J. Zhang
Nagoya Math. J.
Vol. 158 (2000), 73–86

FIXED POINT THEOREMS OF DISCONTINUOUS INCREASING OPERATORS AND APPLICATIONS TO NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS

JINQING ZHANG

Abstract. In this paper, we obtain some new existence theorems of the maximal and minimal fixed points for discontinuous increasing operators in $C[I,E]$, where E is a Banach space. As applications, we consider the maximal and minimal solutions of nonlinear integro-differential equations with discontinuous terms in Banach spaces.

§1. Introduction and preliminaries

For the sake of clarity, we first give some notations and concepts. Let E be a real Banach space with norm $\|\cdot\|$, $I=[a,b]\subset R^1$ with $a<b$, and $C[I,E]$ denote the set of all continuous functions defined on I with values in E. Clearly $C[I,E]$ is a Banach space with the norm $\|x\|_C = \max_{t\in I}\|x(t)\|$. For any $p\geq 1$, set

$$L_p[I,E]=\left\{x(t):I\to E\;\middle|\;\begin{array}{l}x(t)\text{ is strongly measurable and}\\ \int_I\|x(t)\|^p\,dt<\infty\end{array}\right\},$$

then $L_p[I,E]$ is a Banach space with the norm $\|x\|_p=\big(\int_I\|x(t)\|^p\,dt\big)^{1/p}$. Let a nonempty convex closed set P be a cone in E. The cone P defines an ordering in E given by $x\leq y$ iff $y-x\in P$. The orderings in $C[I,E]$ and $L_p[I,E]$ are induced by the cone P as follows, respectively, for $u,v\in C[I,E]$, $u\leq v$ iff $u(t)\leq v(t)$ for any $t\in I$; for $u,v\in L_p[I,E]$, $u\leq v$ iff $u(t)\leq v(t)$ for almost all $t\in I$. Obviously, $C[I,E]$ is an ordered additive group which is additive by the common addition and the ordering induced by the cone of P of E, i.e., $u_1,u_2,v_1,v_2\in C[I,E]$ and $u_1\leq v_1$, $u_2\leq v_2$ imply $u_1+u_2\leq v_1+v_2$. For details on strongly measure functions and cone theory, see [9] and [4] respectively.

Received November 16, 1998.
2000 *Mathematics Subject Classification*: 47H10.
Research supported by NNSF-China and Educ. Comm. DF-China.

It is common knowledge that fixed point theorems on increasing operators are used widely in nonlinear equations and other fields in mathematics (see [1]–[7]). But in most well-known documents, it is assumed generally that increasing operators possess stronger continuity and compactness (see [1]–[6]). In this paper, different from the increasing operators mapping ordering intervals of E into E , A is an increasing operator from an ordering interval D of $C[I,E]$ into $C[I,E]$, and may be expressed as the form $\sum_{i=1}^{m} K_iF_i$. We do not assume any continuity on A. It is only required that $(F_iD)(t)$ (almost all $t \in I$) and $(K_iD_i)(t)$ $(t \in I)$ possess very weak compactness, where $(F_iD)(t)$ and $(K_iD_i)(t)$ can be found in §2, $i = 1, 2, \ldots, m$. In addition, if we use the results in [1]–[7] to study integral equations and differential equations in Banach spaces, we have to verify the compactness or weak compactness in such spaces as $C[I,E]$ or $L_p[I,E]$. But it is very difficult to examine the compactness type conditions in $C[I,E]$ or $L_p[I,E]$. So there is some difficulty in applying the results in [1]–[7] to nonlinear equations in Banach spaces. By using the conclusions of this paper, we may avoid the difficulty and only need to verify the compactness in E rather than $C[I,E]$ or $L_p[I,E]$, whereas the compactness in E is satisfied naturally in many cases (see §3).

As applications, we show the existence of the maximal and minimal solutions of nonlinear integro-differential equations with discontinuous terms in Banach spaces.

§2. Fixed point theorems of increasing operators

Let $u_0, v_0 \in C[I,E]$, $u_0 \le v_0$, $D = [u_0, v_0] = \{u \in C[I,E] \mid u_0 \le u \le v_0\}$. For any $i \in \{i = 1, 2, \ldots, m\}$, $1 \le p_1, p_2, \ldots, p_m < +\infty$, let $F_i : D \to L_{p_i}[I,E]$ be an increasing operator, $D_i = \{w \in L_{p_i}[I,E] \mid F_iu_0 \le w \le F_iv_0\}$, and $K_i : D_i \to C[I,E]$ an increasing operator. Define operator A by $A = \sum_{i=1}^{m} K_iF_i$, thus A is also an increasing operator from D into $C[I,E]$.

In the following, for $t \in I$, set

$$(F_iD)(t) = \{u(t) \in E \mid u \in F_i(D)\},$$
$$(K_iD_i)(t) = \{u(t) \in E \mid u \in K_i(D_i)\},$$

obviously,

$$(F_iD)(t), (K_iD_i)(t) \subset E,$$

here $i = 1, 2, \ldots, m$.

Lemma 1. *Let E be a Banach space, P a cone in E, $x_n, y_n \in E$, and $x_n \leq y_n$ $(n = 1, 2, \ldots)$. Then $x_n \xrightarrow{w} x^*$ and $y_n \xrightarrow{w} y^*$ imply $x^* \leq y^*$, where the notation $\xrightarrow{w}$ means that a sequence converges weakly to some element.*

Proof. It is easy to follow from the assumptions that $y_n - x_n \in P$ $(n = 1, 2, \ldots)$, $y_n - x_n \xrightarrow{w} y^* - x^*$. Since the convex closed set P is weakly closed, $y^* - x^* \in P$, i.e., $x^* \leq y^*$. Thus Lemma 1 holds. ∎

Theorem 1. *Let increasing operators $F_i : D \to L_{p_i}[I, E]$ $(i = 1, 2, \ldots, m$, which is the same sense in the following), increasing operators $K_i : D_i \to C[I, E]$ and $A = \sum_{i=1}^m K_i F_i$. Assume*

(i) *for almost all $t \in I$, any complete ordered subset of $(F_i D)(t)$ is relatively weakly compact in E; for any $t \in I$, any complete ordered subset of $(K_i D_i)(t)$ is also relatively weakly compact in E;*

(ii) *$F_i(D)$ are bounded sets in $L_{p_i}[I, E]$;*

(iii) *$u_0 \leq Au_0$, $Av_0 \leq v_0$;*

Then A has at least one fixed point in D.

Proof. It follows from the monotonicity of A and condition (iii) that $A : D \to D$. Set $R = \{u \in A(D) \mid u \leq Au\}$. By $Au_0 \in R$, $R \neq \emptyset$. Taking any complete ordered set N in R , we set $M = A(N)$, $M(t) = \{u(t) \in E \mid u \in M\}$. Clearly M is also a complete ordered set in R due to the definition of R and the monotonicity of A, so is $M(t)$ in E for any $t \in I$. The following proof will be divided into cases: (a) there exists a $t^* \in I$ such that any element of $M(t^*)$ is not an upper bound of $M(t^*)$, and (b) for any $t \in I$, there exists an $x \in M(t)$ such that x is an upper bound of $M(t)$.

In case of (a): Obviously $M(t^*) = (AN)(t^*) = \sum_{i=1}^m (K_i F_i(N))(t^*)$. Since $N \subset R \subset D$, and N is a complete ordered set of R, $(K_i F_i(N))(t^*)$ are complete ordered sets of $(K_i D_i)(t^*)$ $(i = 1, 2, \ldots, m)$. Now we show that $M(t^*)$ is relatively weakly compact in E. For any $\{z_n\} \subset M(t^*)$, it follows from $M(t^*) = \sum_{i=1}^m (K_i F_i(N))(t^*)$ that there exists a subsequence $\{w_n\} \subset N$ such that $z_n = \sum_{i=1}^m (K_i F_i w_n)(t^*)$. Let $y_{i,n} = (K_i F_i w_n)(t^*)$, clearly $y_{i,n} \subset (K_i F_i(N))(t^*) \subset (K_i D_i)(t^*)$ and $z_n = \sum_{i=1}^m y_{i,n}$, thus $\{y_{i,n}\}$ is complete ordered subset in $(K_i D_i)(t^*)$ $(i = 1, 2, \ldots, m)$. By condition (i), $\{y_{1,n}\}$ has a weakly convergent subsequence $\{y_{1,n}^{(1)}\} \subset \{y_{1,n}\}$. Evidently $\{y_{i,n}^{(1)}\} \subset \{y_{i,n}\}$ $(i = 1, 2, \ldots, m)$. Then we can choose a weakly convergent subsequence $\{y_{2,n}^{(2)}\}$ in $\{y_{2,n}^{(1)}\}$, and we have $\{y_{i,n}^{(2)}\} \subset \{y_{i,n}^{(1)}\}$ $(i = 1, 2, \ldots, m)$. Using the same arguments and going on with the process, we can obtain a weakly convergent subsequence $\{y_{m,n}^{(m)}\}$ in $\{y_{m,n}^{(m-1)}\}$, and $\{y_{i,n}^{(m)}\} \subset \{y_{i,n}^{(m-1)}\}$ $(i = 1, 2, \ldots, m)$. By above discussions we know that

$$\{y_{i,n}^{(m)}\} \subset \{y_{i,n}^{(m-1)}\} \subset \cdots \subset \{y_{i,n}^{(1)}\} \subset \{y_{i,n}\}, \quad i = 1, 2, \ldots, m.$$

and $\{y_{i,n}^{(m)}\}$ is a weakly convergent sequence of $\{y_{i,n}\}$. Obviously we may get $z_n^{(m)} = \sum_{i=1}^m y_{i,n}^{(m)}$ corresponding to $z_n = \sum_{i=1}^m y_{i,n}$, hence $\{z_n^{(m)}\}$ is also a weakly convergent subsequence of $\{z_n\}$. Observing that $\{z_n\} \subset M(t^*)$ is arbitrary, we know that $M(t^*)$ is relatively weakly compact.

Let $\overline{M(t^*)}^w$ denote the closure of $M(t^*)$ in E in the sense of weak topology. Then $\overline{M(t^*)}^w$ is a compact set of $M(t^*) \subset E$ in the sense of weak topology. For $x \in M(t^*)$, set $B(x) = \{y \in \overline{M(t^*)}^w \mid x \le y\}$. It is easy to know from Lemma 1 that $\{y \in E \mid x \le y\}$ is weak closed in E, thus $B(x) = \overline{M(t^*)}^w \cap \{y \in E \mid x \le y\}$ is also weak closed in E. Taking any finite members $\{B(x_i) \mid x_i \in M(t^*), i = 1, 2, \ldots, k\}$, we set $\overline{x} = \max\{x_i \mid i = 1, 2, \ldots, k\}$. Since $M(t^*)$ is a complete ordered set, $\overline{x}$ makes sense, $\overline{x} \in M(t^*)$ and $x_i \le \overline{x}$ $(i = 1, 2, \ldots, k)$. Thus $\overline{x} \in \bigcap_{i=1}^k B(x_i)$, that is, $\bigcap_{i=1}^k B(x_i) \ne \emptyset$. Since $\overline{M(t^*)}^w$ is a compact set in the sense of weak topology, it follows from the finite intersection property of compact set (see [10, Chapter 5]) that $\bigcap_{x \in M(t^*)} B(x) \ne \emptyset$. Taking $x^* \in \bigcap_{x \in M(t^*)} B(x)$, we know from the definition of $B(x)$ and $B(x) \subset \overline{M(t^*)}^w$ that $x^* \in \overline{M(t^*)}^w$ and

$$x \le x^*, \quad \forall x \in M(t^*). \tag{2.1}$$

Since any element of $M(t^*)$ is not an upper bound of $M(t^*)$,

$$x \ne x^*, \quad \forall x \in M(t^*). \tag{2.2}$$

By $x^* \in \overline{M(t^*)}^w$ and on account of the famous Eberlein-Shmulyan theorem, there exists a sequence $\{x_n\}$ of $M(t^*)$ such that

$$x_n \xrightarrow{w} x^*. \tag{2.3}$$

It is clear to see from (2.1), (2.2) and (2.3) that for any $x_{n_1} \in \{x_n\}$, there exists $x_{n_2} \in \{x_n\}$ such that $x_{n_1} \le x_{n_2}$ and $x_{n_1} \ne x_{n_2}$. Similarly, we can choose a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that

$$x_{n_1} \le x_{n_2} \le \cdots \le x_{n_i} \le \cdots, \quad x_{n_1} \ne x_{n_2} \ne \cdots \ne x_{n_i} \ne \cdots.$$

Without loss of generality, we may assume that $\{x_n\}$ satisfies

$$x_1 \le x_2 \le \cdots \le x_n \le \cdots, \quad x_1 \ne x_2 \ne \cdots \ne x_n \ne \cdots. \tag{2.4}$$

Otherwise, we may replace $\{x_n\}$ with $\{x_{n_i}\}$. By (2.1) and (2.2),

$$x_n \le x^*, \quad x_n \ne x^*, \quad n = 1, 2, \ldots. \tag{2.5}$$

Take $u_n \in M$ such that $u_n(t^*) = x_n$. Obviously $\{u_n\}$ is a complete ordered set of $C[I, E]$, which, together with (2.4), implies

$$u_1 \le u_2 \le \cdots \le u_n \le \cdots. \tag{2.6}$$

Letting $v_{i,n} = F_i u_n$ for any n, we know from the monotonicity of F_i that

$$v_{i,1} \le v_{i,2} \le \cdots \le v_{i,n} \le \cdots, \quad i = 1, 2, \ldots, m. \tag{2.7}$$

Thus for almost all $t \in I$, we have

$$v_{i,1}(t) \le v_{i,2}(t) \le \cdots \le v_{i,n}(t) \le \cdots. \tag{2.8}$$

By condition (i), there exist $I_0 \subset I$ and $\mathrm{mes}(I \backslash I_0) = 0$ such that for any $t \in I_0$, $\{v_{n,i}(t)\}$ is relatively weakly compact and (2.8) holds. Thus there exists a subsequence $\{v_{i,nk}(t)\}$ of $\{v_{i,n}(t)\}$ and $v_{i,t} \in \overline{\{v_{i,n}(t)\}}^w$ such that

$$v_{i,nk}(t) \xrightarrow{w} v_{i,t}, \quad t \in I_0. \tag{2.9}$$

For any n_{k_0}, by (2.8) we know that $v_{i,n_{k_0}}(t) \le v_{i,nk}(t)$ when $k_0 \le k$. By Lemma 1 and (2.9), $v_{i,n_{k0}}(t) \le v_{i,t}$. Hence we get

$$v_{i,n}(t) \le v_{i,t}, \quad n = 1, 2, \ldots, \ t \in I_0 \tag{2.10}$$

since n_{k_0} is arbitrary. In view of standard arguments (such as the proof of Theorem 6.1 in [3]), by (2.8) and (2.9) we can prove

$$v_{i,n}(t) \xrightarrow{w} v_{i,t}, \quad t \in I_0. \tag{2.11}$$

Define $v_i^* : I \to E$ as follows: when $t \in I_0$, $v_i^*(t) = v_{i,t}$; when $t \in I \backslash I_0$, $v_i^*(t) = 0$. Then (2.10) and (2.11) imply that

$$v_{i,n}(t) \le v_i^*(t), \ n = 1, 2, \ldots, \quad v_{i,n}(t) \xrightarrow{w} v_i^*(t), \ \forall t \in I_0. \tag{2.12}$$

Since $v_{i,n}$ is strongly measurable because of $v_{i,n} = F_i u_n \in L_{p_i}[I, E]$ ($i = 1, 2, \ldots, m$), by (2.12) and according to Pettis theorem and its proof (see Chapter V of [9]) $v_i^*(t)$ is also strongly measurable. In view of the second formula of (2.12) and the weakly lower semi-continuity of norm, we have

$$\|v_i^*(t)\| \le \varliminf_{n\to\infty} \|v_{i,n}(t)\|, \quad \forall t \in I_0.$$

By Fatou Lemma, we get

$$\int_I \|v_i^*(t)\|^{p_i}\, dt \le \int_I \varliminf_{n\to\infty} \|v_{i,n}(t)\|^{p_i}\, dt \le \varliminf_{n\to\infty} \int_I \|v_{i,n}(t)\|^{p_i}\, dt,$$

which, by $v_{i,n} = F_i u_n \in F_i(D) \subset L_{p_i}[I, E]$ and condition (ii), implies $v_i^* \in L_{p_i}[I, E]$. By (2.12) and according to the weak closeness of the cone P, $v_i^* \in D_i = \{w \in L_{p_i}[I, E] \mid F_i u_0 \le w \le F_i v_0\}$. Let $u^* = \sum_{i=1}^m K_i v_i^*$. Clearly $K_i v_i^* \in C[I, E]$, i.e., $u^* \in C[I, E]$. Now we prove

$$u_n \le u^*, \quad n = 1, 2, \ldots; \tag{2.13}$$

$$u^* \le Au^*. \tag{2.14}$$

For any n_0, by (2.7) $v_{i,n_0} \le v_{i,n}$ when $n_0 \le n$. Hence

$$F_i u_{n_0} = v_{i,n_0} \le v_{i,n} \le v_i^* \tag{2.15}$$

due to the first formula of (2.12). Since $u_{n_0} \le Au_{n_0}$ because of $u_{n_0} \in M \subset R$, it follows from (2.15) and the monotonicity of K_i, that

$$u_{n_0} \le Au_{n_0} = \sum_{i=1}^m K_i F_i u_{n_0} \le \sum_{i=1}^m K_i v_{i,n} \le \sum_{i=1}^m K_i v_i^* = u^*,$$

thus (2.13) holds. By (2.13), $v_{i,n} = F_i u_n \le F_i u^*$, that is, $v_{i,n}(t) \le (F_i u^*)(t)$ for almost all $t \in I$. Letting $n \to \infty$ and observing the second formula of (2.12), by Lemma 1 we know $v_i^*(t) \le (F_i u^*)(t)$ for almost all $t \in I$, i.e., $v_i^* \le F_i u^*$. So, by the definition of u^*, $u^* = \sum_{i=1}^m K_i v_i^* \le \sum_{i=1}^m K_i F_i u^* = Au^*$, i.e., (2.14) holds.

For any $u \in M$, if $u_n \le u$ holds for any n, we have $x_n = u_n(t^*) \le u(t^*)$. Observing (2.3) and using Lemma 1, we get $x^* \le u(t^*)$, which contradicts (2.1) and (2.2). The contradiction and (2.13) mean that for $\forall u \in M$, there exists some n_0 such that

$$u \le u_{n_0} \le u^*. \tag{2.16}$$

By (2.14), $Au^* \le A(Au^*)$, thus $Au^* \in R$. (2.14) and (2.16) imply

$$u \le u^* \le Au^*, \quad \forall u \in M. \tag{2.17}$$

For any $v \in N$, it is clear that $v \le Av$ and $Av \in M$ because of $N \subset R$ and $M = A(N)$. Thus, by (2.17) we get $v \le Av \le Au^*$ ($\forall v \in N$). Therefore Au^* is an upper bound of N in R, that is, N has an upper bound in R.

In case of (b): Take $\{t_n\} \subset I$ such that $\{t_n\}$ is dense in I. In this case, there must exist an $x_1 \in M(t_1)$ such that x_1 is an upper bound of $M(t_1)$. Then we can select $u_1 \in M$ such that $u_1(t_1) = x_1$. If $u_1(t_2)$ is an upper bound of $M(t_2)$, let $u_2 = u_1$; if $u_1(t_2)$ is not an upper bound of $M(t_2)$, select $u_2 \in M$ such that $u_2(t_2)$ is an upper bound of $M(t_2)$. Since M is a complete ordered set, it is obvious that $u_1 \le u_2$ and $u_2(t_1) = u_1(t_1)$. Using the same arguments, we can select a sequence $\{u_n\}$ such that

$$u_1 \le u_2 \le \cdots \le u_n \le \cdots,$$

$u_n(t_n)$ is an upper bound of $M(t_n)$ and $u_n(t_i) = u_i(t_i)$ $(1 \le i \le n-1)$. Let $v_{i,n} = F_i u_n$ $(i = 1, 2, \dots, m)$. Evidently (2.7) holds and there exists $v_i^* \in L_{p_i}[I, E]$ such that (2.12) holds. Let $u^* = \sum_{i=1}^m K_i v_i^*$. Then (2.13) and (2.14) hold. In the following, we shall show $u \le u^*$ for any $u \in M$. If otherwise, there exists some $u \in M$ such that $u \not\le u^*$, i.e., there exists $\overline{t} \in I$ such that $u(\overline{t}) \not\le u^*(\overline{t})$. Since $u, u^* \in C[I, E]$, there exists $\delta > 0$ such that when $t \in I$ and $|t - \overline{t}| < \delta$, $u(t) \not\le u^*(t)$ holds. Selecting $t_{n_0} \in \{t_n\}$ such that $|t_{n_0} - \overline{t}| < \delta$, we can get $u(t_{n_0}) \not\le u^*(t_{n_0})$. By (2.13), $u_{n_0} \le u^*$, that is, $u_{n_0}(t_{n_0}) \le u^*(t_{n_0})$. Hence $u(t_{n_0}) \not\le u_{n_0}(t_{n_0})$, which contradicts that $u_{n_0}(t_{n_0})$ is an upper bound of $M(t_{n_0})$. The contradiction means that for any $u \in M$, $u \le u^*$. Using the same arguments as in the final proof of (a), we know that N has an upper bound in R.

By the above discussions, we know that N has one upper bound in R under various conditions. It follows from Zorn's lemma that R has a maximal element. It is clear that any maximal element of R is a fixed point of A. The proof is completed. ∎

THEOREM 2. *If the conditions in Theorem* 1 *are satisfied, then A has the minimal fixed point and the maximal fixed point in D.*

Proof. Set $\mathrm{Fix} A = \{u \in D \mid u = Au\}$. By Theorem 1, $\mathrm{Fix} A \ne \emptyset$. Set

$$S = \{u \in A(D) \mid u \le Au \text{ and } u \le \overline{u}, \forall \overline{u} \in \mathrm{Fix} A\}.$$

Obviously $S \neq \emptyset$ due to $Au_0 \in S$. Take any complete ordered set N in R and let $M = A(N)$. It is clear that $M \subset S$. In the same way as in the proof of Theorem 1, we need to consider two cases separately. In the first case, by the same method of proving Theorem 1 we may find $\{u_n\}$, $\{v_{i,n}\}$, v_i^* and u^*. Thus (2.6), (2.7), (2.8), (2.12), (2.13), (2.14) and (2.16) still hold. For any $\overline{u} \in \mathrm{Fix}A$, it follows from $u_n \in M \subset S$ that $u_n \leq \overline{u}$. Letting $\overline{v_i} = F_i\overline{u}$ and observing $v_{i,n} = F_i u_n$, we know that $v_{i,n} \leq \overline{v_i}$ $(i = 1, 2, \ldots)$, thus $v_{i,n}(t) \leq \overline{v}(t)$ for almost all $t \in I$. By (2.12) and in view of Lemma 1, $v_i^*(t) \leq \overline{v_i}(t)$ for almost all $t \in I$, i.e., $v_i^* \leq \overline{v_i}$. Since $\overline{u}$ is a fixed point of $A = \sum_{i=1}^m K_i F_i$,

$$u^* = \sum_{i=1}^m K_i v_i^* \leq \sum_{i=1}^m K_i \overline{v_i} = \sum_{i=1}^m K_i F_i \overline{u} = A\overline{u} = \overline{u},$$

thus $Au^* \leq A\overline{u} = \overline{u}$. By (2.14) and (2.16), we get

$$Au^* \leq A(Au^*), \quad u \leq u^* \leq Au^*, \quad \forall u \in M. \tag{2.18}$$

The above discussions show that $Au^* \in S$. For any $v \in N$, by $M = A(N)$, we know $Av \in M$. Observing $v \leq Av$ due to $N \subset S$, by (2.18) we have

$$v \leq Av \leq Au^*, \quad \forall v \in N,$$

which implies that N has an upper bound in S. In the second case, we can use similar arguments to show that N has an upper bound in S. Hence it follows from Zorn's lemma that S has a maximal element $w \in S$. Clearly

$$w \leq Aw, \quad w \leq \overline{u}, \quad \forall \overline{u} \in \mathrm{Fix}A, \tag{2.19}$$

which means $Aw \leq A(Aw)$ and $Aw \leq A\overline{u} = \overline{u}$ $(\forall \overline{u} \in \mathrm{Fix}A)$. So $Aw \in S$. Since w is a maximal element of S, by (2.19) we get $w = Aw$. Observing (2.19) again, we know that w is a minimal fixed point of A in D. Similarly, A has a maximal fixed point in D. The proof is completed. ∎

Remark 1. It is clear to see from the proof of Theorem 1 and Theorem 2 that if I is a measurable closed subset of non-zero measure in R^n, the two theorems still hold.

Remark 2. Comparing with some results in [1]–[7], we easily see that Theorem 1 and Theorem 2 are their generalizations and improvements.

§3. Applications

We first list for convenience the following assumptions:

(H_1) E is sequentially weakly complete, P a normal cone in E.

(H_2) $f_i(t,x) : J \times E \to E$ $(i = 1,2,3,\ J = [0,1]$, we do not suppose that $f_i(t,x)$ are continuous), and the Nemytskii operators

$$f_1 u = f_1(t,u(t)), \quad F_i u = f_i(t,u(t)), \quad i = 2,3 \tag{3.1}$$

map continuous functions into strongly measurable functions.

(H_3) There exists $M > 0$ such that for $x, y \in E$, $y \le x$,

$$f_1(t,x) - f_1(t,y) \ge -M(x-y),$$

and $f_i(t,x)$ $(i = 2,3)$ are increasing on x for $t \in J$.

Consider the nonlinear integro-differential equation

$$\begin{cases} u'(t) = f_1(t,u(t)) + \displaystyle\int_0^t k_1(t,s) f_2(s,u(s))\,ds \\ \qquad\qquad + \displaystyle\int_J k_2(t,s) f_3(s,u(s))\,ds, \\ u(0) = x_0, \end{cases} \tag{3.2}$$

where $t \in J$, $k_1(t,s) : \{(t,s) \in J \times J \mid s \le t\} \to R^1$ and $k_2(t,s) : J \times J \to R^1$ are nonnegative and continuous. By the direct proof, it is easy to follow that the initial value problem (3.2) is equivalent to the equation

$$\begin{aligned} u(t) = e^{-Mt} x_0 + \int_0^t e^{-M(t-s)} \Big[&(f_1(s,u(s)) + Mu(s)) \\ &+ \int_0^s k_1(s,\tau) f_2(\tau,u(\tau))\,d\tau + \int_J k_2(s,\tau) f_3(\tau,u(\tau))\,d\tau \Big] ds, \end{aligned} \tag{3.3}$$

if $f_1(t,x)$ is continuous, where M is a constant given by (H_3) (also see Theorem 1.5.1 in [1]). Hence, when $f_1(t,x)$ is not continuous, we define the solutions of integral equation (3.3) as the solutions of the equation (3.2).

THEOREM 3. *Suppose that the assumptions* (H_1)–(H_3) *are fulfilled and there exist* $u_0, v_0 \in C^1[J,E] = \{u \in C[J,E] \mid u(t)$ *is differentiable*$\}$, $u_0 \le v_0$, $1 \le p_1, p_2, p_3 < \infty$, *such that*

$$f_1 u_0, f_1 v_0 \in L_{p_1}[J,E], \quad F_i u_0, F_i v_0 \in L_{p_i}[J,E], \quad i = 2,3, \tag{3.4}$$

$$
(3.5)\qquad \begin{cases} u_0'(t) \le f_1(t, u_0(t)) + \displaystyle\int_0^t k_1(t,s) f_2(s, u_0(s))\, ds \\ \qquad\qquad + \displaystyle\int_J k_2(t,s) f_3(s, u_0(s)) ds, \\ u_0(0) \le x_0, \end{cases}
$$

$$
(3.6)\qquad \begin{cases} v_0'(t) \ge f_1(t, v_0(t)) + \displaystyle\int_0^t k_1(t,s) f_2(s, v_0(s))\, ds \\ \qquad\qquad + \displaystyle\int_J k_2(t,s) f_3(s, v_0(s))\, ds, \\ v_0(0) \ge x_0. \end{cases}
$$

Then Eq. (3.2) *has the maximal solution and minimal solution in* $D = [u_0, v_0] = \{u \in C[J,E] \mid u_0 \le u \le v_0\}$.

Proof. For any $u \in C[J,E]$, by (3.3) we can define the mapping

$$
(3.7)\quad Au = e^{-Mt} x_0 + \int_0^t e^{-M(t-s)} \Big[(f_1(s, u(s)) + Mu(s)) + \int_0^s k_1(s,\tau) f_2(\tau, u(\tau))\, d\tau + \int_J k_2(s,\tau) f_3(\tau, u(\tau))\, d\tau \Big] ds.
$$

$$
(3.8)\qquad K_1 h_1 = e^{-Mt} x_0 + \int_0^t e^{-M(t-s)} h_1(s)\, ds, \quad \forall\, h_1 \in L_{p_1}[J,E],
$$

$$
(3.9)\quad K_2 h_2 = \int_0^t ds \int_0^s e^{-M(t-s)} k_1(s,\tau) h_2(\tau)\, d\tau, \quad \forall\, h_2 \in L_{p_2}[J,E],
$$

$$
(3.10)\quad K_3 h_3 = \int_0^t ds \int_J e^{-M(t-s)} k_2(s,\tau) h_3(\tau)\, d\tau, \quad \forall\, h_3 \in L_{p_3}[J,E].
$$

By the nonnegativity of $k_1(t,s)$ and $k_2(t,s)$, it is easy to show that K_i are increasing from $L_{p_i}[J,E]$ into $C[J,E]$ ($i = 1,2,3$). Set

$$
(3.11)\qquad F_1 u = f_1 u + Mu, \quad u \in C[J,E].
$$

By (H_2), F_1 maps elements of $C[J,E]$ into strongly measurable functions. For any $u \in [u_0, v_0]$, by (H_3) we get $F_1 u_0 \le F_1 u \le F_1 v_0$. Hence for almost all $t \in J$, $0 \le (F_1 u)(t) - (F_1 u_0)(t) \le (F_1 v_0)(t) - (F_1 u_0)(t)$. On account of the normality of P, there exists a constant $L > 0$ such that

$$
\|(F_1 u)(t) - (F_1 u_0)(t)\| \le L \|(F_1 v_0)(t) - (F_1 u_0)(t)\|,
$$

which, by (3.4), (3.11), implies $F_1u \in L_{p_1}[J,E]$. So F_1 is an increasing operator from $[u_0,v_0]$ into $L_{p_1}[J,E]$. Similarly, by (3.1), (3.4) and (H_2), we can prove that $F_i : [u_0,v_0] \to L_{p_i}[J,E]$ $(i=2,3)$ are increasing. So by (3.1), (3.11) and (3.7)–(3.10), we can get

$$A = \sum_{i=1}^{3} K_i F_i. \tag{3.12}$$

In view of the above discussions, we may have that A is an increasing operator from $C[J,E]$ into $C[J,E]$.

Let $D_1 = \{w \in L_{p_1}[J,E] \mid F_1u_0 \le w \le F_1v_0\}$, it is clear to see from the monotonicity of F_1 that

$$F_1(D) \subset D_1, \tag{3.13}$$

and $F_1u_0 \le w \le F_1v_0$ for any $w \in D_1$. By using the normality of P, we can get

$$\|w(t)\| \le \|(F_1u_0)(t)\| + L\|(F_1v_0)(t) - (F_1u_0)(t)\| \tag{3.14}$$

for almost all $t \in J$, here L is a normal constant. For $t \in J$, set $D_1(t) = \{w(t) \mid w \in D_1\}$. By (3.4), (3.11) and (3.14), there exist $J_0 \subset J$ and $\operatorname{mes} J_0 = \operatorname{mes} J$ such that for $t \in J_0$, $D_1(t)$ is a bounded set in E. Now we show that any complete ordered set of $D_1(t)$ $(t \in J_0)$ is relatively weakly compact. Let $N \subset D_1(t)$ $(t \in J_0)$ be a complete ordered set and $\{x_n\}$ a sequence in N. We consider two cases:

(a) There exists an infinite set $\{x^{(k)}\} \subset \{x_n\}$ such that

$$x^{(1)} = \inf\{x_n\},\ \ x^{(k)} = \inf\{\{x_n\}\backslash\{x^{(1)}, x^{(2)}, \dots, x^{(k-1)}\}\},\ \ k=1,2,\dots.$$

Thus

$$(F_1u_0)(t) \le x^{(1)} \le x^{(2)} \le \cdots \le x^{(k)} \le \cdots \le (F_1v_0)(t), \quad t \in J_0. \tag{3.15}$$

Since the cone P is normal, P is reproduced by Proposition 19.4 in [2], that is, for any $\phi \in E^*$, there exist $\phi_i \in P^*$ $(i=1,2)$ such that $\phi = \phi_1 - \phi_2$. By (3.15), we have

$$\phi_i((F_1u_0)(t)) \le \phi_i(x^{(1)}) \le \phi_i(x^{(2)}) \le \cdots \le \phi_i(x^{(k)}) \le \cdots \le \phi_i((F_1v_0)(t)), \quad i=1,2,\ \ t \in J_0,$$

which, together with the boundedness of $\{x^{(k)}\} \subset D_1(t)$ $(t \in J_0)$, shows that $\{\phi_i(x^{(k)})\}$ $(i = 1, 2)$ are Cauchy sequence in R^1. Hence $\{x^{(k)}\}$ is weakly Cauchy sequence in E since $\phi \in E^*$ is arbitrary. Since E is sequentially weakly complete, $\{x^{(k)}\}$ converges weakly to some element in E.

(b) There exists no $x \in \{x_n\}$ such that $x = \inf\{x_n\}$, or there exists a finite set $\{\overline{x}^{(k)}\} \subset \{x_n\}$ such that

$$\overline{x}^{(1)} = \inf\{x_n\}, \;\; \overline{x}^{(k)} = \inf\{\{x_n\} \backslash \{\overline{x}^{(1)}, \overline{x}^{(2)}, \dots, \overline{x}^{(k-1)}\}, \quad k = 2, 3, \dots, k_0,$$

and $x \neq \inf M_1$ for any $x \in M_1$, here $M_1 = \{x_n\} \backslash \{\overline{x}^{(1)}, \overline{x}^{(2)}, \dots, \overline{x}^{(k_0)}\}$. So we can obtain an infinite set $\{x^{(k)}\} \subset M_1$ such that

$$(F_1 u_0)(t) \leq \cdots \leq x^{(k)} \leq \cdots \leq x^{(2)} \leq x^{(1)} \leq (F_1 v_0)(t), \quad t \in J_0. \tag{3.16}$$

Using the same method as in the proof of (a), we know that $\{x^{(k)}\}$ given by (3.16) converges weakly to some element in E.

By above discussions, any sequence $\{x_n\}$ of the complete ordered set $N \subset D_1(t)$ $(t \in J_0)$ has a convergent subsequence of $\{x_n\}$, that is, any complete ordered set of $D_1(t)$ $(t \in J_0)$ is relatively weakly compact. Observing (3.13) and the boundedness of $D_1(t)$ $(t \in J_0)$, we know that for almost all $t \in J$, any complete ordered set $(F_1 D)(t) = \{w(t) \mid w \in F_1(D)\} \subset D_1(t)$ is relatively weakly compact, and $F_1(D)$ is a bounded set in $L_{p_1}[J, E]$. Using the similar arguments, we can show that for almost all $t \in J$, any complete ordered set of $(F_i D)(t) = \{w(t) \mid w \in F_i(D)\}$ $(i = 2, 3)$ is relatively weakly compact in E and $F_i(D)$ are bounded sets in $L_{p_i}[J, E]$ $(i = 2, 3)$; for any $t \in J$, any complete ordered set of $(K_i D_i)(t) = \{u(t) \mid u \in K_i(D_i)\}$ $(i = 1, 2, 3)$ is also relatively weakly compact in E. Thus condition (i) and (ii) in Theorem 1 are satisfied.

We now show that condition (iii) in Theorem 1 is fulfilled. By (3.7) and (3.5), we have

$$\begin{aligned}
(Au_0)(t) - u_0(t) &= \sum_{i=1}^{3} K_i F_i u_0(t) - u_0(t) \\
&= e^{-Mt} x_0 + \int_0^t e^{-M(t-s)} \Big[\big(f_1(s, u_0(s)) + M u_0(s) \big) \\
&\qquad + \int_0^s k_1(s, \tau) f_2(\tau, u_0(\tau))\, d\tau + \int_J k_2(s, \tau) f_3(\tau, u_0(\tau))\, d\tau \Big] ds - u_0(t) \\
&\geq e^{-Mt} x_0 + e^{-Mt} \int_0^t e^{Ms} \big[u_0'(s) + M u_0(s) \big] ds - u_0(t)
\end{aligned}$$

$$= e^{-Mt}x_0 + e^{-Mt}\big(e^{Mt}u_0(t) - u_0(0)\big) - u_0(t)$$
$$= e^{-Mt}\big(x_0 - u_0(0)\big) \geq \theta,$$

which means $u_0 \leq Au_0$. Similarly we can prove that $Av_0 \leq v_0$.

Since all conditions in Theorem 1 are satisfied, by Theorem 1 and Theorem 2, A has the maximal fixed point and the minimal fixed point in D. Noting that fixed points of A are equivalent to solutions of Eq. (3.3), and Eq. (3.3) is equivalent to Eq. (3.2), the conclusions of Theorem 3 hold. The proof is completed. □

Remark 3. In Theorem 1 and Theorem 2, the increasing operator A is divided into $\sum_{i=1}^{m} K_iF_i$ such that $(F_iD)(t)$ (almost all $t \in I$) and $(K_iD_i)(t)$ ($t \in I$) need only weak compact conditions in E ($i = 1, 2, \ldots, m$). It is clear to see from Theorem 3 that these conditions are examined easily. Moreover, some concrete problems possess the form $\sum_{i=1}^{m} K_iF_i$ originally. Hence this is very convenient in applications.

Remark 4. In order to study nonlinear equations in Banach spaces, the compactness type conditions and the dissipative type conditions are widely used (see [1]–[5]). But we do not use any condition of the aspects in Theorem 3 of this paper.

Remark 5. Since many widely used spaces such as Hilbert spaces, reflexive spaces and L_1 spaces are all sequentially weakly complete, Theorem 3 still holds in these spaces.

Acknowledgements. The author wishes to thank Professors Guo Dajun and Sun Jingxian for their help and suggestions. The author is also indebted to the referees for their comments and suggestions.

References

[1] V. Lakshmikanthan, D. Guo and X. Liu, Nonlinear integral equations in abstract space, Kluwer Academic Publisher, Boston, London, 1996.
[2] K. Deiming, Nonlinear functional analysis, Springer-Verlag, New York, Berli, 1985.
[3] H. Aman, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIMA Rev., **18** (1976), 620–709.
[4] D. Guo and V. Lakshmikantham, Nonlinear problems in abstract cones, Academic Press, New York, 1988.

[5] S. Jingxian and S. Yong, *Some fixed point theorems of increasing operator*, Appl. Anal., **23** (1986), 23–27.

[6] S. Jingxian, *Some new compactness criterion and their applications in Banach spaces*, Chin. Ann. of Math. (Chinese), **11A** (1990), 408–412.

[7] S. Jingxian, *ixed point theorems of increasing operators and applications to nonlinear equations with discontinuous terms*, Acta Mathematica Sinica (Chinese), **31** (1989), 101–107.

[8] V. Lakshmikantham and S. Leela, Differential and integral inequality, Academic Press, New York, 1969.

[9] K. Yosida, Functional analysis, Springer, Berlin, 1978.

[10] J. L. Kelley, General topology, Van Nostrand, New York, 1955.

Department of Basic Courses
Shandong Finance Institute
Jinan
Shandong, 250014
China