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Nagoya Math. J.
Vol. 157 (2000), 211–227

THE JOINT UNIVERSALITY AND THE FUNCTIONAL INDEPENDENCE FOR LERCH ZETA-FUNCTIONS

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Abstract. The joint universality theorem for Lerch zeta-functions $L(\lambda_l, \alpha_l, s)$ $(1 \leq l \leq n)$ is proved, in the case when λ_ls are rational numbers and α_ls are transcendental numbers. The case $n = 1$ was known before ([12]); the rationality of λ_ls is used to establish the theorem for the "joint" case $n \geq 2$. As a corollary, the joint functional independence for those functions is shown.

1. Introduction

Let $s = \sigma + it$ be a complex variable, and let $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ denote the set of all natural numbers, integers, rational numbers, real numbers and complex numbers, respectively. The Lerch zeta-function $L(\lambda, \alpha, s)$, for $\sigma > 1$, is defined by

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s}.$$

Here $\alpha, \lambda \in \mathbb{R}$, $0 < \alpha \leq 1$, are fixed parameters. When $\lambda \in \mathbb{Z}$ the Lerch zeta-function $L(\lambda, \alpha, s)$ reduces to the Hurwitz zeta-function $\zeta(s, \alpha)$. If $\lambda \notin \mathbb{Z}$, then the function $L(\lambda, \alpha, s)$ is analytically continuable to an entire function. Clearly, in this case we may suppose that $0 < \lambda < 1$. In what follows we will deal with this case only.

The Lerch zeta-function is one of the classical objects in number theory, introduced by M. Lerch [16] in 1887.

In recent years the value-distribution of the Lerch zeta-function was studied by D. Klusch, R. Garunkštis, M. Katsurada, W. Zhang, by the authors and other mathematicians. In [12] the universality theorem for the function $L(\lambda, \alpha, s)$ was proved. In order to state it we need some notation.

Received February 18, 1999.
[1]Partially supported by Grant from Lithuanian Foundation of Studies and Science.

By meas$\{A\}$ we denote the Lebesgue measure of the set A, and, for $T > 0$, we use the notation

$$\nu_T(\dots) = \frac{1}{T} \operatorname{meas}\{\tau \in [0,T], \dots\},$$

where in place of dots some condition satisfied by τ is to be written. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. Then the result of [12] is as follows.

Let α be a transcendental number. Let K be a compact subset of the strip D with the connected complement, $f(s)$ be a continuous function on K which is analytic in the interior of K. Then for any $\varepsilon > 0$ it holds that

$$\liminf_{T\to\infty} \nu_T\left(\sup_{s\in K} |L(\lambda, \alpha, s + i\tau) - f(s)| < \varepsilon \right) > 0.$$

The universality for $L(\lambda, \alpha, s)$ was also studied in [5], [13].

It is the purpose of the present paper to obtain a joint universality theorem for Lerch zeta-functions. Suppose $n \geq 2$.

Theorem 1. *Let $\alpha_1, \dots, \alpha_n$ be transcendental numbers, $\lambda_1 = a_1/q_1, \dots, \lambda_n = a_n/q_n$, $(a_1, q_1) = 1, \dots, (a_n, q_n) = 1$, where $q_1, \dots, q_n$ are distinct positive integers and $a_1, \dots, a_n$ are positive integers with $a_1 < q_1, \dots, a_n < q_n$. Let $K_1, \dots, K_n$ be compact subsets of the strip D with connected complements, and, for $1 \leq l \leq n$, let $f_l(s)$ be a continuous function on K_l which is analytic in the interior of K_l. Then for every $\varepsilon > 0$ it holds that*

$$\liminf_{T\to\infty} \nu_T\left(\sup_{1\leq l\leq n} \sup_{s\in K_l} |L(\lambda_l, \alpha_l, s + i\tau) - f_l(s)| < \varepsilon \right) > 0.$$

Joint universality theorems for Dirichlet L-functions were obtained by B. Bagchi [2], [3], S.M. Gonek [7], and S.M. Voronin [18], [19]. For more general Dirichlet series such theorems were proved in [8], [9], [14].

The proof of Theorem 1 is based on Bagchi's method [2], [3], but some new ideas are necessary for the proof of Lemmas 5 and 6 below.

In the case of the aforementioned universality theorem [12] for a single zeta-function, the arithmetic nature of λ is irrelevant. However, in the proof of Theorem 1, the fact that $\lambda_l \in \mathbb{Q}$ $(1 \leq l \leq n)$ is used essentially. In Section 4 we will discuss briefly the case when $\lambda_l \notin \mathbb{Q}$.

As an application of Theorem 1, we will show the joint functional independence.

THEOREM 2. *Let $\alpha_l, \lambda_l = a_l/q_l$ be as in Theorem 1, and F_j $(0 \le j \le k)$ be continuous functions on $\mathbb{C}^{Nn}$. Suppose*

$$\sum_{j=0}^{k} s^j F_j(L(\lambda_1, \alpha_1, s), \dots, L(\lambda_n, \alpha_n, s),\ L'(\lambda_1, \alpha_1, s), \dots,$$
$$L'(\lambda_n, \alpha_n, s), \dots, L^{(N-1)}(\lambda_1, \alpha_1, s), \dots, L^{(N-1)}(\lambda_n, \alpha_n, s)) = 0$$

identically for all $s \in \mathbb{C}$. Then $F_j \equiv 0$ $(0 \le j \le k)$.

This theorem gives a generalization of the result proved in Garunkštis-Laurinčikas [6]. A quite different approach to this type of problems has recently been developed by Amou-Katsurada [1].

2. A joint limit theorem for Lerch zeta-functions

For the proof of Theorem 1 we will apply a joint limit theorem in the sense of weak convergence of probability measures for the Lerch zeta-functions $L(\lambda_1, \alpha_1, s), \dots L(\lambda_n, \alpha_n, s)$ in the space of analytic functions. Denote by $H(D)$ the space of analytic on D functions equipped with the topology of uniform convergence on compacta. Let $\mathcal{B}(S)$ stand for the class of Borel sets of the space S. Define on $(H^n(D), \mathcal{B}(H^n(D)))$ the probability measure

$$P_T(A) = \nu_T\Big((L(\lambda_1, \alpha_1, s+i\tau), \dots, L(\lambda_n, \alpha_n, s+i\tau)) \in A\Big),\ \ A \in \mathcal{B}(H^n(D)).$$

What we need is a limit theorem in the sense of weak convergence of probability measures for P_T as $T \to \infty$, with an explicit form of the limit measure. Denote by γ the unit circle on $\mathbb{C}$, i.e. $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and let

$$\Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m = 0, 1, 2, \dots$. With the product topology and pointwise multiplication the infinite dimensional torus Ω is a compact topological Abelian group. Denoting by m_H the probability Haar measure on $(\Omega, \mathcal{B}(\Omega))$, we obtain the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\omega(m)$ be the projection of $\omega \in \Omega$ to the coordinate space γ_m, and define on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ the $H^n(D)$-valued random element $L(s, \omega)$ by

$$L(s, \omega) = (L(\lambda_1, \alpha_1, s, \omega), \dots, L(\lambda_n, \alpha_n, s, \omega)),$$

where

$$L(\lambda_l, \alpha_l, s, \omega) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda_l m} \omega(m)}{(m+\alpha_l)^s}, \qquad s \in D,\ \omega \in \Omega,\ l = 1, \dots, n.$$

The proof that $L(\lambda_l, \alpha_l, s, \omega)$ is an $H(D)$-valued random element can be found in [11]. Let P_L stand for the distribution of the random element $\underline{L}(s, \omega)$, i.e.

$$P_L(A) = m_H(\omega \in \Omega : \underline{L}(s, \omega) \in A), \qquad A \in \mathcal{B}(H^n(D)).$$

LEMMA 1. *The probability measure P_T converges weakly to P_L as $T \to \infty$.*

Proof. Let $D_0 = \{s \in \mathbb{C} : \sigma > 1/2\}$. Then in [15] the result of the lemma was proved in the case of the space $H^n(D_0)$. Obviously, from this the lemma follows.

3. The support of the random element $\underline{L}$

In this section we will consider the support of the measure P_L. We recall that the minimal closed set $S_{P_L} \subseteq H^n(D)$ such that $P_L(S_{P_L}) = 1$ is called the support of P_L. The set S_{P_L} consists of all $\underline{f} \in H^n(D)$ such that for every neighbourhood $\mathcal{G}$ of $\underline{f}$ the inequality $P_L(\mathcal{G}) > 0$ is satisfied.

The support of the distribution of the random element X is called the support of X and is denoted by S_X.

LEMMA 2. *Let $\{\underline{X}_m\}$ be a sequence of independent $H^n(D)$-valued random elements, and suppose that the series*

$$\sum_{m=1}^{\infty} \underline{X}_m$$

converges almost surely. Then the support of the sum of this series is the closure of the set of all $\underline{f} \in H^n(D)$ which may be written as a convergent series

$$\underline{f} = \sum_{m=1}^{\infty} \underline{f}_m, \qquad \underline{f}_m \in S_{\underline{X}_m}.$$

Proof of the lemma in the case $n = 1$ is given in [10], Theorem 1.7.10. The proof when $n > 1$ is similar to that of the case $n = 1$.

Let $\underline{f}(s) = (f_1(s), \dots, f_n(s)) \in H^n(D)$. Then we write

$$|\underline{f}(s)|^2 = \sum_{l=1}^{n} |f_l(s)|^2.$$

LEMMA 3. *Let* $\{\underline{f}_m = (f_{1m}, \dots, f_{nm}),\ m \geq 1\}$ *be a sequence in* $H^n(D)$ *which satisfies:*

a) *If* $\mu_1, \dots, \mu_n$ *are complex measures on* $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ *with compact supports contained in* D *such that*

$$\sum_{m=1}^{\infty} \left| \sum_{l=1}^{n} \int\limits_{\mathbb{C}} f_{lm}\, d\mu_l \right| < \infty,$$

then

$$\int\limits_{\mathbb{C}} s^r\, d\mu_l(s) = 0$$

for all $l = 1, \dots, n$, $r = 0, 1, 2, \dots$.

b) *The series*

$$\sum_{m=1}^{\infty} \underline{f}_m$$

converges in $H^n(D)$.

c) *For any compact* $K \subseteq D$

$$\sum_{m=1}^{\infty} \sup_{s \in K} |\underline{f}_m(s)|^2 < \infty.$$

Then the set of all convergent series

$$\sum_{m=1}^{\infty} a_m \underline{f}_m$$

with $a_m \in \gamma$ *is dense in* $H^n(D)$.

Proof. This lemma is Lemma 5.2.9 of [2], see also [3]. In [10] the proof in the case $n = 1$ is given, see Theorem 6.3.10. The proof of the general case is obtained in a similar way.

Now we state two lemmas on entire functions of exponential type. Recall that an entire function $f(s)$ is of exponential type if

$$\limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r} < \infty$$

uniformly in θ, $|\theta| \le \pi$.

LEMMA 4. *Let μ be a complex measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with the compact support contained in the half-plane $\sigma > \sigma_0$, and let*

$$f(z) = \int\limits_{\mathbb{C}} e^{sz}\, d\mu(s), \qquad z \in \mathbb{C}.$$

If $f(z) \not\equiv 0$, then

$$\limsup_{x\to\infty} \frac{\log|f(x)|}{x} > \sigma_0.$$

This lemma is due to B. Bagchi [2]. For the proof see Lemma 6.4.10 of [10].

Let $\mathcal{M}$ be a set of natural numbers having a positive density, i.e.

$$\lim_{x\to\infty} \frac{1}{x}\, \#\{m \in \mathcal{M} : m \le x\} = d > 0. \tag{1}$$

LEMMA 5. *Let $f(s)$ be an entire function of exponential type, and let*

$$\limsup_{r\to\infty} \frac{\log|f(r)|}{r} > -1.$$

Then

$$\sum_{m\in\mathcal{M}} |f(\log m)| = \infty.$$

Proof. Let $\alpha > 0$ be such that

$$\limsup_{y\to\infty} \frac{\log|f(\pm iy)|}{y} \le \alpha. \tag{2}$$

Let us fix a positive number β such that $\alpha\beta < \pi$, and suppose, on the contrary, that

$$\sum_{m\in\mathcal{M}} |f(\log m)| < \infty. \tag{3}$$

Consider the set $A = \{m \in \mathbb{N} : \exists r \in ((m-1/4)\beta,\ (m+1/4)\beta]$ and $|f(r)| \le e^{-r}\}$. Let, for brevity,

$$m_{\mathcal{M}}(x) = \sum_{\substack{m \le x \\ m \in \mathcal{M}}} 1.$$

Clearly, we have

$$\sum_{m \in \mathcal{M}} |f(\log m)| \ge \sum_{m \notin A} {\sum_m}' |f(\log k)| \ge \sum_{m \notin A} {\sum_m}' \frac{1}{k}, \tag{4}$$

where ${\sum'}_m$ denotes the sum extended over all natural numbers $k \in \mathcal{M}$ satisfying $(m-1/4)\beta < \log k \le (m+1/4)\beta$. If we denote

$$a = \exp\left\{\left(m - \frac{1}{4}\right)\beta\right\}, \qquad b = \exp\left\{\left(m + \frac{1}{4}\right)\beta\right\},$$

then we have that

$${\sum_m}' \frac{1}{k} = \sum_{\substack{k \in \mathcal{M} \\ a < k \le b}} \frac{1}{k}.$$

Summing by parts, we find

$$\sum_{\substack{k \in \mathcal{M} \\ a < k \le b}} \frac{1}{k} = \frac{1}{b} \sum_{\substack{k \in \mathcal{M} \\ a < k \le b}} 1 + \int_a^b \Bigg(\sum_{\substack{k \in \mathcal{M} \\ a < k \le u}} 1 \Bigg) \frac{du}{u^2}. \tag{5}$$

Obviously,

$$\sum_{\substack{k \in \mathcal{M} \\ a < k \le u}} 1 = m_{\mathcal{M}}(u) - m_{\mathcal{M}}(a).$$

The assumption (1) implies

$$m_{\mathcal{M}}(x) = dx(1 + o(1)), \qquad x \to \infty,$$

hence, for any $\varepsilon > 0$, there exists a number $x_0 = x_0(\varepsilon)$ such that

$$m_{\mathcal{M}}(u) \ge du(1-\varepsilon),$$
$$m_{\mathcal{M}}(a) \le da(1+\varepsilon)$$

if $a \ge x_0$. Therefore

$$\sum_{\substack{k \in \mathcal{M} \\ a < k \le u}} 1 \ge d((u-a) - \varepsilon(a+u)). \tag{6}$$

Let η satisfy the inequality $1 < \eta < \exp\{\beta/2\}$, and consider the case $u \geq \eta a$. Then we have

$$\begin{aligned}\frac{1}{2}(u-a) - \varepsilon(a+u) &\geq \frac{1}{2}\Big(u - \frac{u}{\eta}\Big) - \varepsilon\Big(\frac{u}{\eta} + u\Big)\\ &= u\bigg(\Big(\frac{1}{2} - \varepsilon\Big) - \frac{1}{\eta}\Big(\frac{1}{2} + \varepsilon\Big)\bigg) > 0\end{aligned}$$

if we choose ε sufficiently small. Hence and from (6) we obtain

$$\sum_{\substack{k\in\mathcal{M}\\ a<k\leq u}} 1 \geq \frac{d}{2}(u-a), \qquad u \geq \eta a.$$

Combining this with (5), and using partial summation again, we have

$$\begin{aligned}(7)\ \sum_{\substack{k\in\mathcal{M}\\ a<k\leq b}} \frac{1}{k} &\geq \frac{d}{2b}(b-a) + \frac{d}{2}\int_{\eta a}^{b} (u-a)\,\frac{du}{u^2}\\ &\geq \frac{d}{2}\Big\{\frac{1}{b}([b]-[\eta a]) + \int_{\eta a}^{b} ([u]-[\eta a])\frac{du}{u^2} + \frac{B}{a}\Big\} = \frac{d}{2}\sum_{\eta a<k\leq b}\frac{1}{k} + \frac{B}{a}\,,\end{aligned}$$

where $[x]$ denotes the integer part of x, and B is a number (not always the same) bounded by a constant. Clearly,

$$\begin{aligned}\sum_{\eta a<k\leq b}\frac{1}{k} &= \log b - \log(\eta a) + \frac{B}{\eta a}\\ &= \Big(m + \frac{1}{4}\Big)\beta - \log\eta - \Big(m - \frac{1}{4}\Big)\beta + Be^{-m\beta}\\ &= \frac{\beta}{2} - \log\eta + Be^{-m\beta}\,.\end{aligned}$$

From the choice of η it follows that

$$\frac{\beta}{2} - \log\eta > 0.$$

Now (7) shows

$$\sum_{\substack{k\in\mathcal{M}\\ a<k\leq b}} \frac{1}{k} \geq \frac{d}{2}\Big(\frac{\beta}{2} - \log\eta\Big) + Be^{-m\beta}\,.$$

This together with (3) and (4) implies

$$\sum_{m\notin A}\left(\frac{d}{2}\Big(\frac{\beta}{2}-\log\eta\Big)+Be^{-m\beta}\right)\le\sum_{m\in\mathcal{M}}|f(\log m)|<\infty,$$

hence

$$\sum_{m\notin A}1<\infty. \tag{8}$$

Let $A=\{a_m:\ a_1<a_2<\ldots\}$. Then (8) gives that

$$\lim_{m\to\infty}\frac{a_m}{m}=1. \tag{9}$$

By the definition of the set A, there exists a sequence $\{\lambda_m\}$ such that

$$\Big(a_m-\frac{1}{4}\Big)\beta<\lambda_m\le\Big(a_m+\frac{1}{4}\Big)\beta,$$

and $|f(\lambda_m)|\le e^{-\lambda_m}$. Hence, in view of (9),

$$\lim_{m\to\infty}\frac{\lambda_m}{m}=\beta, \tag{10}$$

and

$$\limsup_{m\to\infty}\frac{\log|f(\lambda_m)|}{\lambda_m}\le-1.$$

Now we apply Theorem 6.4.12 of [10]. The assumptions of that theorem are satisfied by (10), (2), and the condition $\alpha\beta<\pi$. Hence by that theorem it follows that

$$\limsup_{r\to\infty}\frac{\log|f(r)|}{r}\le-1.$$

This contradicts the assumption of the lemma, and Lemma 5 is proved.

LEMMA 6. *The support of the measure P_L is the whole of $H^n(D)$.*

Proof. It follows from the definition of Ω that $\{\omega(m)\}$ is a sequence of independent random variables with respect to the measure m_H. Hence $\{\underline{f}_m(s,\omega(m)), m\in\mathbb{N}\cup\{0\}\}$ is a sequence of independent $H^n(D)$-valued random elements, where

$$\underline{f}_m(s,\omega(m))=\left(\frac{e^{2\pi i\lambda_1 m}\omega(m)}{(m+\alpha_1)^s},\ldots,\frac{e^{2\pi i\lambda_n m}\omega(m)}{(m+\alpha_n)^s}\right).$$

The support of each $\omega(m)$ is the unit circle γ. Therefore the set $\{\underline{f}_m(s,a) : a \in \gamma\}$ is the support of the random element $\underline{f}_m(s,\omega(m))$. Consequently, by Lemma 2 the closure of the set of all convergent series

$$\sum_{m=0}^{\infty} \underline{f}_m(s, a_m), \qquad a_m \in \gamma,$$

is the support of the random element $L(s,\omega)$. It remains to check that the latter set is dense in $H^n(D)$.

Let $\mu_1, \dots, \mu_n$ be complex measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact supports contained in D such that

$$\sum_{m=0}^{\infty} \left| \sum_{l=1}^{n} \int\limits_{\mathbb{C}} \frac{e^{2\pi i \lambda_l m}}{(m+\alpha_l)^s}\, d\mu_l(s) \right| < \infty. \tag{11}$$

It is well known that for all $s \in \mathbb{C}$

$$e^s = 1 + B|s|e^{|s|}.$$

Therefore, for $m \geq 2$,

$$\begin{aligned}(m+\alpha_l)^{-s} &= m^{-s}\Big(1+\frac{\alpha_l}{m}\Big)^{-s} = m^{-s}\exp\Big\{ -s\log\Big(1+\frac{\alpha_l}{m}\Big)\Big\} \\ &= m^{-s}\exp\Big\{\frac{B|s|}{m}\Big\} = m^{-s}\Big(1+\frac{B|s|}{m}e^{B|s|}\Big) \\ &= m^{-s} + Bm^{-1-\sigma}|s|e^{B|s|}.\end{aligned}$$

Hence, taking into account the properties of the measures $\mu_1, \dots, \mu_n$, we deduce from (11) that

$$\sum_{m=0}^{\infty} \left| \sum_{l=1}^{n} \int\limits_{\mathbb{C}} \frac{e^{2\pi i \lambda_l m}}{m^s}\, d\mu_l(s) \right| < \infty,$$

which can be rewritten in the form

$$\sum_{\substack{m=0 \\ m \equiv r (\mathrm{mod}\, q)}}^{\infty} \left| \sum_{l=1}^{n} \int\limits_{\mathbb{C}} \frac{e^{2\pi i \lambda_l r}}{m^s}\, d\mu_l(s) \right| < \infty, \qquad 1 \leq r \leq q, \tag{12}$$

where $q = [q_1, \dots, q_n]$. Now let

$$\nu_r(A) = \sum_{l=1}^{n} e^{2\pi i \lambda_l r}\mu_l(A), \qquad A \in \mathcal{B}(\mathbb{C}),\ 1 \leq r \leq q.$$

Note that the measures $\nu_1, \dots, \nu_q$ have the same properties as $\mu_1, \dots, \mu_n$. Using this notation, we may write the relation (12) as follows:

$$\sum_{\substack{m=0 \\ m \equiv r \pmod q}}^{\infty} \left| \int_{\mathbb{C}} m^{-s} \, d\nu_r(s) \right| < \infty, \qquad 1 \le r \le q. \tag{13}$$

Let

$$\tilde{\varrho}_r(z) = \int_{\mathbb{C}} e^{-sz} \, d\nu_r(s), \qquad z \in \mathbb{C}.$$

Then (13) becomes the following condition

$$\sum_{\substack{m=0 \\ m \equiv r \pmod q}}^{\infty} |\tilde{\varrho}_r(\log m)| < \infty, \qquad 1 \le r \le q. \tag{14}$$

By Lemma 4 we obtain that $\tilde{\varrho}_r(z) \equiv 0$, or

$$\limsup_{x \to \infty} \frac{\log |\tilde{\varrho}_r(x)|}{x} > -1, \qquad 1 \le r \le q.$$

Lemma 5 shows that the last inequality contradicts (14). Hence

$$\tilde{\varrho}_r(z) \equiv 0 \tag{15}$$

for $1 \le r \le q$. Let

$$\varrho_l(z) = \int_{\mathbb{C}} e^{-sz} \, d\mu_l(s), \qquad z \in \mathbb{C}, \ l = 1, \dots, n.$$

Then by the definitions of ν_r and $\tilde{\varrho}_r$ we have

$$\begin{aligned} \tilde{\varrho}_r(z) &= \int_{\mathbb{C}} e^{-sz} \sum_{l=1}^{n} e^{2\pi i \lambda_l r} \, d\mu_l(s) = \sum_{l=1}^{n} e^{2\pi i \lambda_l r} \int_{\mathbb{C}} e^{-sz} \, d\mu_l(s) \\ &= \sum_{l=1}^{n} e^{2\pi i \lambda_l r} \varrho_l(z), \end{aligned}$$

which is identically equal to zero by (15). Multiplying by $e^{-2\pi i \lambda_j}$, we have

$$\sum_{l=1}^{n} e^{2\pi i (\lambda_l - \lambda_j) r} \varrho_l(z) \equiv 0, \qquad 1 \le r \le q. \tag{16}$$

Taking into account that

$$\sum_{r=1}^{q} e^{2\pi i(\lambda_l-\lambda_j)r} = \begin{cases} q & \text{if } (\lambda_l-\lambda_j)\in\mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

and the fact that $\lambda_l - \lambda_j \in \mathbb{Z}$ only if $l = j$, and summing (16) over $r = 1, \ldots, q$, we find that

$$\varrho_j(z) = \int_{\mathbb{C}} e^{-sz}\, d\mu_j(s) \equiv 0, \qquad j = 1, 2, \ldots, n.$$

Differentiating this equality r times and then putting $z = 0$, we find that

$$\int_{\mathbb{C}} s^r\, \mu_j(s) = 0$$

for all $j = 1, \ldots, n$, $r = 0, 1, 2, \ldots$. Thus the condition a) of Lemma 3 for the sequence $\{\underline{f}_m(s,1),\ m \geq 1\}$ is satisfied.

Let, for a natural number N,

$$S(\lambda, N) = \sum_{m=0}^{N} e^{2\pi i\lambda m}.$$

If $\lambda \notin \mathbb{Z}$, then we have

$$S(\lambda, N) = \frac{1 - e^{2\pi i\lambda(N+1)}}{1 - e^{2\pi i\lambda}} \tag{17}$$

which is uniformly bounded for all $N \geq 1$. Summing by parts, we find

$$\sum_{m=0}^{N} \frac{e^{2\pi i\lambda m}}{(m+\alpha)^s} = S(\lambda, N)(N+\alpha)^{-s} + s\int_0^N S(\lambda, u)\,\frac{du}{(u+\alpha)^{s+1}}\,.$$

Taking $N \to \infty$ we obtain

$$\sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}}{(m+\alpha)^s} = s\int_0^\infty S(\lambda, u)\,\frac{du}{(u+\alpha)^{s+1}}\,,$$

which converges for $\sigma > 0$ in view of (17). Consequently, the series

$$\sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}}{(m+\alpha)^s} \tag{18}$$

with $\lambda \notin \mathbb{Z}$ converges (Corollary 2.1.3 of [10]) uniformly on compacta in the half-plane $\sigma > \sigma_0$ for any $\sigma_0 > 0$. This shows that the series

$$\sum_{m=0}^{\infty} \underline{f}_m(s,1)$$

converges in $H^n(D)$, i.e. the condition b) of Lemma 3 holds for the sequence $\{\underline{f}_m(s,1),\ m \geq 1\}$. The condition c) of Lemma 3 is also satisfied clearly, since for $s \in K$ we have that $\sigma > 1/2$.

Now, applying Lemma 3, we have that the set of all convergent series

$$\sum_{m=0}^{\infty} a_m \underline{f}_m(s,1) = \sum_{m=0}^{\infty} \underline{f}_m(s,a_m)$$

with $a_m \in \gamma$ is dense in $H^n(D)$. This completes the proof of the lemma.

4. Proof of Theorem 1

The following deduction of Theorem 1 from the above lemmas is standard (cf. Section 6.5 of [10]), but we present it for the convenience of readers.

We begin with the Mergelyan theorem.

Lemma 7. *Let K be a compact subset of $\mathbb{C}$ whose complement is connected. Then any continuous function $f(s)$ on K which is analytic in the interior of K is approximable uniformly on K by polynomials of s.*

Proof is given, for example, in [20].

Proof of Theorem 1. First suppose that functions $f_l(s)$, $l = 1, \dots, n$, can be continued analytically to the whole of D. Denote by $\mathcal{G}$ the set of all $(g_1, \dots, g_n) \in H^n(D)$ such that

$$\sup_{1 \leq l \leq n} \sup_{s \in K_l} |g_l(s) - f_l(s)| < \frac{\varepsilon}{4}.$$

Let P_n and P be probability measures defined on $(S, \mathcal{B}(S))$. It is well known (see [4], Theorem 2.1) that P_n converges weakly to P as $n \to \infty$ if and only if

$$\liminf_{n \to \infty} P_n(G) \geq P(G)$$

for all open sets G.

The set $\mathcal{G}$ is open, and, by Lemma 1, the measure P_T converges weakly to P_L as $T \to \infty$. Therefore, using the above property of the weak convergence of probability measures and Lemma 6, we obtain

$$\liminf_{T\to\infty} \nu_T\left(\sup_{1\le l\le n} \sup_{s\in K_l} |L(\lambda_l, \alpha_l, s+i\tau) - f_l(s)| < \frac{\varepsilon}{4}\right) = P_L(\mathcal{G}) > 0. \tag{19}$$

Now let the functions $f_l(s)$, $l = 1, \dots, n$, be the same as in the statement of Theorem 1. By Lemma 7 there exist polynomials $p_l(s)$, $l = 1, \dots, n$, such that

$$\sup_{1\le l\le n} \sup_{s\in K_l} |p_l(s) - f_l(s)| < \frac{\varepsilon}{2}\,. \tag{20}$$

By the first part of the proof we have that

$$\liminf_{T\to\infty} \nu_T\left(\sup_{1\le l\le n} \sup_{s\in K_l} |L(\lambda_l, \alpha_l, s+i\tau) - p_l(s)| < \frac{\varepsilon}{2}\right) > 0. \tag{21}$$

Obviously, for $l = 1, \dots, n$

$$\sup_{s\in K_l} |L(\lambda_l, \alpha_l, s+i\tau)| \le \sup_{s\in K_l} |L(\lambda_l, \alpha_l, s+i\tau) - p_l(s)| + \sup_{s\in K_l} |f_l(s) - p_l(s)|.$$

Therefore by (20) it is easily seen that

$$\left\{\tau : \sup_{1\le l\le n} \sup_{s\in K_l} |L(\lambda_l, \alpha_l, s+i\tau) - f_l(s)| < \varepsilon\right\}$$
$$\supseteq \left\{\tau : \sup_{1\le l\le n} \sup_{s\in K_l} |L(\lambda_l, \alpha_l, s+i\tau) - p_l(s)| < \frac{\varepsilon}{2}\right\}.$$

This and (21) yield the assertion of Theorem 1.

Now we discuss briefly the case that $1, \lambda_1, \dots, \lambda_n$ are linearly independent over $\mathbb{Q}$. Then the sequence

$$\{(\lambda_1 m, \dots, \lambda_n m),\ m \in \mathbb{N}\}$$

is uniformly distributed $\bmod 1$ in $\mathbb{R}^n$ (see Kuipers-Niederreiter [17], Section 1.6, Example 6.1), hence the set

$$N_\varepsilon = \{m \in \mathbb{N} :\ (\lambda_1 m, \dots, \lambda_n m) \in (-\varepsilon, \varepsilon)^n \bmod 1\}$$

has the positive density $(2\varepsilon)^n$. From (11) we have

$$\sum_{m\in N_\varepsilon}\left|\sum_{l=1}^{n} e^{2\pi i\lambda_l m}\varrho_l(\log m)\right| < \infty, \tag{22}$$

which suggests that

$$\sum_{m\in N_\varepsilon}\left|\sum_{l=1}^{n} \varrho_l(\log m)\right| < \infty \tag{23}$$

might be also true. If (23) would be true, then using Lemmas 4 and 5 we obtain

$$\sum_{l=1}^{n} \varrho_l(z) \equiv 0$$

for any $z \in \mathbb{C}$. We could prove

$$-\varrho_1(z) + \sum_{l=2}^{n} \varrho_l(z) \equiv 0$$

in the same way, hence $\varrho_1(z) \equiv 0$, and similarly $\varrho_l(z) \equiv 0$, $l = 2, 3, \ldots, n$. From this fact we could deduce the joint universality theorem in this case. If we could prove the above conclusion $\varrho_l(z) \equiv 0$ not only from (23), but also from (22), then this argument would be complete.

5. Proof of Theorem 2

It is sufficient to give a sketch, because the proof is a direct generalization of that in [6]. Define the mapping $h : \mathbb{R} \to \mathbb{C}^{Nn}$ by

$$\begin{aligned} h(t) = (&L(\lambda_1, \alpha_1, \sigma + it), \ldots, L(\lambda_n, \alpha_n, \sigma + it),\\ &L'(\lambda_1, \alpha_1, \sigma + it), \ldots, L'(\lambda_n, \alpha_n, \sigma + it), \ldots,\\ &L^{(N-1)}(\lambda_1, \alpha_1, \sigma + it), \ldots, L^{(N-1)}(\lambda_n, \alpha_n, \sigma + it)). \end{aligned}$$

For any $\varepsilon > 0$ and any $s_{\nu l} \in \mathbb{C}$ $(0 \le \nu \le N-1,\ 1 \le l \le n)$, we can find $\tau \in \mathbb{R}$ such that

$$|L^{(\nu)}(\lambda_l, \alpha_l, \sigma + i\tau) - s_{\nu l}| < \varepsilon \qquad (0 \le \nu \le N-1,\ 1 \le l \le n).$$

This can be shown by the same way as in Lemma 3 of [6], by taking the polynomial

$$p_{lN}(s) = \sum_{\nu=0}^{N-1} \frac{s_{\nu l} s^\nu}{\nu!} \qquad (1 \le l \le n)$$

and applying Theorem 1. Hence the image of $\mathbb{R}$ by the mapping h is dense in $\mathbb{C}^{Nn}$. From this, similarly to [6] (or Section 6.6 of [10]), we can deduce the conclusion of Theorem 2.

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