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NONEXISTENCE OF REAL ANALYTIC LEVI FLAT HYPERSURFACES IN $\mathbb{P}^2$

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Abstract. A real hypersurface M in a complex manifold X is said to be Levi flat if it separates X locally into two Stein pieces. It is proved that there exist no real analytic Levi flat hypersurfaces in $\mathbb{P}^2$.

1. Let X be a complex manifold of dimension n. A closed subset $M \subset X$ is called a real hypersurface of class C^α, for $0 \leq \alpha \leq \infty$ or $\alpha = \omega$, if M is locally expressed, with respect to real analytic coordinates of X, as the graph of some function of class C^α in $2n-1$ real variables.

2. If a real hypersurface $M \subset X$ divides X locally into Stein domains, M is said to be Levi flat. In case M is of class C^2, this condition is equivalent to that the complex Hessian of a defining function of M is identically zero on the analytic tangent bundle $T_M^{1,0} := (T_X^{1,0}|M) \cap (T_M \otimes \mathbb{C})$ of M.

3. There is an open question whether or not there exists a Levi flat real hypersurface in $\mathbb{P}^2$. One of the motivations for asking this comes from the theory of Pfaff forms (cf. [C]).

4. The purpose of the present note is to prove the following.

THEOREM. *There exist no real analytic Levi flat hypersurfaces in* $\mathbb{P}^2$.

5. Proof will be done by contradiction.

6. Suppose that there existed such $M \subset \mathbb{P}^2$. We may assume that M is connected.

7. The following observations have long been known.

(i) The holomorphic normal bundle $N_M^{1,0} := (T_{\mathbb{P}^2}^{1,0} \mid M)/T_M^{1,0}$ admits a fiber metric whose curvature form is positive along $T_M^{1,0}$.

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(ii) $\mathbb{P}^2 \setminus M$ is a Stein manifold with two connected components.

8. (i) is true because $N_M^{1,0}$ is, as a CR line bundle, a quotient of a Griffiths-positive bundle say $T_{\mathbb{P}^2}^{1,0} \,|\, M$. (That $T_{\mathbb{P}^2}^{1,0}$ is Griffiths-positive is equivalent to saying that $\mathbb{P}^2$ admits a metric whose holomorphic bisectional curvature is positive. It is straightforward that the Fubini-Study metric has this property.)

9. (ii) is true in virtue of A. Takeuchi [T]. (For a somewhat simplified proof of Takeuchi's theorem, see [O].) That $\mathbb{P}^2 \setminus M$ has two components follows from the fact that $\mathbb{P}^2$ is simply connected. It follows in particular that M is orientable.

10. As in [O-S] we put

$$N_M := T_M \otimes \mathbb{C}/(T_M^{1,0} + \overline{T_M^{1,0}}).$$

Since the projection $T_{\mathbb{P}^2} \otimes \mathbb{C} \to T_{\mathbb{P}^2}^{1,0}$ induces an isomorphism between N_M and $N_M^{1,0}$, we shall not distinguish them.

11. Since M is Levi flat and of class C^ω, N_M admits a system of real analytic local frames such that the transition functions between them are real valued.

12. Moreover, since M is orientable, these transition functions can be chosen to be positive.

13. More explicitly, real analytic defining functions of the Levi flat hypersurface M are of the form $\operatorname{Re} f$ for holomorphic f. Since M is orientable, open sets of $\mathbb{P}^2$, say U_i ($i = 1, 2, \ldots, m$), can be chosen in such a way that $M \subset \bigcup_{i=1}^m U_i$ and that one has a holomorphic function f_i on U_i such that $\operatorname{Re} f_i$ is a defining function of $M \cap U_i$ and

$$e_{ij} := \frac{df_i}{df_j}\Big|_{M\cap U_i \cap U_j} \left(= \frac{d(\operatorname{Im} f_i|M \cap U_i \cap U_j)}{d(\operatorname{Im} f_j|M \cap U_i \cap U_j)} \right) > 0.$$

14. This implies that, for any positive integer k, there exists a holomorphic line bundle, say L_k, over some neighbourhood of M in $\mathbb{P}^2$ satisfying

$$L_k^{\otimes k}|M = N_M.$$

15. We shall prove that, for each k, at least one of such L_k can be extended to a holomorphic line bundle over $\mathbb{P}^2$.

16. For that, because of the Steinness of $\mathbb{P}^2 \setminus M$ and because $\dim(\mathbb{P}^2 \setminus M) > 1$, it suffices to show that some L_k is an analytic subsheaf of a coherently extendable locally free sheaf.

17. We note that this criterion of extendability is a corollary of a more general extension theorem due to S. Ivashkovitch [I] , which asserts that, given any connected Stein manifold S of dimension at least 2 and any compact Kähler manifold Y, holomorphic maps from the complement of any compact subset of S to Y extend meromorphically to S.

18. By this criterion, the extendability of L_1 is immediate because $L_1^* \subset (T_{\mathbb{P}}^{1,0})^*$. Hence the sheaf of the germs of holomorphic sections of L_1, denoted by $\mathcal{O}(L_1)$, is the restriction of some invertible sheaf $\mathcal{L} \to \mathbb{P}^2$.

19. By (i), the degree of $\mathcal{L}$ must be positive. This means that there exist two sections $s_1, s_2 \in \Gamma(\mathbb{P}^2, \mathcal{L})$ such that $s_1^{-1}(0) \cup s_2^{-1}(0)$ consists of $2d$ complex lines of general position, where $d = \deg \mathcal{L}$, and that

$$\mathrm{Sing}(s_1^{-1}(0) \cup s_2^{-1}(0)) \cap M = \emptyset.$$

20. Let $\pi_k : X_k \to \mathbb{P}^2$ be the k-sheeted ramified covering that makes the k-th root of s_2/s_1 univalent outside the set of indeterminancy.

21. Then one can find a holomorphic section $\tau_k \in \Gamma(\pi_k^{-1}(U), \mathcal{O}(\pi_k^* L_k))$ for a sufficiently small neighbourhood U of M, such that $\tau_k^k = \pi_k^* s_1$, *for some* L_k.

22. Hence there exists L_k such that $\pi_k^* L_k$ is the line bundle associated to the divisor $\frac{1}{k}\pi_k^{-1}(s_1^{-1}(0))$.

23. Since $\pi_k^* L_k$ is extendable to X_k, $\pi_{k*}\pi_k^* L_k$ is extendable to $\mathbb{P}^2$. But L_k is a subbundle of $\pi_{k*}\pi_k^* L_k$, so that L_k is extendable, too.

24. Therefore $\deg \mathcal{L}$ is a positive integer which is divisible by any positive integer k, which is absurdity. □

25. The following is an immediate consequence of the theorem.

COROLLARY 1. *If* $\mathbb{P}^n$ *admits a Levi flat real analytic hypersurface,* $n \leq 1$.

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