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Nagoya Math. J.
Vol. 158 (2000), 63–72

TANGENT LOCI AND CERTAIN LINEAR SECTIONS OF ADJOINT VARIETIES

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Abstract. An *adjoint variety* $X(\mathfrak{g})$ associated to a complex simple Lie algebra $\mathfrak{g}$ is by definition a projective variety in $\mathbb{P}_*(\mathfrak{g})$ obtained as the projectivization of the (unique) non-zero, minimal nilpotent orbit in $\mathfrak{g}$. We first describe the tangent loci of $X(\mathfrak{g})$ in terms of $\mathfrak{sl}_2$-triples. Secondly for a graded decomposition of contact type $\mathfrak{g} = \bigoplus_{-2 \le i \le 2} \mathfrak{g}_i$, we show that the intersection of $X(\mathfrak{g})$ and the linear subspace $\mathbb{P}_*(\mathfrak{g}_1)$ in $\mathbb{P}_*(\mathfrak{g})$ coincides with the cubic Veronese variety associated to $\mathfrak{g}$.

Introduction

The purpose of this article is to study tangent loci and certain linear sections of adjoint varieties.

Let $\mathfrak{g}$ be a complex simple Lie algebra, G the inner automorphism of $\mathfrak{g}$, λ the highest root of $\mathfrak{g}$ with respect to some Cartan subalgebra and to some basis of the roots, and $X_{\pm\lambda}$ the root vectors such that $(X_\lambda, H, X_{-\lambda})$ forms an $\mathfrak{sl}_2$-triple for some $H \in \mathfrak{g}$. Consider the adjoint orbit $G \cdot X_\lambda \subseteq \mathfrak{g}$, which is the (unique) non-zero, minimal nilpotent orbit. We call its projectivization $\pi(G \cdot X_\lambda) \subseteq \mathbb{P}_*(\mathfrak{g})$ the *adjoint variety* associated to $\mathfrak{g}$, and set

$$X(\mathfrak{g}) := \pi(G \cdot X_\lambda),$$

where $\pi : \mathfrak{g} \setminus \{0\} \to \mathbb{P}_*(\mathfrak{g})$ is the canonical projection with $\mathbb{P}_*(\mathfrak{g}) := (\mathfrak{g} \setminus \{0\})/\mathbb{C}^\times$ (see, for example, [KOY]).

For a smooth projective variety $X \subseteq \mathbb{P}^N$, the *tangent locus* Θ_z with respect to a point $z \in \mathbb{P}^N$ is defined by

$$\Theta_z := \{x \in X \mid T_x X \ni z\},$$

where $T_x X$ denotes the embedded tangent space to X at x, that is, the unique linear subspace L of $\mathbb{P}^N$ such that the (abstract) tangent spaces

Received March 25, 1999.
1991 *Mathematics Subject Classification*: Primary 14M17, 17B70; Secondary 17B20, 14N05.

to X and to L at x coincide in that of $\mathbb{P}^N$ as vector subspaces (see, for example, [FR]).

The first result here describes tangent loci of adjoint varieties as follows:

Theorem A. *For $x, y \in X(\mathfrak{g})$ in general position, we have*

$$\Theta_{[x,y]} = \{x, y\},$$

where we set $[x, y] := \pi([\pi^{-1}x, \pi^{-1}y])$.

Let $\operatorname{Sec} X(\mathfrak{g})$ be the *secant variety* of $X(\mathfrak{g}) \subseteq \mathbb{P}_*(\mathfrak{g})$, that is, the closure of the union of all projective lines which contain two or more points of $X(\mathfrak{g})$. According to [KOY, Proposition 5.3], the adjoint orbit $G \cdot \pi H$ is dense in $\operatorname{Sec} X(\mathfrak{g})$. Therefore from Theorem A it turns out that *for $z \in \operatorname{Sec} X(\mathfrak{g})$ in general position, Θ_z consists of exactly two points and if $\Theta_z = \{x, y\}$, then there exists an $\mathfrak{sl}_2$-triple (X, K, Y) such that $\pi X = x$, $\pi Y = y$ and $\pi K = z$.* Note that $\operatorname{Sec} X(\mathfrak{g})$ coincides with the tangential variety, that is, the union of all embedded tangent spaces of $X(\mathfrak{g})$ (see [KOY, §5]).

Next, we set

$$\mathfrak{g}_i := \{Y \in \mathfrak{g} \mid (\operatorname{ad} H)Y = iY\},$$
$$M := \{Y \in \mathfrak{g}_1 \mid Y \neq 0, (\operatorname{ad} Y)^2 \mathfrak{g}_{-2} = 0\}.$$

We obtain a linear subspace $\mathbb{P}_*(\mathfrak{g}_1)$ of $\mathbb{P}_*(\mathfrak{g})$. The second result is

Theorem B. *We have*

$$X(\mathfrak{g}) \cap \mathbb{P}_*(\mathfrak{g}_1) = \pi M.$$

The projective varieties $\pi M \subseteq \mathbb{P}_*(\mathfrak{g}_1)$ appeared above are known as the *cubic Veronese varieties*, while M are known as *Freudenthal's varieties of planes* (see, for example, [F], [M]).

§1. Preliminaries

Lemma 1. (cf. [KOY, §3]) *We have*

$$G \cdot X_\lambda = \{Y \in \mathfrak{g} \mid Y \neq 0, \ (\operatorname{ad} Y)^2 \mathfrak{g} \subseteq \mathbb{C} \cdot Y\}.$$

Proof. For the inclusion $\subseteq$, it suffices to show that $(\operatorname{ad} X_\lambda)^2\mathfrak{g} \subseteq \mathbb{C}\cdot X_\lambda$, and this is clear since X_λ is a highest root vector.

For the converse, let $Y \in \mathfrak{g}$ be a non-zero element such that $(\operatorname{ad} Y)^2\mathfrak{g} \subseteq \mathbb{C}\cdot Y$. Since Y is nilpotent with $(\operatorname{ad} Y)^3 = 0$, according to a theorem of Jacobson-Morozov (see, for example, [CM, §3.3]), there exist $K, Z \in \mathfrak{g}$ such that (Y, K, Z) forms an $\mathfrak{sl}_2$-triple with semi-simple element K. Set $\mathfrak{g}'_i := \{X \in \mathfrak{g} \mid (\operatorname{ad} K)X = iX\}$. Then $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}'_i$, and $\mathfrak{g}'_i = 0$ if $|i| > 2$ (see, for example, [CM, §§3.4–3.5]). Moreover, it follows from $(\operatorname{ad} Y)^2\mathfrak{g} \subseteq \mathbb{C}\cdot Y$ that

$$\mathfrak{g}'_2 = \mathbb{C}\cdot Y.$$

Indeed, we have $(\operatorname{ad} Y)^2 \circ (\operatorname{ad} Z)^2|_{\mathfrak{g}'_2} = 4\operatorname{id}_{\mathfrak{g}'_2}$, whose image is contained in $\mathbb{C}\cdot Y$. This implies that Y is a highest root vector with respect to some Cartan subalgebra $\mathfrak{h}'$ containing K and to the lexicographic order on the roots defined by a basis of $\mathfrak{h}'$ of the form, $H_1 := K, H_2, \ldots, H_l$ with $\operatorname{rk}\mathfrak{g} = l$. Thus, we have $Y \in G\cdot X_\lambda$. ∎

Lemma 2. *We have*

$$G\cdot X_\lambda \cap \mathfrak{g}_1 \subseteq M.$$

Proof. If $Y \in G\cdot X_\lambda \cap \mathfrak{g}_1$, then it follows from Lemma 1 that

$$(\operatorname{ad} Y)^2 X_{-\lambda} \in \mathbb{C}\cdot Y \cap \mathfrak{g}_0 \subseteq \mathfrak{g}_1 \cap \mathfrak{g}_0 = \{0\}.$$

Therefore $(\operatorname{ad} Y)^2 X_{-\lambda} = 0$, that is, $Y \in M$. ∎

Following [A1], [A2], we introduce a skew-symmetric form

$$\langle\ ,\ \rangle : \mathfrak{g}_1 \times \mathfrak{g}_1 \longrightarrow \mathbb{C}$$

and a symmetric bi-linear product

$$\times : \mathfrak{g}_1 \times \mathfrak{g}_1 \longrightarrow \mathfrak{g}_0,$$

which are respectively defined by

$$\begin{aligned} 2\langle P, Q\rangle X_\lambda &:= [P, Q], \\ -2P\times Q &:= [P[Q, X_{-\lambda}]] + [Q[P, X_{-\lambda}]], \end{aligned}$$

for $P, Q, R \in \mathfrak{g}_1$. Note that using this notation we have

$$M = \{P \in \mathfrak{g}_1 \mid P \neq 0,\ P\times P = 0\}.$$

PROPOSITION 1. (a) *For $P, Q \in \mathfrak{g}_1$, we have*

$$P \times Q = 0,\ P \in M \Longrightarrow \langle P, Q \rangle = 0.$$

(b) *For $P \in \mathfrak{g}_1$, $Z \in \mathfrak{g}_0$, set $Z^{\#} := [P, Z] \in \mathfrak{g}_1$. Then we have*

$$P \in M \Longrightarrow P \times Z^{\#} = 0,$$

hence $\langle P, Z^{\#} \rangle = 0$.

Proof. (a) Since $P \in M$, using the Jacobi identity we have

$$\begin{aligned}[P[[P, X_{-\lambda}]Q]] &= -[Q[P[P, X_{-\lambda}]]] - [[P, X_{-\lambda}][Q, P]] \\ &= -[Q, 0] + 2\langle P, Q \rangle [[P, X_{-\lambda}]X_\lambda] \\ &= 2\langle P, Q \rangle P.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}[P[[P, X_{-\lambda}]Q]] &= -[P[[Q, P]X_{-\lambda}]] - [P[[X_{-\lambda}, Q]P]] \\ &= -2\langle Q, P \rangle [P, H] - [P, (-2P \times Q - [Q[P, X_{-\lambda}]])] \\ &= -2\langle P, Q \rangle P + 2[P, P \times Q] + [P[Q[P, X_{-\lambda}]]],\end{aligned}$$

so that $[P[[P, X_{-\lambda}]Q]] = -\langle P, Q \rangle P$ since $P \times Q = 0$. Therefore it follows $3\langle P, Q \rangle P = 0$, hence $\langle P, Q \rangle = 0$ whether $P = 0$ or not.

(b) Using the Jacobi identity and the assumption $P \in M$, since $[Z, X_{-\lambda}] \in \mathfrak{g}_{-2}$, we have

$$\begin{aligned}[P[Z^{\#}, X_{-\lambda}]] &= [P[[P, Z]X_{-\lambda}]] \\ &= -[P[[Z, X_{-\lambda}]P]] - [P[[X_{-\lambda}, P]Z]] \\ &= -[P[[X_{-\lambda}, P]Z]], \\ [Z^{\#}[P, X_{-\lambda}]] &= [[P, Z], [P, X_{-\lambda}]] \\ &= -[[Z[P, X_{-\lambda}]]P] - [[[P, X_{-\lambda}]P]Z] \\ &= -[[Z[P, X_{-\lambda}]]P].\end{aligned}$$

Thus we obtain $P \times Z^{\#} = -\frac{1}{2}\{[P[Z^{\#}, X_{-\lambda}]] + [Z^{\#}[P, X_{-\lambda}]]\} = 0.$ ∎

Next we consider a subalgebra of $\mathfrak{g}_0$ as follows:

$$\mathfrak{D}_0 := \{Z \in \mathfrak{g}_0 \mid (\operatorname{ad} Z)\mathfrak{g}_{-2} = 0\}.$$

LEMMA 3. $[\mathfrak{g}_0, \mathfrak{g}_0] \subseteq \mathfrak{D}_0$.

Proof. Since $[\mathfrak{g}_0, H] = 0$, we have $[\mathfrak{g}_0, \mathfrak{g}_0] = [\mathfrak{D}_0 \oplus \mathbb{C} \cdot H, \mathfrak{D}_0 \oplus \mathbb{C} \cdot H] = [\mathfrak{D}_0, \mathfrak{D}_0] \subseteq \mathfrak{D}_0$. ∎

PROPOSITION 2. (a) $\mathfrak{g}_1 \times \mathfrak{g}_1 \subseteq \mathfrak{D}_0$.
(b) *For* $Y \in \mathfrak{g}_{-1}$, $P \in \mathfrak{g}_1$, *we have*

$$[Y, P] = -Y^+ \times P - \langle Y^+, P \rangle H,$$

where we set $Y^+ := [X_\lambda, Y]$.

Proof. (a) It follows from the Jacobi identity that for $P_1, P_2 \in \mathfrak{g}_1$ we have

$$\begin{aligned} [[P_i[P_j, X_{-\lambda}]]X_\lambda] &= -[[[P_j, X_{-\lambda}]X_\lambda]P_i] - [[X_\lambda, P_i], [P_j, X_{-\lambda}]] \\ &= -[P_j, P_i] - [0, [P_j, X_{-\lambda}]] \\ &= [P_i, P_j], \end{aligned}$$

where $[X_\lambda, P_i] \in \mathfrak{g}_3 = 0$. Therefore we have

$$-2[P_1 \times P_2, X_\lambda] = [([P_1[P_2, X_\lambda]] + [P_2[P_1, X_\lambda]]), X_\lambda] = [P_1, P_2] + [P_2, P_1] = 0,$$

so that $P_1 \times P_2 \in \mathfrak{D}_0$.

(b) Dividing into two, applying the Jacobi identity to the latter term below, we have

$$\begin{aligned} [Y, P] &= [[X_{-\lambda}, Y^+]P] \\ &= \frac{1}{2}[[X_{-\lambda}, Y^+]P] + \frac{1}{2}[[X_{-\lambda}, Y^+]P] \\ &= \frac{1}{2}[[X_{-\lambda}, Y^+]P] + \frac{1}{2}\big(- [[Y^+, P]X_{-\lambda}] - [[P, X_{-\lambda}]Y^+]\big) \\ &= \frac{1}{2}\big([[X_{-\lambda}, Y^+]P] + [[X_{-\lambda}, P]Y^+]\big) - \langle Y^+, P \rangle [X_\lambda, X_{-\lambda}] \\ &= -Y^+ \times P - \langle Y^+, P \rangle H. \end{aligned}$$

∎

§2. Tangent loci

Proof of Theorem A. We first show that

$$\Theta_{\pi H} = \{\pi X_\lambda, \pi X_{-\lambda}\}.$$

Since $T_{\pi P}X(\mathfrak{g}) = \mathbb{P}_*([\mathfrak{g}, P])$ for $P \in G \cdot X_\lambda$ (see [KOY, Lemma 2.1]), in terms of Lie algebra, this is equivalent to showing that

$$\{P \in G \cdot X_\lambda \mid [\mathfrak{g}, P] \ni H\} = \mathbb{C}^\times \cdot X_\lambda \sqcup \mathbb{C}^\times \cdot X_{-\lambda}.$$

Since the inclusion $\supseteq$ is trivial, it suffices to show that for $g \in G$ and $Y \in \mathfrak{g}$ we have

$$H = [Y, gX_\lambda] \Longrightarrow gX_\lambda \in \mathfrak{g}_2 \cup \mathfrak{g}_{-2}.$$

Here we have

$$gX_\lambda \in \mathfrak{g}_i$$

for some i with $-2 \leq i \leq 2$: Indeed, it follows from Lemma 1 that

$$[H, gX_\lambda] = [[Y, gX_\lambda]gX_\lambda] = (\operatorname{ad} gX_\lambda)^2 Y \in \mathbb{C} \cdot gX_\lambda,$$

so that gX_λ is an eigenvector of $\operatorname{ad} H$.

If we write $Y = \sum_{j=-2}^{2} Y_j$ with $Y_j \in \mathfrak{g}_j$, then we have

$$H = [Y, gX_\lambda] = \sum_{j=-2}^{2} [Y_j, gX_\lambda].$$

Since $H \in \mathfrak{g}_0$ and $[Y_j, gX_\lambda] \in \mathfrak{g}_{i+j}$, by taking the component of degree 0 we obtain

$$H = [Y_{-i}, gX_\lambda].$$

Thus taking $Y := Y_{-i}$, we may assume $Y \in \mathfrak{g}_{-i}$.

Now we first claim that $i \neq 0$. Suppose $i = 0$: it follows from Lemma 3 that

$$H = [Y, gX_\lambda] \in [\mathfrak{g}_0, \mathfrak{g}_0] \subseteq \mathfrak{D}_{\mathfrak{o}},$$

that is, $H \in \mathfrak{D}_0$. This contradicts to $[H, X_\lambda] = 2X_\lambda \neq 0$. Thus we have $i \neq 0$.

Next we claim that $i \neq \pm 1$. Suppose $i = 1$: we have $Y \in \mathfrak{g}_{-1}$, $gX_\lambda \in \mathfrak{g}_1$, and it follows from Proposition 2 (b) that

$$H = [Y, gX_\lambda] = -Y^+ \times gX_\lambda - \langle Y^+, gX_\lambda \rangle H.$$

Taking account of the decomposition $\mathfrak{g}_0 = \mathfrak{D}_0 \oplus \mathbb{C} \cdot H$ and Proposition 2 (a), comparing both sides above, we obtain two equalities,

$$Y^+ \times gX_\lambda = 0 \quad \text{and} \quad \langle Y^+, gX_\lambda \rangle = -1.$$

Now it follows from Lemma 2 that $gX_\lambda \in M$. Therefore, by Proposition 1 (a) we obtain from the former equality that $\langle Y^+, gX_\lambda \rangle = 0$. But this contradicts to the latter equality. Thus, $i \neq 1$. Similarly we obtain $i \neq -1$.

Therefore $i = 2$ or $i = -2$, and this completes the proof of our claim.

Now the statement for general case follows from the claim above. Indeed, there exists $g \in G$ such that

$$([x, y], x, y) = g \cdot (h, x_+, x_-),$$

since the orbit $G \cdot (x_+, x_-)$ is dense in $X(\mathfrak{g}) \times X(\mathfrak{g})$, where we set $h := \pi H$ and $x_\pm := \pi X_{\pm\lambda}$. The density is checked by counting the dimension of the orbit $G \cdot (x_+, x_-)$. Indeed, in terms of the stabilizers $C_G(x_\pm)$ of $x_\pm$, respectively, the stabilizer of (x_+, x_-) is given by $C_G(x_+) \cap C_G(x_-)$, whose Lie algebra is $\mathfrak{g}_0$ since the Lie algebras of $C_G(x_\pm)$ are respectively equal to $\mathfrak{g}_0 \oplus \mathfrak{g}_{\pm 1} \oplus \mathfrak{g}_{\pm 2}$. Therefore,

$$\dim G \cdot (x_+, x_-) = \dim \bigoplus_{i \neq 0} \mathfrak{g}_i = 2 \dim X(\mathfrak{g}).$$

□

§3. Cubic veronese varieties

Proof of Theorem B. The claim obviously follows from

$$G \cdot X_\lambda \cap \mathfrak{g}_1 = M,$$

and we here show the inclusion $\supseteq$: the converse is just Lemma 2. By virtue of Lemma 1, it suffices to show that if $Y \in M$, then

$$(\operatorname{ad} Y)^2 Z \in \mathbb{C} \cdot Y$$

for all $Z \in \mathfrak{g}_i$ with $-2 \leq i \leq 2$.

In case of $i = -2$, this is obvious from the definition of M. If $i > 0$, then the claim follows since $(\operatorname{ad} Y)^2 Z \in \mathfrak{g}_{i+2} = 0$ with $i + 2 > 2$.

In case of $i = 0$, set $Z^{\#} := [Y, Z]$. According to Proposition 1 (b), we have $\langle Y, Z^{\#}\rangle = 0$, that is, $[Y, Z^{\#}] = 0$ and the claim follows.

In case of $i = -1$, set $Z^{+} := [X_{\lambda}, Z]$. We have $(\operatorname{ad} Y)^2 Z = 4\langle Y, Z^{+}\rangle Y$. Indeed, applying the Jacobi identity twice, we have

$$\begin{aligned}(\operatorname{ad} Y)^2 Z &= [Y[Y[X_{-\lambda}, Z^{+}]]] \\ &= -[Y[X_{-\lambda}[Z^{+}, Y]]] - [Y[Z^{+}[Y, X_{-\lambda}]]] \\ &= -2\langle Z^{+}, Y\rangle[Y[X_{-\lambda}, X_{\lambda}]] \\ &\qquad + \big\{[Z^{+}[[Y, X_{-\lambda}]Y]] + [[Y, X_{-\lambda}], [Y, Z^{+}]]\big\} \\ &= -2\langle Z^{+}, Y\rangle[Y, -H] + [Z^{+}, 0] + 2\langle Y, Z^{+}\rangle[[Y, X_{-\lambda}]X_{\lambda}] \\ &= 2\langle Y, Z^{+}\rangle Y + 0 + 2\langle Y, Z^{+}\rangle Y \\ &= 4\langle Y, Z^{+}\rangle Y.\end{aligned}$$

□

We finally give a few examples where, using Theorem B, one can easily as well as geometrically determine cubic Veronese varieties.

Example 1. The cubic Veronese variety $\pi M \subseteq \mathbb{P}_*(\mathfrak{g}_1)$ is $\mathbb{P}^{l-2} \sqcup \mathbb{P}^{l-2}$, a disjoint union of two linear subspaces in $\mathbb{P}^{2l-3} \simeq \mathbb{P}_*(\mathfrak{g}_1)$ if $\mathfrak{g}$ is of type A_l. Indeed, in this case, $X(\mathfrak{g})$ is realized as the projectivization of the set of traceless matrices $[z_{ij}]_{0\le i,j\le l}$ with rank 1 (see, for example [FH, p. 389]). On the other hand, taking $H := \operatorname{diag}(1, 0, \dots, 0, -1)$, we have that $\mathfrak{g}_1$ is the subspace given by $z_{00} = z_{0l} = z_{ll} = 0$ and $z_{ij} = 0$ for all i, j with $i > 0$ and $j < l$. Therefore the intersection $X(\mathfrak{g}) \cap \mathbb{P}_*(\mathfrak{g}_1)$ is the (disjoint) union of linear subspaces defined by $z_{00} = z_{0l} = z_{ij} = 0$ for all i, j with $i > 0$ and by $z_{0l} = z_{ll} = z_{ij} = 0$ for all i, j with $j < l$.

Example 2. The cubic Veronese variety $\pi M \subseteq \mathbb{P}_*(\mathfrak{g}_1)$ is *empty* if $\mathfrak{g}$ is of type C_l. Indeed, in this case, $X(\mathfrak{g})$ is the Veronese embedding of $\mathbb{P}^{2l-1}$ of degree 2 (see, for example [KOY, §5]), then a simple calculation shows that

$$X(\mathfrak{g}) \cap T_{\pi X_\lambda} X(\mathfrak{g}) = \{\pi X_\lambda\}.$$

On the other hand, for any adjoint variety $X(\mathfrak{g})$ we have

$$T_{\pi X_\lambda} X(\mathfrak{g}) \supseteq \mathbb{P}_*(\mathfrak{g}_1) \not\ni \pi X_\lambda.$$

Therefore the intersection $X(\mathfrak{g}) \cap \mathbb{P}_*(\mathfrak{g}_1)$ is empty.

Acknowledgements. The authors would like to thank the referee for his/her invaluable advice: In particular, the formulation of Lemma 1 is due to the referee. This work was done mainly from April 1997 through March 1998, and in that period the first author was partly supported by TOKUTEI-KADAI #97A–331, Waseda University. The results here were announced in a symposium "Lie Groups and Geometry" (organized by S. Kaneyuki, December 3–5, 1997 at Yamaguchi University) and in a symposium "School on Commutative Algebra and Projective Varieties" (organized by C. Miyazaki and T. Fujisawa, March 11–13, 1998 at Hotel Mielparque Nagano).

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