&\lesssim \int_{B^n}|1-\langle z,w\rangle|^{(1-\alpha-n-q)}\omega(B(w))^A r(w)^B\,dm(w)\\
&\lesssim r(z)^{(1-\alpha-n-q)+B+n+1}\omega(B(z))^A.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\int_{B^n}|\mathcal{R}^\alpha f(z)|^p\frac{\omega(B(w))^A}{r(w)^{-\alpha p+n+1}}\,dm(w)\\
&\lesssim \int_{B^n}|\mathcal{R}f(z)|^p r(z)^{(1-\alpha-n-q)+B+n+1+\delta(p-1)+q-1}\omega(B(z))^A\,dm(z)\\
&\lesssim \int_{B^n}|\mathcal{R}f(z)|^p r(z)^{p-n-1}\omega(B(z))^A\,dm(z).
\end{aligned}
$$

This completes the proof. □

§4. Derivative estimates

Let $Q(\delta)=\{z\in\mathbb{C}^n : |z_1|<\delta,\ |z_j|<\sqrt{\delta},\ j=2,\dots,n\}$. The following lemma follows from Corollary 3.3 of [B].

LEMMA 4.1. *If f is holomorphic in a neighborhood of $\overline{Q(\delta)}$, then for $p\geq 2$ we have*

$$
\left|\delta\frac{\partial f}{\partial z_1}(0)\right|^p \lesssim \frac{\delta^2}{|Q(\delta)|}\int_{Q(\delta)}|f|^{p-2}\left|\frac{\partial f}{\partial z_1}\right|^2 dm,
$$

$$
\left|\delta^{1/2}\frac{\partial f}{\partial z_j}(0)\right|^p \lesssim \frac{\delta}{|Q(\delta)|}\int_{Q(\delta)}|f|^{p-2}\left|\frac{\partial f}{\partial z_j}\right|^2 dm \quad \text{for } j=2,\dots,n.
$$

Define the complex-tangential vector fields T_{ij} by

$$
T_{ij}=\overline{z}_i\frac{\partial}{\partial z_j}-\overline{z}_j\frac{\partial}{\partial z_i},\quad i,j=1,\dots,n,\ i\neq j.
$$

And define the complex-tangential derivative of order k by

$$|\nabla_T^k f(z)| = \sum |T_{i_1 j_1} \cdots T_{i_k j_k} f(z)|$$

where the sum is over all possible combinations of indices.

LEMMA 4.2. *If f is holomorphic in B^n, then for $p \geq 2$, $k \geq 1$, and $z \in B^n$ with $|z| > 1/2$, we have*

$$|r(z)^k \nabla^k f(z)|^p \lesssim \frac{r(z)^2}{|P_\epsilon(z)|} \int_{P_\epsilon(z)} |f|^{p-2} |\nabla f|^2 \, dm$$

$$|r(z)^{k/2} \nabla_T^k f(z)|^p \lesssim \frac{r(z)^2}{|P_\epsilon(z)|} \int_{P_\epsilon(z)} |f|^{p-2} |\nabla f|^2 \, dm + \frac{r(z)}{|P_\epsilon(z)|} \int_{P_\epsilon(z)} |f|^{p-2} |\nabla_T f|^2 \, dm.$$

Proof. Let $z \in B^n$ and $|z| > 1/2$. Choose $c > 0$ so that $Q_{c\epsilon}(z) \subset P_\epsilon(z)$ as in Lemma 2.1 (iv). Then by Lemma 4.1, we have

$$|r(z) \nabla f(z)|^p \lesssim \frac{r(z)^2}{|Q_{c\epsilon}(z)|} \int_{Q_{c\epsilon}(z)} |f|^{p-2} |\nabla f|^2 \, dm.$$

It also follows from the Cauchy estimates over polydisc $Q_{c\epsilon}(z)$ that

$$|r(z)^k \nabla^k f(z)|^p \lesssim \frac{r(z)^p}{|Q_{c\epsilon}(z)|} \int_{Q_{c\epsilon}(z)} |\nabla f(w)|^p \, dm(w).$$

Note that if $w \in P_\epsilon(z)$, then $r(w) \approx r(z)$ and $Q_{c\epsilon}(w) \subset P_{C\epsilon}(z)$ by Lemma 2.1 (iv). Therefore,

$$\begin{aligned} &|r(z)^k \nabla^k f(z)|^p \\ &\quad \lesssim \frac{r(z)^2}{|Q_{c\epsilon}(z)|} \int_{Q_{c\epsilon}(z)} \frac{1}{|Q_{c\epsilon}(w)|} \int_{Q_{c\epsilon}(w)} |f(\zeta)|^{p-2} |\nabla f(\zeta)|^2 \, dm(\zeta) dm(w) \\ &\quad \lesssim \frac{r(z)^2}{|P_\epsilon(z)|} \int_{P_{C\epsilon}(z)} |f(\zeta)|^{p-2} |\nabla f(\zeta)|^2 \, dm(\zeta). \end{aligned}$$

This completes the proof for the first inequality.

For the second inequality, by Lemma 3.2 of [KK] we have

$$\begin{aligned} &|r(z)^{k/2} \nabla_T^k f(z)|^p \\ &\quad \lesssim \frac{1}{|P_\epsilon(z)|} \int_{P_\epsilon(z)} \left(|r(w) \nabla f(w)|^p + |r(w)^{1/2} \nabla_T f(w)|^p \right) dm(w). \end{aligned} \tag{4.1}$$

Now fix $w \in P_\epsilon(z)$ and by unitary change of coordinates, we may assume that $w = (w_1, 0, \ldots, 0)$. Then we have

$$|\nabla_T f(w)| \lesssim \sum_{j=2}^{n} \left| \frac{\partial f}{\partial z_j}(w) \right|. \tag{4.2}$$

And for $\eta \in P_\epsilon(w) \subset P_{C\epsilon}(z)$ we have

$$\left| 1 - \frac{\eta_1}{|\eta|} \right| \lesssim r(z), \quad |\eta_j| \lesssim r(z)^{1/2} \quad \text{for } j = 2, \ldots, n.$$

Thus we have

$$\begin{aligned} \left| \frac{\partial f}{\partial z_j}(\eta) \right| &\le \frac{1}{|\eta|} \left| \overline{\eta}_1 \frac{\partial f}{\partial z_j}(\eta) - \overline{\eta}_j \frac{\partial f}{\partial z_1}(\eta) \right| + \frac{1}{|\eta|} \left| \overline{\eta}_j \frac{\partial f}{\partial z_1}(\eta) \right| \\ &\quad + \left| 1 - \frac{\overline{\eta}_1}{|\eta|} \right| \left| \frac{\partial f}{\partial z_j}(\eta) \right| \\ &\lesssim |\nabla_T f(\eta)| + r(z)^{1/2} |\nabla f(\eta)|. \end{aligned} \tag{4.3}$$

Then from (4.3) and Lemma 4.1, we have

$$\begin{aligned} &|r(w)|^{p/2} \left| \frac{\partial f}{\partial z_j}(w) \right|^p \\ &\quad \lesssim \frac{r(w)}{|P_\epsilon(w)|} \int_{P_\epsilon(w)} |f(\eta)|^{p-2} \left| \frac{\partial f}{\partial z_j}(\eta) \right|^2 dm(\eta) \\ &\quad \lesssim \frac{r(w)}{|P_\epsilon(w)|} \int_{P_{5\epsilon}(w)} |f(\eta)|^{p-2} \big(|\nabla_T f(\eta)|^2 + r(z) |\nabla f(\eta)|^2 \big)\, dm(\eta). \end{aligned} \tag{4.4}$$

By substituting (4.4) into (4.1) and using Fubini's theorem together with the first inequality, we obtain the second inequality. ∎

§5. Proofs of Theorems

We now prove Theorems A and B in this section. We first prove the sufficiency in them and then prove the necessity. We begin by defining a maximal function and an area integral. For $\zeta \in \partial B^n$ and a positive number a, we define a nontangential admissible approach region by $\Gamma_a(\zeta) = \{z \in B^n : |1 - \langle z, \zeta \rangle| < a r(z)\}$. Then the area integral $\mathcal{S}_a(f)$ of a function f holomorphic on B^n is defined by

$$\big(\mathcal{S}_a(f)(\zeta)\big)^2 = \int_{\Gamma_a(\zeta)} \big(|r(z) \nabla f(z)| + |r^{1/2}(z) \nabla_T f(z)| \big)^2 \frac{dm(z)}{r(z)^{n+1}}.$$

For $\zeta \in \partial B^n$ the nontangential maximal function is defined by

$$\mathcal{N}_a(f)(\zeta) = \sup_{\Gamma_a(\zeta)} \big(|f(z)| + r(z)|\nabla f(z)| + r^{1/2}(z)|\nabla_T f(z)| \big).$$

The L^p-boundedness of these operators are well-known [S1]. Using the arguments in [ST], one can derive the following weighted version whose proof is given at the end of this paper.

THEOREM 5.1. *If $1 < p < \infty$ and $\omega \in \mathcal{A}_p$, then for any holomorphic function $f \in H^p(\omega)$, we have*

$$\|\mathcal{N}_a(f)\|_{L^p(\omega\, d\sigma)} \lesssim \|f\|_{H^p(\omega)}$$

and

$$\|\mathcal{S}_a(f)\|_{L^p(\omega\, d\sigma)} \lesssim \|f\|_{H^p(\omega)}.$$

We also need the following theorem of Gu [G].

THEOREM 5.2. *Let $p = q \geq 2$ or $1 < p < q < \infty$. Then $\mu(\hat{B}) \lesssim \omega(B)^{q/p}$ if and only if*

$$\left(\int_{B^n} |f|^q \, d\mu \right)^{1/q} \lesssim \|f\|_{H^p(\omega)} \quad \textit{for all } f \in H^p(\omega).$$

In fact, Gu proved this theorem when $n = 1$. However, the exactly same arguments work for the higher dimensions.

Sufficiency in Theorem A. Suppose that μ is a positive measure satisfying

$$\mu(\hat{B}) \lesssim \omega(B)^{q/p} \sigma(B)^{q\alpha/n}.$$

We first deal with the case when $p = q \geq 2$. By the mean value property of a holomorphic function $\mathcal{R}^\alpha f$, we have

$$\begin{aligned}
&\int_{B^n} |\mathcal{R}^\alpha f(z)|^p \, d\mu(z) \\
&\quad \lesssim \int_{B^n} \frac{1}{|P_\epsilon(z)|} \int_{P_\epsilon(z)} |\mathcal{R}^\alpha f(w)|^p \, dm(w) d\mu(z) \\
&\quad \lesssim \int_{B^n} |\mathcal{R}^\alpha f(w)|^p \int_{B^n} \frac{1}{|P_\epsilon(z)|} \mathcal{X}_{P_\epsilon(z)}(w) \, d\mu(z) dm(w) \\
&\quad \lesssim \int_{B^n} |\mathcal{R}^\alpha f(w)|^p \mu(P_{C\epsilon}(w)) |P_{C\epsilon}(w)|^{-1} \, dm(w) \\
&\quad \lesssim \int_{B^n} |\mathcal{R}^\alpha f(w)|^p \omega(B(w)) r(w)^{p\alpha - n - 1} \, dm(w).
\end{aligned}$$

It then follows from Lemma 3.1 that

$$\begin{aligned}\tag{5.1} &\int_{B^n} |\mathcal{R}^\alpha f(z)|^p \, d\mu(z) \\ &\quad\lesssim \int_{B^n} |\mathcal{R} f(w)|^p \omega(B(w)) r(w)^{p-n-1} \, dm(w) \\ &\quad\lesssim \int_{B^n} \big(|f(w)|^p + |\nabla f(w)|^p\big)\omega(B(w)) r(w)^{p-n-1} \, dm(w).\end{aligned}$$

From Lemma 4.2 and Theorem 5.1, we have

$$\begin{aligned}&\int_{B^n} |\nabla f(w)|^p \omega(B(w)) r(w)^{p-n-1} \, dm(w) \\ &\quad\lesssim \int_{B^n} |f(z)|^{p-2} (r(z)|\nabla f(z)|)^2 \omega(B(z)) r(z)^{-n-1} \, dm(z) \\ &\quad= \int_{B^n} \left(\int_{\partial B^n} \mathcal{X}_{B(z)}(\zeta)\omega(\zeta) \, d\sigma(\zeta) \right) |f(z)|^{p-2} \\ &\qquad\qquad \times (r(z)|\nabla f(z)|)^2 r(z)^{-n-1} \, dm(z) \\ &\quad\lesssim \int_{\partial B^n} (\mathcal{N}_2 f)^{p-2}(\zeta)(\mathcal{S}_2 f)^2(\zeta)\omega(\zeta) \, d\sigma(\zeta) \\ &\quad\lesssim \|\mathcal{N}_2 f\|_{L^p(\omega)}^{p-2} \|\mathcal{S}_2 f\|_{L^p(\omega)}^2 \\ &\quad\lesssim \|f\|_{H^p(\omega)}^p.\end{aligned}$$

For the first part in the last quantity of the inequalities (5.1), we have from Lemma 2.2

$$\int_{\hat{B}(\zeta,\delta)} \omega(B(z)) r(z)^{p-n-1} dm(z) \lesssim \omega(B(\zeta,\delta))\delta^p \lesssim \omega(B(\zeta,\delta)).$$

Thus by Theorem 5.2

$$\int_{B^n} |f(z)|^p \omega(B(z)) r(z)^{p-n-1} \, dm(z) \lesssim \int_{\partial B^n} |f(\zeta)|^p \omega(\zeta) \, d\sigma(\zeta).$$

This completes the proof for the case $p = q \geq 2$.

Suppose now that $1 < p < q < \infty$. For this case we use Theorem 5.2. As before,

$$\int_{B^n} |\mathcal{R}^\alpha f(z)|^q \, d\mu(z) \lesssim \int_{B^n} |\mathcal{R}^\alpha f(z)|^q \omega(B(z))^{q/p} r(z)^{\alpha q-n-1} \, dm(z).$$

Thus by Lemma 3.1 and the mean value property of $\mathcal{R}f$,

$$\int_{B^n} |\mathcal{R}^\alpha f(z)|^q \, d\mu(z) \lesssim \int_{B^n} |\mathcal{R}f(z)|^q \omega(B(z))^{q/p} r(z)^{q-n-1} \, dm(z)$$
$$\lesssim \int_{B^n} |f(z)|^q \omega(B(z))^{q/p} r(z)^{-n-1} \, dm(z).$$

Thus from Lemma 2.4 and Theorem 5.2, we have the inequality (1.5). ∎

Sufficiency in Theorem B. Suppose that μ is a positive measure satisfying

$$\mu(\hat{B}) \lesssim \omega(B)^{q/p} \sigma(B)^{qk/2n}.$$

When $p = q \geq 2$, by Lemma 4.2 and the same argument as in the previous proof, we have

$$\int_{B^n} |\nabla_T^k f(z)|^p \, d\mu(z)$$
$$\lesssim \int_{B^n} |f(z)|^{p-2} \big(|r(z) \nabla f(z)|^2 + |r(z)^{1/2} \nabla_T f(z)|^2 \big) \omega(B(z)) r(z)^{-n-1} \, dm(z).$$

Thus

$$\int_{B^n} |\nabla_T^k f(z)|^p \, d\mu(z) \lesssim \int_{\partial B^n} (\mathcal{N}_a f)^{p-2}(z) (\mathcal{S}_a f)^2(z) \omega(\zeta) \, d\sigma(\zeta) \lesssim \|f\|_{H^p(\omega)}^p.$$

If $1 < p < q < \infty$, we again use Theorem 5.2. Note that from [KK, Lemma 3.1] we have

$$|r(z)^{k/2} \nabla_T^k f(z)|^q \lesssim \frac{1}{|P_\epsilon(z)|} \int_{P_\epsilon(z)} |f(w)|^q \, dm(w).$$

Thus by Fubini's theorem and the given condition, we have

$$\int_{B^n} |\nabla_T^k f(z)|^q \, d\mu(z) \lesssim \int_{B^n} |f(z)|^q \omega(B(z))^{q/p} r(z)^{-n-1} \, dm(z).$$

Thus by Lemma 2.4 and Theorem 5.2, the proof is complete. ∎

Necessity in Theorem A and B. Let B be a Koranyi ball in ∂B^n. By a unitary change of coordinates, we may assume that $B = B(\zeta_0, \delta)$ where $\zeta_0 = (1, 0, \ldots, 0)$. Suppose that μ is a positive measure satisfying either (1.5) or (1.7). Then one can easily see that μ is a finite measure by taking $f \equiv 1$ in (1.5) or $f(z) = z_j^k$ with $j = 1, \ldots, n$ in (1.7). Thus we may assume δ is sufficiently small. Let $C_1 > 1$ be a number independent of δ satisfying

$$(1 - C_1\delta)^{1/\delta} < e^{-C_1/2}. \tag{5.2}$$

We will specify C_1 later. Let C_ω be the doubling constant of ω and N be an integer such that $C_\omega < 2^{pN}$. Now define f by

$$f(z) := \frac{1}{(1 - (1 - C_1\delta)z_1)^N} = \frac{1}{(1 - \langle z, (1 - C_1\delta)\zeta_0\rangle)^N}. \tag{5.3}$$

Then for $\zeta \in B(\zeta_0, 2^{k+1}C_1\delta) \setminus B(\zeta_0, 2^k C_1\delta)$, we have

$$|f(\zeta)| \approx (2^k C_1\delta)^{-N}, \tag{5.4}$$

and for $\zeta \in B(\zeta_0, C_1\delta)$, (5.4) holds with $k = 0$. Now let $z = (z_1, \ldots, z_n) \in \hat{B}$ and $z_1 = re^{i\theta}$. Then $|1 - z_1| < 2\delta$ and hence

$$1 - 2\delta < r < 1, \quad |\theta| < \sin^{-1}(2\delta) \le C_2\delta \tag{5.5}$$

for some constant C_2. Let M be the integral part of $\pi/(8|\theta|)$ when $\theta \neq 0$. Then

$$M \ge \frac{C_3}{\delta} \tag{5.6}$$

for some constant C_3 independent of δ. Now we choose C_1 so that $C_1C_3 > 2(N + \alpha + 2)$. Let

$$A(k) = \frac{(1+k)^\alpha (k + N - 1)!}{k!(N-1)!}(1 - C_1\delta)^k r^k.$$

Then from (5.2) and (5.6), when $k \ge M$ we get

$$\begin{aligned}\frac{A(k+M)}{A(k)} &\le r^M (1 - C_1\delta)^M 2^\alpha \Big(1 + \frac{N}{M}\Big)^M \\ &\le 2^\alpha e^{N - C_1C_3/2} \\ &\le 1/4.\end{aligned}$$

Thus, for $k \ge M$ and $j = 1, 2, \ldots$ we have

$$A(k + jM) \le \frac{1}{4^j} A(k)$$

and hence

$$\sum_{k=2M}^{\infty} A(k) \le \frac{1}{3} \sum_{k=M}^{2M} A(k). \tag{5.7}$$

Now for $z = (z_1, \ldots, z_n) \in \hat{B}$, we have

$$\begin{aligned} |\mathcal{R}^\alpha f(z)| &= \left| \sum_{k=0}^{\infty} (1+k)^\alpha f_k(z) \right| \\ &= \left| \sum_{k=0}^{\infty} (1+k)^\alpha \frac{(k+N-1)!}{k!(N-1)!} ((1-C_1\delta)z_1)^k \right| \\ &\geq \sum_{k=0}^{\infty} A(k) \cos(k\theta). \end{aligned}$$

If $k \leq 2M$, then $k|\theta| \leq \pi/4$ by the definition of M and hence

$$|\mathcal{R}^\alpha f(z)| \geq \frac{1}{2} \sum_{k=0}^{2M} A(k) - \sum_{k=2M}^{\infty} A(k).$$

Thus it follows from (5.7) that

$$|\mathcal{R}^\alpha f(z)| \geq \frac{1}{12} \sum_{k=0}^{\infty} A(k). \tag{5.8}$$

Note that (5.8) holds trivially when $\theta = 0$. Since by Stirling's formula

$$\frac{(1+k)^\alpha \Gamma(k+N)\Gamma(N+\alpha)}{\Gamma(k+N+\alpha)\Gamma(N)} \gtrsim 1,$$

we have

$$A(k) \gtrsim \frac{\Gamma(k+N+\alpha)}{\Gamma(k+1)\Gamma(N+\alpha)} r^k (1-C_1\delta)^k.$$

Thus it follows from (5.5) and (5.8) that, for $z \in \hat{B}$,

$$\begin{aligned} |\mathcal{R}^\alpha f(z)| &\gtrsim \sum_{k=0}^{\infty} \frac{\Gamma(k+N+\alpha)}{\Gamma(k+1)\Gamma(N+\alpha)} r^k (1-C_1\delta)^k \\ &= (1 - r(1-C_1\delta))^{-(N+\alpha)} \\ &\geq (1 - (1-2\delta)(1-C_1\delta))^{-(N+\alpha)} \\ &\gtrsim \delta^{-(N+\alpha)}. \end{aligned} \tag{5.9}$$

For $\nabla_T f$, let $\zeta_0' = (\sqrt{1-16(n-1)C_1^2\delta}, 4C_1\sqrt{\delta}, \ldots, 4C_1\sqrt{\delta}) \in \partial B^n$. Then by (iii) of Lemma 2.1, if $z \in \hat{B}(\zeta_0', C_1\delta)$, $z = (1+t)\zeta_0' + s\eta$ where

$|t| < 2C_1\delta$, $|s| < \sqrt{2C_1\delta}$ and $\eta \in T(\zeta_0')$. Hence, for $j = 2, \dots, n$, $z_j = (1+t)4C_1\sqrt{\delta} + s\eta_j$. Since $C_1 > 1$, $|z_j| \approx \delta^{1/2}$ for $j = 2, \dots, n$. Also note that by definition of $T_{i,j}$, for $i, j \in \{2, \dots, n\}$

$$T_{1,j}f(z) = \frac{N(1-C_1\delta)\overline{z}_j}{(1-(1-C_1\delta)z_1)^{N+1}}, \quad T_{i,1}f(z) = \frac{-N(1-C_1\delta)\overline{z}_i}{(1-(1-C_1\delta)z_1)^{N+1}}$$

and $T_{i,j}f(z) = 0$ if $i, j \in \{2, \dots, n\}$. Hence, for $z \in \hat{B}(\zeta_0', C_1\delta)$, we have

$$|\nabla_T^k f(z)| \approx \sum_{2 \le n_j \le n} \frac{|z_{n_1} \cdots z_{n_k}|}{|1-(1-C_1\delta)z_1|^{N+k}} \approx \delta^{-(N+k/2)}. \tag{5.10}$$

We will only prove the necessity for the fractional derivative case with $p = q \ge 2$. The fractional derivative case with $1 < p < q < \infty$ is similar to this case if one uses (5.4) and (5.9). And for the tangential derivative cases, if we use $\hat{B}(\zeta_0', C_1\delta)$ instead of $\hat{B}(\zeta_0, C_1\delta)$, then proof follows with the same arguments using from (5.4) and (5.10). Recall that we defined N so that $C_\omega < 2^{pN}$ where C_ω is the doubling constant of $\omega\, d\sigma$. Then, from (5.4) and (5.9), we have

$$\begin{aligned}
\mu(\hat{B}) &\lesssim \delta^{(N+\alpha)p} \int_{B^n} |\mathcal{R}^\alpha f(z)|^p \, d\mu(z) \\
&\lesssim \delta^{(N+\alpha)p} \int_{\partial B^n} |f|^p \omega \, d\sigma \\
&\lesssim \delta^{(N+\alpha)p} \left(\int_{B(\zeta_0, C_1\delta)} |f|^p \omega \, d\sigma + \sum_{k=0}^{\infty} \int_{B(\zeta_0, 2^{k+1}C_1\delta) \setminus B(\zeta_0, 2^k C_1\delta)} |f|^p \omega \, d\sigma \right) \\
&\lesssim \delta^{(N+\alpha)p} \sum_{k=0}^{\infty} \omega(B(\zeta_0, 2^k C_1\delta))/(2^k C_1\delta)^{Np} \\
&\lesssim \delta^{\alpha p} \sum_{k=0}^{\infty} C_\omega^k \omega(B(\zeta_0, C_1\delta))/2^{Npk} \\
&\lesssim \delta^{p\alpha} \omega(B(\zeta_0, C_1\delta)) \\
&\approx \omega(B(\zeta_0, C_1\delta)) \sigma(B(\zeta_0, C_1\delta))^{p\alpha/n}
\end{aligned}$$

This completes the proof. □

§6. Appendix – Proof of Theorem 5.1

Here we give a proof of Theorem 5.1. The proof uses the arguments in [S1] and [ST] and their variants.

For $f \in H^p(\omega)$ define the classical maximal function $\mathcal{N}_a^{\#}(f)(\zeta)$ at $\zeta \in \partial B^n$ by

$$\mathcal{N}_a^{\#}(f)(\zeta) := \sup_{w \in \Gamma_a(\zeta)} |f(w)|.$$

Then it is well known that $\mathcal{N}_a^{\#}(f)(\zeta) \lesssim \mathcal{M}f(\zeta)$ where $\mathcal{M}f$ is the Hardy-Littlewood maximal function on ∂B^n and hence $\|\mathcal{N}_a^{\#}(f)\|_{L^p(\omega)} \lesssim \|f\|_{H^p(\omega)}$. (See [S2].) Thus the following lemma leads us to the first inequality of Theorem 5.1. To make notations short, put

$$|\mathcal{D}f(z)|^2 = r(z)^2|\nabla f(z)|^2 + r(z)|\nabla_T f(z)|^2.$$

LEMMA 6.1. *Let f be a holomorphic function on B^n and $\zeta \in \partial B^n$. Then there exists a constant C independent of f and ζ such that*

$$\mathcal{N}_a(f)(\zeta) \lesssim \mathcal{N}_{Ca}^{\#}(f)(\zeta).$$

Proof. Note that there are constants C and ϵ such that if $z \in \Gamma_a(\zeta)$, then $P_\epsilon(z) \subset \Gamma_{Ca}(\zeta)$. By Theorem A and Theorem B of [Gr], we have

$$|\mathcal{D}f(z)|^2 \lesssim \frac{1}{|P_\epsilon(z)|} \int_{P_\epsilon(z)} |f(w)|^2 \, dm(w) \lesssim \mathcal{N}_{Ca}^{\#}(f)(\zeta). \tag{6.1}$$

Hence Lemma 6.1 follows. ∎

To derive the second inequality we need a lemma.

LEMMA 6.2. *Let f be a holomorphic function in a neighborhood of $\overline{B^n}$. Let $B(\eta, \delta)$ be a Koranyi ball in ∂B^n and λ a positive number. Assume that there is $\zeta_0 \in \overline{B(\eta, \delta)}$ such that $\mathcal{S}_a(f)(\zeta_0) \le \lambda$. For each $0 < t < 1$, there exists $\epsilon > 0$ independent of λ, η, and δ such that*

$$|\{\zeta \in B(\eta, \delta) : \mathcal{S}_a(f)(\zeta) \ge 2\lambda, \, \mathcal{N}_{Ca}^{\#}(f)(\zeta) \le \epsilon\lambda\}| \le t|B(\eta, \delta)|$$

where C is the constant in Lemma 6.1.

Proof. Let $B = B(\eta, \delta)$ and $D = \{\zeta \in B : \mathcal{S}_a(f)(\zeta) \ge 2\lambda, \, \mathcal{N}_{Ca}^{\#}(f)(\zeta) \le \epsilon\lambda\}$. If $\zeta \in B$, then

$$\begin{aligned}\Gamma_a(\zeta) &\subset \Gamma_a(\zeta_0) \cup \{z \in \Gamma_a(\zeta) \setminus \Gamma_a(\zeta_0) : \delta < ar(z)\} \cup \{z \in \Gamma_a(\zeta) : ar(z) \le \delta\} \\ &:= \Gamma_a(\zeta_0) \cup E_1(\zeta) \cup E_2(\zeta).\end{aligned}$$

Note that since $|1-\langle\zeta,\eta\rangle|\le\delta$, $|E_1(\zeta)\cap\{r(z)=c\}|\lesssim\delta^n$ for all $0<c<1$. If $\zeta\in D$, we have from (6.1)

$$\begin{aligned}\mathcal{S}_a(f)^2(\zeta)&\le\mathcal{S}_a(f)^2(\zeta_0)+C_1\mathcal{N}^{\#}_{Ca}(f)^2(\zeta)\int_{E_1(\zeta)}\frac{dm(z)}{r(z)^{n+1}}\\&\quad+\int_{E_2(\zeta)}|\mathcal{D}f(z)|^2\frac{dm(z)}{r(z)^{n+1}}\\&\le\lambda^2+C_1(\epsilon\lambda)^2+C_1\int_{E_2(\zeta)}|\mathcal{D}f(z)|^2\frac{dm(z)}{r(z)^{n+1}}\end{aligned}$$

for some constant C_1. Thus

$$\begin{aligned}(2\lambda)^2|D|&\le\int_D\mathcal{S}_a(f)^2(\zeta)\,d\sigma(\zeta)\\&\le\lambda^2|D|+C_1\epsilon^2\lambda^2|B|+C_1\int_D\int_{E_2(\zeta)}|\mathcal{D}f(z)|^2\frac{dm(z)}{r(z)^{n+1}}d\sigma(\zeta).\end{aligned}$$

Let $\Omega:=\{z\in B^n : z\in\Gamma_a(\zeta)\text{ for some }\zeta\in D,\ r(z)\le\delta/a\}=\bigcup_{\zeta\in D}E_2(\zeta)$. If $z\in E_2(\zeta)$, then $\zeta\in B(\pi(z),\alpha r(z))$ for some α independent of z. It thus follows that

$$\begin{aligned}I&:=\int_D\int_{E_2(\zeta)}|\mathcal{D}f(z)|^2\frac{dm(z)}{r(z)^{n+1}}d\sigma(\zeta)\\&\le\int_\Omega\int_{B(\pi(z),\alpha r(z))}d\sigma(\zeta)|\mathcal{D}f(z)|^2\frac{dm(z)}{r(z)^{n+1}}\\&\lesssim\int_\Omega r(z)^n|\mathcal{D}f(z)|^2\frac{dm(z)}{r(z)^{n+1}}.\end{aligned}$$

It is known that $|\mathcal{D}f(z)|^2\approx|\tilde\nabla f(z)|^2$ and $dm(z)/r(z)^{n+1}\approx dV(z)$ where $\tilde\nabla f(z)$ and $dV(z)$ are the gradient and the volume element induced by the Bergman metric on B^n. See Chapter 3 of [S1] for the proof of these facts. If we follow the argument in pp. 65–68 of [S1], we can see that

$$\int_\Omega r(z)^n|\mathcal{D}f(z)|^2\frac{dm(z)}{r(z)^{n+1}}\lesssim(\epsilon\lambda)^2\int_D d\sigma.$$

In fact, it is proved in [S1] that the left-hand side of above inequality is finite when $|f|$ is bounded in Ω. However, the exactly same proof gives the above inequality.

It thus follows that

$$I \lesssim (\epsilon\lambda)^2 |D| \leq (\epsilon\lambda)^2 |B|.$$

Combining all the relevant inequalities together, we have for some constant C_2,

$$|D| \leq C_2 \epsilon^2 |B|.$$

For a given t, choose ϵ so that $t = C_2\epsilon^2$. This completes the proof. ∎

We now continue to prove Theorem 5.1.

We first assume that f is holomorphic in a neighborhood of $\overline{B}$. Note that

$$\|\mathcal{S}_a(f)\|^p_{L^p(\omega\, d\sigma)} = p\int_0^\infty \lambda^{p-1}\omega(\{\mathcal{S}_a(f) > \lambda\})\, d\lambda.$$

By a covering lemma, we can find a mutually disjoint sequence (finite or infinite) $\{B(\zeta_j, \delta_j)\}$ of Koranyi balls maximal with respect to inclusion such that $B(\zeta_j, \delta_j) \subset \{\mathcal{S}_a(f) > \lambda\}$ and $\{\mathcal{S}_a(f) > \lambda\} \subset \bigcup_{j=1}^\infty B(\zeta_j, C\delta_j)$ for some constant C. Since $B(\zeta_j, \delta_j)$ is maximal, there exists $\zeta_0 \in \partial B(\zeta_j, \delta_j)$ such that $\mathcal{S}_a(f) \leq \lambda$. For given t, choose ϵ as in Lemma 6.2. Then

$$\begin{aligned}&\{\mathcal{S}_a(f) > 2\lambda\}\\ &\quad\subset \bigcup_{j=1}^\infty \Big[\{\mathcal{S}_a(f) > 2\lambda,\, \mathcal{N}^\#_{Ca}(f) \leq \epsilon\lambda\} \cap B(\zeta_j, C\delta_j)\Big] \cup \{\mathcal{N}^\#_{Ca}(f) > \epsilon\lambda\}.\end{aligned}$$

Since $w \in \mathcal{A}_p$, there exists a constant β such that

$$\left(\frac{|D|}{|B|}\right)^p \leq \beta\,\frac{\omega(D)}{\omega(B)} \quad \text{for any } D \subset B \text{ and for any ball } B.$$

See [S2] for this fact. Set $B = B(\zeta_j, C\delta_j)$ and $D = B \setminus \{\zeta \in B(\zeta_j, C\delta_j) : \mathcal{S}_a(f) > 2\lambda,\, \mathcal{N}^\#_{Ca}(f) \leq \epsilon\lambda\}$. Then it follows from Lemma 6.2 and the doubling property of ω that

$$\omega(\{\zeta \in B(\zeta_j, C\delta_j) : \mathcal{S}_a(f) > 2\lambda,\, \mathcal{N}^\#_{Ca}(f) \leq \epsilon\lambda\}) \leq C(t)\omega(B(\zeta_j, \delta_j))$$

where $C(t) \to 0$ as $t \to 0$. Since $\{B(\zeta_j, \delta_j)\}$ is mutually disjoint and $\bigcup_{j=1}^\infty B(\zeta_j, \delta_j) \subset \{\mathcal{S}_a(f) > \lambda\}$, we have

$$\omega(\{\mathcal{S}_a(f) > 2\lambda\}) \leq C(t)\omega(\{\mathcal{S}_a(f) > \lambda\}) + \omega(\{\mathcal{N}^\#_{Ca}(f) > \epsilon\lambda\}).$$

Then

$$2^{-p}\|\mathcal{S}_a(f)\|^p_{L^p(\omega\, d\sigma)} \le C(t)\|\mathcal{S}_a(f)\|^p_{L^p(\omega d\sigma)} + \epsilon^{-p}|\mathcal{N}^{\#}_{Ca}(f)\|^p_{L^p(\omega\, d\sigma)}.$$

By taking t small enough, we have

$$\|\mathcal{S}_a(f)\|^p_{L^p(\omega\, d\sigma)} \lesssim \|\mathcal{N}^{\#}_{Ca}(f)\|^p_{L^p(\omega\, d\sigma)} \lesssim \|f\|^p_{L^p(\omega\, d\sigma)}.$$

If $f \in H^p(\omega)$, one can consider the function $f_r(z) := f(rz)$ $(0 < r < 1)$ and pass to the limit $r \to 1$. ∎

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