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ON THE DIRICHLET PROBLEM OF PRESCRIBED MEAN CURVATURE EQUATIONS WITHOUT H-CONVEXITY CONDITION

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Abstract. The Dirichlet problem of prescribed mean curvature equations is well posed, if the boundery is H-convex. In this article we eliminate the H-convexity condition from a portion Γ of the boundary and prove the existence theorem, where the boundary condition is satisfied on Γ in the weak sense.

§1. Introduction

Let Ω be a bounded domain in R^n, $n \geqq 2$, with its boundary $\partial\Omega$. We denote by $(x_1, \ldots, x_n)$ the coordinates in R^n and write $D = (D_1, \ldots, D_n)$, where $D_i = \partial/\partial x_i$.

We consider the Dirichlet problem

$$D \cdot \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) = nH \quad \text{in } \Omega \tag{1.1}$$

with

$$u = \phi \quad \text{on } \partial\Omega, \tag{1.2}$$

which was studied by many authors. The equation (1.1) is called by the prescribed mean curvature equation.

Let $|\Omega|$ and ω_n be two volumes of Ω and the unit ball in R^n, respectively. Throughout this article we assume

$$\sup_{\Omega} |H| < \frac{1}{n} \left(\frac{\omega_n}{|\Omega|} \right)^{1/n}, \tag{1.3}$$

which may be replaced in some weaker conditions. That is, (1.3) means that

$$\int_{\Omega} |H|^n \, dx < \omega_n \tag{1.4}$$

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and for some $\varepsilon_0 > 0$

$$\left| \int_\Omega H\eta \, dx \right| \le \frac{1-\varepsilon_0}{n} \int_\Omega |D\eta| \, dx, \quad \eta \in C_0^1(\Omega). \tag{1.5}$$

But we impose (1.3) on this article for the sake of simplicity.

Serrin [17] solved first the Dirichlet problem (1.1) with (1.2). His result is as follows: Suppose that $\partial\Omega \in C^{2,\alpha}$, $\phi \in C^{2,\alpha}(\overline{\Omega})$ for some α with $0 < \alpha < 1$, and $H \in C^1(\overline{\Omega})$. If (1.3) and

$$\frac{n}{n-1}|H| \le \Lambda \tag{1.6}$$

are assumed on $\partial\Omega$, then the problem (1.1) with (1.2) is uniquely solvable for $u \in C^{2,\alpha}(\overline{\Omega})$, where Λ is the boundary mean curvature of Ω.

The condition (1.6) is called by H-convexity. Afterward Serrin's result was extended to the generalized mean curvature equation of higher order by Ivochkina [6], whose study is closely related to the fully nonlinear elliptic equation. Recently, Gregori [5] studied the relation between BV solutions and viscosity solutions for (1.1).

By weakening the above assumptions except for (1.6), many authors solved the problem (1.1) with (1.2), where the required solutions u are in $C^2(\Omega) \cap C(\overline{\Omega})$ (see e.g., [1], [2], [4], [18], [20]). The starting point is in two ways. One is to estimate the generalized BV solution (see e.g., [4]). Another is to estimate the approximating solution of each perturbed uniformly elliptic equation (see e.g., [18]). In either case it needs to construct the barrier functions. Further there are a few papers which prove $u - \phi \in W_0^{1,1}(\Omega)$ (see e.g., [15], [20]). Their method is also to construct the barrier function. So, it is difficult to drop the condition (1.6).

Suppose that (1.6) is not assumed. Let $H = 0$, namely (1.1) be the minimal surface equation. Then the problem (1.1) with (1.2) is solvable, if ϕ is small concerning some norm (see [14], [22]). When $H \neq 0$, there is the result of Schulz and Williams [16]. Lancaster [12] showed the non-existence of solutions for some domain having a reentrant corner. Recently, Jin and Lancaster [8] investigated the behavior near a reentrant corner of a solution to a quasilinear elliptic equation in a two dimensional domain. And Tersenov [21] proved the existence of $C^{2,\alpha}(\overline{\Omega})$ solutions of the Dirichlet problem with zero boundary conditions for quasilinear elliptic equations in some non convex domains Ω. In [21] the condition on Ω is complicated to be stated. On the other hand, when $H = 0$, Jenkins and Serrin [7] showed previously that a necessary and sufficient condition on $\partial\Omega$ for the solvability of the Dirichlet problem (1.1) with (1.2) for arbitrary continuous ϕ is that $\Lambda \geqq 0$ everywhere. Afterward Williams [23] constructed a C^∞-domain in R^2 and a C^∞-function ϕ such that the limit of the generalized solution at a point on $\partial\Omega$ from inward does not exist.

We consider the case of $H = 0$. Let u be the generalized BV solution. The assumption (1.6) guarantees that $u = \phi$ on $\partial\Omega$. We suppose that $\phi \in C^{0,1}(\partial\Omega)$ and $\partial\Omega$ is of class C^4. Let $A = \{x \in \partial\Omega \mid \Lambda(x) < 0,\, u(x) \neq \phi(x)\}$. Then by Lau and Lin [13] it was proved that u is Hölder continuous near A with exponent exactly $1/2$, and the trace of u over A is regular according to the regularity of $\partial\Omega$. More precise results were obtained by Korevaar and Simon [9] and Simon [19].

In this article our aim is as follows: Let Γ be a portion of $\partial\Omega$, where (1.6) is not assumed. Instead we assume that Γ is transformed into a hyperplane by an orthogonal coordinates mapping (see the Definition in the beginning of Section 2). Then we shall show that there exists a solution u of the Dirichlet problem (1.1) with (1.2) such that for some $\alpha > 0$, $(u-\phi)/(1+|Du|^2)^\alpha$ belongs to $W^{1,2}$ near Γ and its trace over Γ equals 0 (see Theorem 1). This statement means that $u = \phi$ on Γ in the weak sense. In fact the equality $u = \phi$ on Γ, is equivalent to that the trace of $(u-\phi)/(1+|Du|^2)^\alpha$ vanishes there, if u is smooth. Next we shall give a sufficient condition in order that $u - \phi \in W^{1,1}(\Omega)$ and its trace vanishes on Γ (see Theorem 2).

There is the result of Ladyzhenskaya and Ural'ceva [10] and [11] concerning the local interior estimate of approximating solutions. In [11] particularly, the gradient bound in the interior domain was proved. In this article, by using the method in [10], we prepare some boundary estimates in order to prove our theorems.

§2. Result

From now on, let Ω be a bounded domain, and $\partial\Omega$ be locally Lipschitz-continuous. We denote by $B_\delta(P)$ the open ball in R^n with its center P and with its radius δ. We set the following

DEFINITION. We say that $P \in \partial\Omega$ has property (A), if the following holds:

There exist a positive number δ and an one-to-one mapping Φ

$$\Phi : B_\delta(P) \ni (x_1, \dots, x_n) \longmapsto (\xi_1, \dots, \xi_n) \in R^n$$

satisfying

(I) Φ and Φ^{-1} are both of class C^3 such that

$$\frac{D(\xi_1,\dots,\xi_n)}{D(x_1,\dots,x_n)} > 0 \quad \text{in } B_\delta(P).$$

(II) $\Phi(P) = O$, $\Phi(B_\delta(P)\cap\Omega) \subset \{\xi_n > 0\}$ and

$$\Phi(B_\delta(P)\cap\partial\Omega) \subset \{\xi_n = 0\}.$$

(III) $D_x\xi_i \cdot D_x\xi_j = 0$ in $B_\delta(P)$, if $i \neq j$.

We denote by $\mathbf{n}$ and τ the inward normal vector and the tangent vector at $\partial\Omega \cap B_{\delta/2}(P)$, respectively. Then from the above (III) we have

$$\frac{\partial}{\partial \mathbf{n}} = a_n D_{\xi_n} \quad \text{and} \quad \frac{\partial}{\partial \tau} = \sum_{i=1}^{n-1} a_i D_{\xi_i},$$

where a_i are C^2 functions such that $a_n > 0$ and $\sum_{i=1}^{n-1} |a_i| \neq 0$ on $\partial\Omega \cap B_{\delta/2}(P)$. Let $1 \le p < \infty$ and Γ be an open set on $\partial\Omega$. We define

$$\begin{aligned} W_0^{1,p}(\Omega;\Gamma) = \{u \mid\, & u \in W^{1,p}(\Omega') \text{ and the trace of } u \text{ over } \partial\Omega'\cap\Gamma \\ & \text{vanishes for any subdomain } \Omega' \text{ of } \Omega \text{ such that} \\ & \overline{\Omega}' \cap (\partial\Omega - \Gamma) = \phi \text{ and } \overline{\partial\Omega'\cap\Gamma} \subset \Gamma\}. \end{aligned}$$

Throughout this article we set the following assumptions:

We take two relatively open subsets Γ_1 and Γ_2 of $\partial\Omega$, where Γ_1 is of class C^3 and each point of Γ_1 has property (A). It is not assumed that $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Let H be a function in $C^{0,1}(\overline{\Omega})$ satisfying (1.3). Let ϕ be a function in $C^{2,1}(\Omega \cup \overline{\Gamma}_1) \cap C^0(\overline{\Omega}) \cap W^{1,1}(\Omega)$. Further (1.6) is imposed only on Γ_2, namely it is not assumed on Γ_1.

Then we solve the Dirichlet problem (1.1) with (1.2). Our first aim is to prove

THEOREM 1. *There exists a solution* $u \in C^2(\Omega)\cap C^0(\Omega\cup\Gamma_2)\cap W^{1,1}(\Omega)$ *of* (1.1) *such that*

$$u = \phi \ \text{ on } \Gamma_2, \quad \text{and} \quad \frac{u-\phi}{(1+|Du|^2)^{5/4}} \in W_0^{1,2}(\Omega;\Gamma_1).$$

As stated at the end of the previous section, the last relation in Theorem 1 is regarded as $u = \phi$ on Γ_1, in the weak sense. We take a sequence of domains $\{\Omega_j\}$ as follows: Each $\partial\Omega_j$ is of class C^3 and $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_j \to \Omega$ $(j \to \infty)$. Further for any compact set K in Γ_1, it holds that $\partial\Omega_{j_0} \supset K$ for some j_0. Obviously there is a sequence $\{\phi_j\} \subset C^3(\overline{\Omega})$ satisfying

$$\phi_j \to \phi \quad \text{in } C^{2,1}(\Omega \cup \overline{\Gamma}_1) \cap C^0(\overline{\Omega}) \cap W^{1,1}(\Omega)\ (j \to \infty).$$

We take a positive sequence $\{\varepsilon_j\}$ such that $\varepsilon_j \to 0$ $(j \to \infty)$ and $\overline{\lim}_{j\to\infty}\big(\varepsilon_j \int_\Omega |D\phi_j|^2\, dx\big) < \infty$. It is known that for each j there is a solution $u_j \in C^2(\overline{\Omega}_j)$ of

$$\text{(2.1)} \qquad \begin{cases} \varepsilon_j \Delta u_j + D\cdot\left(\dfrac{Du_j}{\sqrt{1+|Du_j|^2}}\right) = nH & \text{in } \Omega_j \\ u_j = \phi_j & \text{on } \partial\Omega_j \end{cases}$$

(see [3]).

According to the result of Simon [18], there is a subsequence $\{u_\nu\}$ of $\{u_j\}$ and a function $u \in C^2(\Omega)$ such that for any compact subset K of Ω

$$\text{(2.2)} \qquad D^\alpha u_\nu \rightrightarrows D^\alpha u \quad \text{in } K\ (\nu \to \infty), \quad |\alpha| \le 2,$$

$u \in C^0(\Omega \cup \Gamma_2)$, $u = \phi$ on Γ_2 and (1.1) holds. The reason for its validity is due to (1.6).

In [18] the following equation was considerd in place of that in (2.1):

$$\varepsilon_j \Delta u_j + (1-\varepsilon_j) D\cdot\left(\frac{Du_j}{\sqrt{1+|Du_j|^2}}\right) = nH.$$

But the situation is quite parallel.

Next we have

THEOREM 2. *Assume that* $\partial u_j/\partial\mathbf{n} \geqq 0$ *on* $\Gamma_1 \cap \partial\Omega_j$, *for each* u_j. *Then there is a positive constant* d_0 *depending only on the shape of* Γ_1 *such that if* $H \geqq d_0$ *on* Γ_1, *the equation* (1.1) *is solvable for* $u \in C^2(\Omega) \cap C^0(\Omega \cup \Gamma_2)$ *satisfying*

$$u = \phi \ \text{ on } \Gamma_2, \quad u - \phi \in W^{1,1}(\Omega) \cap W_0^{1,1}(\Omega; \Gamma_1)$$

and

$$\frac{u-\phi}{(1+|Du|^2)^{1/4}} \in W_0^{1,2}(\Omega;\Gamma_1).$$

The constant d_0 will be concretely given in the proof of Proposition 4.3 (see (4.27)). Though it is almost impossible to verify the assumption in the above theorem, we give two examples to show that Theorem 2 is not meaningless. For this sake we prepare the following.

Let D be a bounded domain in R^n and $\tilde{H}(x)$ be a bounded function in D. For $\varepsilon > 0$ we define the operator Q^ε:

$$\begin{aligned} Q^\varepsilon u = \big(\varepsilon\sqrt{1+|Du|^2}+1\big)\big(1+|Du|^2\big)\Delta u \\ -D_i u \cdot D_j u \cdot D_i D_j u - n\tilde{H}(x)\big(1+|Du|^2\big)^{3/2}. \end{aligned}$$

Then the following assertion holds, which is due to Theorem 10.1 in [3].

Suppose that u, $v \in C^2(D) \cap C^1(\overline{D})$ *and* $Q^\varepsilon v \geqq Q^\varepsilon u$ *in* D. *Then* $v \leqq u$ *in* D, *if* $v \leqq u$ *on* ∂D.

Here we assume that ∂D is of class C^1. Let Γ be an open subset of ∂D. Let $\tilde{\phi} \in C^{2,1}(\overline{D})$ and $\tilde{H} \in C^{0,1}(\overline{D})$. Then we have

Proposition 2.1. *Suppose that* u *is a function in* $C^2(D) \cap C^1(\overline{D})$ *satisfying*

$$Q^\varepsilon u = 0 \ \text{ in } D \quad \text{and} \quad u = \tilde{\phi} \ \text{ on } \partial D.$$

Then $\partial u/\partial \mathbf{n} \geqq 0$ *on* Γ, *if there is a function* $v \in C^2(D) \cap C^1(\overline{D})$ *such that*

$$\begin{aligned} &Q^\varepsilon v \geqq 0 \ \text{ in } D, && v \leqq \tilde{\phi} \ \text{ on } \partial D, \\ &v = \tilde{\phi} \ \text{ on } \Gamma && \text{and} \ \ \partial v/\partial \mathbf{n} \geqq 0 \ \text{ on } \Gamma. \end{aligned}$$

Proof. By the previous assertion we see that $v \leqq u$ in $\overline{D}$. Since $v = u$ on Γ, it holds that $\partial v/\partial \mathbf{n} \leqq \partial u/\partial \mathbf{n}$ on Γ. This completes the proof. ∎

We give the following two examples satisfying the assumptions in Theorem 2.

Example 1. Let Γ_1 be the arc defined in the example given at the end of Section 4, where we put $R = 1$. Let Ω be a bounded domain such that Ω lies up Γ_1 and $\partial\Omega \supset \Gamma_1$ (see Figure 1).

Let $\Gamma_2 = \partial\Omega - \overline{\Gamma}_1$. We assume

$$1 < \sqrt{\frac{\pi}{|\Omega|}}. \tag{2.3}$$

As stated in this section, we take the approximating sequence $\{\Omega_j\}$ of Ω.

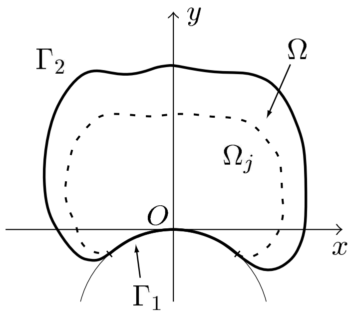

Figure 1.

Let H be a positive function in $C^{0,1}(\overline{\Omega})$ such that

$$\frac{1}{2} \leqq H < \frac{1}{2}\sqrt{\frac{\omega_2}{|\Omega|}}. \tag{2.4}$$

From (2.3) it is possible to take such a function H. Retaking Γ_2, we may assume that (1.6) holds there.

We define two positive numbers d_1 and d_2 as follows:

$$d_1 = -\inf\{y \mid (x,y) \in \Omega\}, \quad d_2 = \sup\{y \mid (x,y) \in \Omega\}.$$

Taking two real numbers A and B, we set

$$v(x,y) = A(y+d_1)^2 + B.$$

Let us impose the following assumptions on A and $d_1 + d_2$:

$$A \geqq (\sup_\Omega |H|)\big(1 + 4A^2(d_1+d_2)^2\big)^{3/2}. \tag{2.5}$$

The two relations (2.4) and (2.5) are not contradictory each other.

Let ϕ be a function such that $\phi \geqq v$ in Ω and $\phi = v$ on Γ_1. Under the above conditions we set $\varepsilon = \varepsilon_j$, $D = \Omega_j$, $\Gamma = \Gamma_1 \cap \partial\Omega_j$, $\tilde{H} = H$ and $\tilde{\phi} = \phi$ in Proposition 2.1. From (2.4) the condition (1.3) on H is satisfied. Let u_j be the solution satisfying

$$Q^{\varepsilon_j} u_j = 0 \text{ in } \Omega_j \quad \text{and} \quad u_j = \phi \text{ on } \partial\Omega_j.$$

Previously we may assume

$$\partial v/\partial \mathbf{n} \geqq 0 \quad \text{on } \Gamma_1.$$

From (2.5) we see that $Q^{\varepsilon_j} v \geqq 0$ in Ω. Therefore it follows from Proposition 2.1 that

$$\partial u_j/\partial \mathbf{n} \geqq 0 \quad \text{on } \Gamma_1 \cap \partial\Omega_j. \tag{2.6}$$

In this case the positive constant d_0 in Theorem 2 will be calculated exactly in the example given at the end of Section 4. That is, we can take $d_0 = 1/2$, which is independent of $d_1 + d_2$. The conditions (2.4) with (2.6), mean that this example satisfies the assumptions in Theorem 2.

Example 2. Next let Ω be the annular domain such as

$$\Omega = \{(x, y) \mid 0 < R_1 < r < R_2\}, \quad r = \sqrt{x^2 + y^2}.$$

We set $\Gamma_1 = \{r = R_1\}$ and $\Gamma_2 = \{r = R_2\}$ (see Figure 2). We put $\Omega_j = \Omega$

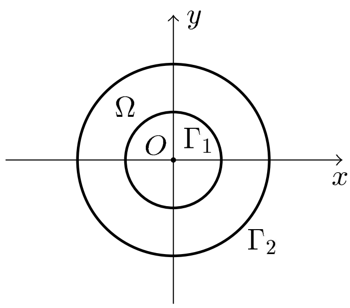

Figure 2.

for all j. Let H be a positive function in $C^{0,1}(\overline{\Omega})$ satisfying (1.3), where $|\Omega| = \pi(R_2^2 - R_1^2)$. It is known that the boundary mean curvature Λ equals $1/R_2$ at each point on Γ_2.

We assume

$$H \geqq \frac{1}{2R_1} \text{ on } \Gamma_1 \quad \text{and} \quad H \leqq \frac{1}{2R_2} \text{ on } \Gamma_2.$$

Then (1.6) is satisfied on Γ_2. Taking two real numbers A and B, we define

$$v(x, y) = Ar^2 + B$$

and assume that

$$2A + 4A^3 r^2 > H(1 + 4A^2 r^2)^{3/2} \quad \text{for } R_1 \leqq r \leqq R_2. \tag{2.7}$$

Then it is seen that $Q^{\varepsilon_j} v \geqq 0$ in Ω. And obviously, $\partial v/\partial \mathbf{n} \geqq 0$ on Γ_1. Let ϕ be a function such that $\phi \geqq v$ in Ω and $\phi = v$ on Γ_1. We take u_j in such a way that

$$Q^{\varepsilon_j} u_j = 0 \ \text{ in } \Omega \quad \text{and} \quad u_j = \phi \ \text{ on } \partial\Omega.$$

Applying Proposition 2.1, we have

$$\partial u_j/\partial \mathbf{n} \geqq 0 \quad \text{on } \Gamma_1.$$

Similarly as in Example 1, we can take $d_0 = 1/2R_1$. The following inequality holds:

$$2t + 4t^3 > \frac{1}{2}\left(1 + 4t^2\right)^{3/2} \quad \text{for } t \geqq \frac{1}{2}.$$

This means that we can take a function H and four numbers A, B, R_1 and R_2 satisfying the above conditions.

§3. Preliminaries

In this section we prepare some known results for solutions u_j in (2.1). We denote by $(\ ,\ )_j$ the $L^2(\Omega_j)$-inner product. Setting $v_j = u_j - \phi_j$, we multiply (2.1) with v_j. Then by integration by parts, we have from (1.5)

$$\begin{aligned}
&\varepsilon_j\left(1, |Dv_j|^2\right)_j + \left(\frac{Du_j}{\sqrt{1+|Du_j|^2}}, Du_j\right)_j \\
&\quad \leqq -\varepsilon_j(Dv_j, D\phi_j)_j + \left(\frac{Du_j}{\sqrt{1+|Du_j|^2}}, D\phi_j\right)_j + (1-\varepsilon_0)(1, |Dv_j|)_j.
\end{aligned}$$

We use the assumption on $\{\varepsilon_j\}$ and the inequality $|t| - 1 \leqq t^2/\sqrt{1+t^2}$ $(t \in R)$. Then it follows that

$$\varepsilon_j\left(1, |Du_j|^2\right)_j + (1, |Du_j|)_j \leqq C \tag{3.1}$$

where C depends on H, ϕ and Ω, but not on j.

Next we verify the uniform boundedness of u_j with respect to j. If $\varepsilon_j = 0$, this is due to Chapter 10 in [3]. So, we can proceed in parallel with it. We rewrite the equation in (2.1) with

$$\begin{aligned}
Q_j u_j = {}& \varepsilon_j\left(1+|Du_j|^2\right)^{3/2}\Delta u_j + \left(1+|Du_j|^2\right)\Delta u_j \\
& - D_k u_j \cdot D_h u_j \cdot D_{kh} u_j - nH\left(1+|Du_j|^2\right)^{3/2} = 0.
\end{aligned} \tag{3.2}$$

We put

$$\begin{aligned} p &= (p_1, \ldots, p_n), \\ a_j^{kh}(p) &= \varepsilon_j \big(1+|p|^2\big)^{3/2}\delta_{kh} + \big(1+|p|^2\big)\delta_{kh} - p_k p_h, \\ b(x,p) &= -nH(x)\big(1+|p|^2\big)^{3/2}, \\ \mathfrak{D}_j &= \det[a_j^{kh}(p)] \quad \text{and} \quad \mathfrak{D}_j^* = \mathfrak{D}_j^{1/n}. \end{aligned}$$

The equation (3.2) becomes $a_j^{kh}(Du_j)D_kD_hu_j+b(x,Du_j)=0$. In virtue of Theorem 10.5 in [3], the following assertion holds:

Let $g(p)$ be in $L^n_{loc}(R^n)$ and $h(x)$ be in $L^n(\Omega_j)$ such that

$$\frac{|b(x,p)|}{n\mathfrak{D}_j^*} \leqq \frac{h(x)}{g(p)} \quad \text{for } (x,p) \in \Omega_j \times R^n$$

and

$$\int_{\Omega_j} h^n\, dx \leqq \int_{R^n} g^n\, dp.$$

Then the solution u_j of (3.2) *satisfies*

$$\sup_{\Omega_j} |u_j| \leqq \sup_{\partial\Omega_j} |u_j| + C_0\, d(\Omega_j), \tag{3.3}$$

where $d(\Omega_j)$ is the diameter of Ω_j and C_0 depends only on g and h.

In our case we easily see that

$$\mathfrak{D}_j = \big(1+\varepsilon_j(1+|p|^2)^{3/2}\big)\big(1+|p|^2+\varepsilon_j(1+|p|^2)^{3/2}\big)^{n-1}.$$

Hence $\mathfrak{D}_j^* \geqq \big(1+|p|^2\big)^{(n-1)/n}$ and

$$\frac{|b(x,p)|}{n\mathfrak{D}_j^*} \leqq |H(x)|\big(1+|p|^2\big)^{-(n+2)/(2n)}.$$

It is enough to set

$$h(x) = |H(x)| \quad \text{and} \quad g(p) = \big(1+|p|^2\big)^{-(n+2)/(2n)}.$$

Since

$$\int_{R^n} g^n\, dp = \omega_n, \qquad \int_{\Omega_j} h^n\, dx < \omega_n \quad (\text{from } (1.4)),$$

the assumptions on the above assertion are satisfied. We define the function $G(t)$, $t \geqq 0$ with

$$G^{-1}(t) = \int_{B_t(O)} g^n \, dp, \tag{3.4}$$

where $B_t(O) = \{|p| < t\}$. Then G is a function from $(0, \omega_n)$ onto $(0, \infty)$. The constant C_0 on (3.3) is given by $G\big(\int_\Omega h^n \, dx\big)$, which is referred to [3].

Let u be the function in (2.2). We show that

$$u \in W^{1,1}(\Omega). \tag{3.5}$$

Hereafter we denote by the same $\{\nu\}$ any subsequence of $\{j\}$. Let Ω' be any fixed subdomain of Ω with $\overline{\Omega}' \subset \Omega$. From (3.1), $\int_{\Omega'} \big(|Du_j|^{1/2}\big)^2 \, dx$ are uniformly bounded with respect to j. Hence

$$|Du_\nu|^{1/2} \longrightarrow g \quad \text{weakly in } L^2(\Omega') \text{ as } \nu \to \infty.$$

Thus

$$\int_{\Omega'} g^2 \, dx \leqq \varliminf_{\nu \to \infty} \int_{\Omega'} |Du_\nu| \, dx$$

The right-hand side is uniformly bounded with respect to Ω' in virtue of (3.1). This means that $g \in L^2(\Omega)$. On the other hand $g = |Du|^{1/2}$ from (2.2). Hence (3.5) is correct.

§4. Main estimate

We suppose the assumptions in the beginning of Section 2. Let u_j be the solution of (2.1). Let P be any fixed point on Γ_1. From our assumption there is the coordinates transformation Φ in the definition of property (A).

We put

$$h_i(x) = |D_x \xi_i|^2, \quad i = 1, \ldots, n.$$

and

$$J(\xi) = \frac{\partial(x_1, \ldots, x_n)}{\partial(\xi_1, \ldots, \xi_n)}.$$

Then $h_i > 0$ in $B_\delta(P)$ and $J = (h_1 \cdots h_n)^{-1/2}$. Obviously

$$D_x u \cdot D_x v = h_i D_{\xi_i} u \cdot D_{\xi_i} v.$$

So, $|D_x u|^2 = h_i (D_{\xi_i} u)^2$, which is written by $|Eu|^2$. The first equation in (2.1) becomes

$$\varepsilon_j \int_{\Omega_j} D_x u_j \cdot D_x \varphi \, dx + \int_{\Omega_j} \frac{D_x u_j \cdot D_x \varphi}{\sqrt{1+|D_x u_j|^2}} \, dx = -n \int_{\Omega_j} H \varphi \, dx, \qquad \varphi \in C_0^\infty(\Omega_j),$$

from which we have

$$\varepsilon_j D_{\xi_i}(J h_i D_{\xi_i} u_j) + D_{\xi_i}\left(\frac{J h_i}{\sqrt{1+|Eu_j|^2}} D_{\xi_i} u_j\right) = nHJ. \tag{4.1}$$

We recall that P is mapped to the origin in $(\xi_1, \ldots, \xi_n)$-space. From now on we consider u_j only for sufficiently large j.

Our first object in this Section is to prove the following

Proposition 4.1. *There is a positive number ρ such that*

$$\int_{B_\rho(O) \cap \{\xi_n \geqq 0\}} \left(\varepsilon_j + \frac{1}{\left(1+|D_\xi u_j|^2\right)^{3/2}}\right) |D_\xi D_{\xi_k} u_j|^2 \, d\xi \leqq C \ (< \infty), \qquad k = 1, \ldots, n-1,$$

where ρ and C are independent of j.

Proof. For simplicity $D_\xi, D_{\xi_i}, \ldots$ are denoted by $D, D_i, \ldots$, respectively. In the defnition of property (A) we take $\rho > 0$ in such a way that $\Phi(B_\delta(P)) \supset B_{2\rho}(O)$. Let ζ be a non-negative function in $C_0^\infty(B_{2\rho}(O))$. It may be assumed that $\zeta^{-1}|D\zeta|^2$ and $|D(|D\zeta|)|$ are bounded. For simplicity we denote u_j and ϕ_j by u and ϕ, respectively. And we denote by $(\ , \ )$ the $L^2(\{\xi_n \geqq 0\})$-inner product. Let k be any fixed integer such as $k = 1, \ldots, n-1$. Setting $v = u - \phi$ $(= u_j - \phi_j)$, we multiply (4.1) with $D_k(\zeta D_k v)$. Then

$$\begin{aligned} &\varepsilon_j (D_i(J h_i D_i u),\, D_k(\zeta D_k v)) + \left(D_i\left(\frac{J h_i D_i u}{\sqrt{1+|Eu|^2}}\right),\, D_k(\zeta D_k v)\right) \\ &\qquad = n(JH,\, D_k(\zeta D_k v)) \end{aligned} \tag{4.2}$$

We estimate the first term on the left-hand side of (4.2). By integration by parts

$$\begin{aligned} (D_i(J h_i D_i u),\, D_k(\zeta D_k v)) &= -(D_i D_k(J h_i D_i u),\, \zeta D_k v) \\ &= (D_k(J h_i D_i u),\, D_i(\zeta D_k v)), \end{aligned}$$

since $D_k v = 0$ on $\{\xi_n = 0\}$. This calculation needs that u is in C^3. But it is avoided, because we can take an approximating sequence of C^3 functions for u. Hence

$$(4.3)\qquad \begin{aligned}&(D_i(Jh_iD_iu),\, D_k(\zeta D_kv))\\ &\quad = (D_k(Jh_iD_iv),\, D_i(\zeta D_kv)) + (D_k(Jh_iD_i\phi),\, D_i(\zeta D_kv))\\ &\quad \equiv I_1 + I_2, \quad \text{say.}\end{aligned}$$

Obviously

$$\begin{aligned}I_1 = {}&(\zeta Jh_i,\, (D_iD_kv)^2) + (Jh_iD_i\zeta,\, D_kv\cdot D_iD_kv)\\ &+(\zeta D_k(Jh_i),\, D_iv\cdot D_iD_kv) + (D_i\zeta\cdot D_k(Jh_i),\, D_iv\cdot D_kv).\end{aligned}$$

From now on we denote by the same C any positive constant independent of j. By Cauchy's inequality we have for $\delta > 0$

$$|(Jh_iD_i\zeta,\, D_kv\cdot D_iD_kv)| \leqq \delta\big(\zeta Jh_i,\, (D_iD_kv)^2\big) + C(\delta)\big(\zeta^{-1}(D_i\zeta)^2,\, (D_kv)^2\big)$$

and

$$|(\zeta D_k(Jh_i),\, D_iv\cdot D_iD_kv)| \leqq \delta\big(\zeta Jh_i,\, (D_iD_kv)^2\big) + C(\delta)\big(\zeta,\, (D_iv)^2\big),$$

where $C(\delta)$ depends on δ but not on j. Hence we obtain

$$(4.4)\qquad \begin{aligned}I_1 \geqq {}&(1-2\delta)\big(\zeta Jh_i,\, (D_iD_kv)^2\big)\\ &-C(\delta)\big(\zeta + |D\zeta| + \zeta^{-1}|D\zeta|^2,\, |Dv|^2\big).\end{aligned}$$

Next we write

$$I_2 = (\zeta D_k(Jh_iD_i\phi),\, D_iD_kv) + (D_i\zeta\cdot D_k(Jh_iD_i\phi),\, D_kv).$$

Let M be a positive constant such that

$$|\phi|,\, |D\phi|,\, |DD_i\phi| \leqq M,$$

where M depends on the support of ζ, but not on j. Then

$$|(\zeta D_k(Jh_iD_i\phi),\, D_iD_kv)| \leqq \delta\big(\zeta Jh_i,\, (D_iD_kv)^2\big) + C(\delta)M^2,$$

and

$$|(D_i\zeta\cdot D_k(Jh_iD_i\phi),\, D_kv)| \leqq C\big[\big(|D\zeta|,\, |Dv|^2\big) + M^2\big].$$

Hence

$$I_2 \geqq -\delta\big(\zeta J h_i,\, (D_i D_k v)^2\big) - C(\delta)\big[\big(|D\zeta|,\, |Dv|^2\big) + M^2\big]. \tag{4.5}$$

Combining (4.4) and (4.5) with (4.3), we obtain

$$\begin{aligned}(4.6)\quad & (D_i(Jh_iD_iu),\, D_k(\zeta D_k v)) \\ & \geqq \frac{1}{2}\big(\zeta J h_i,\, (D_i D_k v)^2\big) - C\big[\big(\zeta + |D\zeta| + \zeta^{-1}|D\zeta|^2,\, |Dv|^2\big) + M^2\big].\end{aligned}$$

Here we note the following: If (4.3) is correct for $k = n$, then (4.6) is so.

Next we estimate the second term on the left-hand side of (4.2). Similarly by integration by parts

$$\begin{aligned}(4.7)\quad & \bigg(D_i\Big(\frac{Jh_iD_iu}{\sqrt{1+|Eu|^2}}\Big),\, D_k(\zeta D_k v)\bigg) \\ & = \bigg(D_k\Big(\frac{Jh_iD_iu}{\sqrt{1+|Eu|^2}}\Big),\, D_i(\zeta D_k v)\bigg) \\ & = \bigg(D_k\Big(\frac{Jh_iD_iu}{\sqrt{1+|Eu|^2}}\Big),\, D_i(\zeta D_k u)\bigg) - \bigg(D_k\Big(\frac{Jh_iD_iu}{\sqrt{1+|Eu|^2}}\Big),\, D_i(\zeta D_k \phi)\bigg) \\ & \equiv I_3 + I_4, \quad \text{say.}\end{aligned}$$

If (4.7) holds for $k = n$, the following argument is also correct for the case of $k = n$, except for the terms I_{32} and K_1. First we estimate I_3. Using the equality

$$\begin{aligned} D_k\bigg(\frac{Jh_iD_iu}{\sqrt{1+|Eu|^2}}\bigg) = {} & \frac{Jh_i}{\sqrt{1+|Eu|^2}}\bigg(D_iD_ku - \frac{h_lD_lu\cdot D_iu\cdot D_kD_lu}{1+|Eu|^2}\bigg) \\ & + \frac{D_k(Jh_i)\cdot D_iu}{\sqrt{1+|Eu|^2}} - \frac{1}{2}\cdot\frac{D_kh_l\cdot(D_lu)^2}{\big(1+|Eu|^2\big)^{3/2}} Jh_iD_iu,\end{aligned}$$

we have

$$\begin{aligned}(4.8)\quad I_3 = {} & \bigg(\frac{J\zeta h_i}{\sqrt{1+|Eu|^2}},\, \big(D_iD_ku\big)^2 - \frac{h_lD_lu\cdot D_iu\cdot D_kD_lu}{1+|Eu|^2} D_iD_ku\bigg) \\ & + \bigg(\frac{\zeta D_k(Jh_i)}{\sqrt{1+|Eu|^2}},\, D_iu\cdot D_iD_ku\bigg) \\ & - \frac{1}{2}\bigg(\frac{J\zeta h_iD_kh_l}{\big(1+|Eu|^2\big)^{3/2}},\, (D_lu)^2 D_iu\cdot D_iD_ku\bigg)\end{aligned}$$

$$
\begin{aligned}
&+\Big(\frac{Jh_iD_i\zeta}{\sqrt{1+|Eu|^2}},\ D_iD_ku\cdot D_ku-\frac{h_lD_lu\cdot D_iu\cdot D_kD_lu}{1+|Eu|^2}D_ku\Big)\\
&+\Big(\frac{D_k(Jh_i)\cdot D_i\zeta}{\sqrt{1+|Eu|^2}},\ D_iu\cdot D_ku\Big)\\
&-\frac{1}{2}\left(\frac{Jh_iD_i\zeta\cdot D_kh_l}{(1+|Eu|^2)^{3/2}},\ (D_lu)^2D_iu\cdot D_ku\right)\\
\equiv{}&\sum_{i=1}^{6}I_{3i},\quad\text{say.}
\end{aligned}
$$

Obviously

$$|I_{35}|,\ |I_{36}|\leqq C(|D\zeta|,|Du|).$$

We estimate I_{34}. The idea of the estimation of I_{34} is due to [10]. Let us set three vectors as follows:

$$\boldsymbol{a}=(\sqrt{h_i}D_i\zeta),\quad \boldsymbol{b}=(\sqrt{h_i}D_iu),\quad \boldsymbol{c}=(\sqrt{h_i}D_iD_ku).$$

Then

$$
\begin{aligned}
&\left|h_iD_i\zeta\cdot D_iD_ku-\frac{h_iD_i\zeta\cdot h_lD_lu\cdot D_iu\cdot D_kD_lu}{1+|Eu|^2}\right|\\
&\quad=\left|\boldsymbol{a}\cdot\boldsymbol{c}-\frac{(\boldsymbol{a}\cdot\boldsymbol{b})(\boldsymbol{b}\cdot\boldsymbol{c})}{1+|\boldsymbol{b}|^2}\right|\\
&\quad\leqq\left(|\boldsymbol{a}|^2-\frac{(\boldsymbol{a}\cdot\boldsymbol{b})^2}{1+|\boldsymbol{b}|^2}\right)^{1/2}\left(|\boldsymbol{c}|^2-\frac{(\boldsymbol{b}\cdot\boldsymbol{c})^2}{1+|\boldsymbol{b}|^2}\right)^{1/2}.
\end{aligned}
$$

Hence we see that

$$|I_{34}|\leqq\left(\frac{J|\boldsymbol{a}||D_ku|}{\sqrt{1+|Eu|^2}},\left(|\boldsymbol{c}|^2-\frac{(\boldsymbol{b}\cdot\boldsymbol{c})^2}{1+|\boldsymbol{b}|^2}\right)^{1/2}\right).$$

By Cauchy's inequality

$$
\begin{aligned}
|I_{34}|\leqq{}&\delta\Big(\frac{J\zeta}{\sqrt{1+|Eu|^2}},\ h_i(D_iD_ku)^2-\frac{h_ih_lD_iu\cdot D_lu\cdot D_iD_ku\cdot D_kD_lu}{1+|Eu|^2}\Big)\\
&+C(\delta)\big(\zeta^{-1}|D\zeta|^2,\ |Du|\big),\quad \delta>0.
\end{aligned}
$$

Hence

$$I_{31}+I_{34}\geqq\frac{1}{2}I_{31}-C\big(\zeta^{-1}|D\zeta|^2,\ |Du|\big).$$

On the other hand

$$\begin{aligned}&\big(1+|Eu|^2\big)h_i(D_iD_ku)^2-h_ih_lD_iu\cdot D_lu\cdot D_iD_ku\cdot D_lD_ku\\&\quad=|\boldsymbol{c}|^2+|\boldsymbol{b}|^2|\boldsymbol{c}|^2-(\boldsymbol{b}\cdot\boldsymbol{c})^2\geqq|\boldsymbol{c}|^2.\end{aligned}$$

Combining the above inequalities with (4.8) we obtain

$$\begin{aligned}I_3\geqq\frac{1}{2}\bigg(\frac{J\zeta h_i}{\big(1+|Eu|^2\big)^{3/2}},\,(D_iD_ku)^2\bigg)+I_{32}+I_{33}\\-C\big(|D\zeta|+\zeta^{-1}|D\zeta|^2,\,|Du|\big).\end{aligned}\tag{4.9}$$

Now we estimate the remained terms I_{32} and I_{33}. By integration by parts

$$I_{32}=-\frac{1}{2}\bigg(D_k\Big(\frac{\zeta D_k(Jh_i)}{\sqrt{1+|Eu|^2}}\Big),\,(D_iu)^2\bigg).$$

Hence

$$\begin{aligned}I_{32}=&-\frac{1}{2}\bigg(\frac{D_k(\zeta D_k(Jh_i))}{\sqrt{1+|Eu|^2}},\,(D_iu)^2\bigg)\\&+\frac{1}{2}\bigg(\frac{\zeta D_k(Jh_i)}{\big(1+|Eu|^2\big)^{3/2}},\,h_lD_lu\cdot D_kD_lu\cdot(D_iu)^2\bigg)\\&+\frac{1}{4}\bigg(\frac{\zeta D_k(Jh_i)}{\big(1+|Eu|^2\big)^{3/2}},\,D_kh_l\cdot(D_lu)^2(D_iu)^2\bigg)\\\equiv&\sum_{i=1}^{3}J_i,\quad\text{say.}\end{aligned}\tag{4.10}$$

Easily

$$|J_1|,\,|J_3|\leqq C(\zeta+|D\zeta|,|Du|).\tag{4.11}$$

Since

$$\begin{aligned}J_2=&\frac{1}{2}\bigg(\frac{\zeta JD_kh_i}{\big(1+|Eu|^2\big)^{3/2}},\,h_lD_lu\cdot D_kD_lu\cdot(D_iu)^2\bigg)\\&+\frac{1}{2}\bigg(\frac{\zeta D_kJ\cdot h_i}{\big(1+|Eu|^2\big)^{3/2}},\,h_lD_lu\cdot D_kD_lu\cdot(D_iu)^2\bigg),\end{aligned}$$

we have

$$\begin{aligned}
J_2 + I_{33} &= \frac{1}{2}\bigg(\frac{\zeta D_k J\cdot h_i}{\big(1+|Eu|^2\big)^{3/2}},\, h_l D_l u\cdot D_k D_l u\cdot (D_i u)^2\bigg)\\
&= \frac{1}{2}\bigg(\frac{\zeta D_k J}{\big(1+|Eu|^2\big)^{3/2}},\, |Eu|^2 h_l D_l u\cdot D_k D_l u\bigg)\\
&= \frac{1}{2}\bigg(\frac{\zeta D_k J}{\sqrt{1+|Eu|^2}},\, h_l D_l u\cdot D_k D_l u\bigg)\\
&\qquad -\frac{1}{2}\bigg(\frac{\zeta D_k J}{\big(1+|Eu|^2\big)^{3/2}},\, h_l D_l u\cdot D_k D_l u\bigg).
\end{aligned}$$

Setting

$$K_1 = \frac{1}{2}\bigg(\frac{\zeta D_k J}{\sqrt{1+|Eu|^2}},\, h_l D_l u\cdot D_k D_l u\bigg),$$

we see that

$$|J_2 + I_{33} - K_1| \leqq C\bigg(\frac{\zeta}{\sqrt{1+|Eu|^2}},\, |DD_k u|\bigg). \tag{4.12}$$

By integration by parts

$$K_1 = -\frac{1}{4}\bigg(D_k\Big(\frac{\zeta D_k J\cdot h_l}{\sqrt{1+|Eu|^2}}\Big),\, (D_l u)^2\bigg).$$

Hence

$$\begin{aligned}
K_1 = -\frac{1}{4}&\bigg(\frac{D_k(\zeta D_k J\cdot h_l)}{\sqrt{1+|Eu|^2}},\, (D_l u)^2\bigg)\\
&+\frac{1}{4}\bigg(\frac{\zeta D_k J\cdot h_l}{\big(1+|Eu|^2\big)^{3/2}},\, h_i D_i u\cdot D_i D_k u\cdot (D_l u)^2\bigg)\\
&+\frac{1}{8}\bigg(\frac{\zeta D_k J\cdot D_k h_i\cdot h_l}{\big(1+|Eu|^2\big)^{3/2}},\, (D_i u)^2 (D_l u)^2\bigg).
\end{aligned}$$

Here we write with K_2 the second term on the right-hand side. Then

$$K_2 = \frac{1}{4}\bigg(\frac{\zeta D_k J}{\big(1+|Eu|^2\big)^{3/2}},\, |Eu|^2 h_i D_i u\cdot D_i D_k u\bigg)$$

$$
\begin{aligned}
&= \frac{1}{4}\left(\frac{\zeta D_k J}{\sqrt{1+|Eu|^2}},\, h_i D_i u \cdot D_i D_k u\right) \\
&\qquad -\frac{1}{4}\left(\frac{\zeta D_k J}{\left(1+|Eu|^2\right)^{3/2}},\, h_i D_i u \cdot D_i D_k u\right) \\
&= \frac{1}{2}K_1 - \frac{1}{4}\left(\frac{\zeta D_k J}{\left(1+|Eu|^2\right)^{3/2}},\, h_i D_i u \cdot D_i D_k u\right).
\end{aligned}
$$

From the above we have

$$
\begin{aligned}
|K_1 - K_2| &\leqq C(\zeta + |D\zeta|,\, |Du|), \\
\left|K_2 - \frac{1}{2}K_1\right| &\leqq C\left(\frac{\zeta}{\sqrt{1+|Eu|^2}},\, |DD_k u|\right).
\end{aligned}
$$

Writing $\frac{1}{2}K_1 = (K_1 - K_2) + \left(K_2 - \frac{1}{2}K_1\right)$, we obtain from these inequalities

$$
|K_1| \leqq C\left[\left(\frac{\zeta}{\sqrt{1+|Eu|^2}},\, |DD_k u|\right) + (\zeta + |D\zeta|,\, |Du|)\right].
$$

Therefore it follows from (4.12) that

$$
(4.13) \quad |J_2 + I_{33}| \leqq C\left[\left(\frac{\zeta}{\sqrt{1+|Eu|^2}},\, |DD_k u|\right) + (\zeta + |D\zeta|,\, |Du|)\right].
$$

Combining (4.10), (4.11) and (4.13) with (4.9), we conclude that

$$
\begin{aligned}
(4.14)\ I_3 \geqq{}& \frac{1}{2}\left(\frac{J\zeta h_i}{\left(1+|Eu|^2\right)^{3/2}},\, (D_i D_k u)^2\right) \\
&- C\left[\left(\zeta + |D\zeta| + \zeta^{-1}|D\zeta|^2,\, |Du|\right) + \left(\frac{\zeta}{\sqrt{1+|Eu|^2}},\, |DD_k u|\right)\right].
\end{aligned}
$$

Lastly we estimate I_4. We can write

$$
\begin{aligned}
-I_4 ={}& \left(\frac{J\zeta h_i}{\sqrt{1+|Eu|^2}},\, D_i D_k u \cdot D_i D_k \phi - \frac{h_l D_l u \cdot D_i u \cdot D_k D_l u}{1+|Eu|^2} D_i D_k \phi\right) \\
&+ \left(\frac{\zeta D_k (J h_i)}{\sqrt{1+|Eu|^2}},\, D_i u \cdot D_i D_k \phi\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\bigg(\frac{J\zeta h_i D_k h_l}{\big(1+|Eu|^2\big)^{3/2}},\ (D_l u)^2 D_i u\cdot D_i D_k\phi\bigg)\\
&+\bigg(\frac{J h_i D_i\zeta}{\sqrt{1+|Eu|^2}},\ D_i D_k u\cdot D_k\phi-\frac{h_l D_l u\cdot D_i u\cdot D_k D_l u}{1+|Eu|^2}D_k\phi\bigg)\\
&+\bigg(\frac{D_k(Jh_i)\cdot D_i\zeta}{\sqrt{1+|Eu|^2}},\ D_k\phi\cdot D_i u\bigg)\\
&-\frac{1}{2}\bigg(\frac{J h_i D_i\zeta\cdot D_k h_l}{\big(1+|Eu|^2\big)^{3/2}},\ D_k\phi\cdot(D_l u)^2 D_i u\bigg).
\end{aligned}
$$

Hence

$$
|I_4|\leqq CM\bigg[1+\bigg(\frac{\zeta+|D\zeta|}{\sqrt{1+|Eu|^2}},\ |DD_k u|\bigg)\bigg].
$$

Therefore, from (4.7) and (4.14), it follows that

$$
\begin{aligned}
&\bigg(D_i\Big(\frac{J h_i D_i u}{\sqrt{1+|Eu|^2}}\Big),\ D_k(\zeta D_k v)\bigg)\geqq\frac{1}{2}\bigg(\frac{J\zeta h_i}{\big(1+|Eu|^2\big)^{3/2}},\ (D_i D_k u)^2\bigg)\\
&-C(1+M)\bigg[1+\big(\zeta+|D\zeta|+\zeta^{-1}|D\zeta|^2,\ |Du|\big)+\bigg(\frac{\zeta+|D\zeta|}{\sqrt{1+|Eu|^2}},\ |DD_k u|\bigg)\bigg].
\end{aligned}
$$

By Cauchy's inequality

$$
\begin{aligned}
\text{(4.15)}\quad\bigg(\frac{\zeta+|D\zeta|}{\sqrt{1+|Eu|^2}},\ |DD_k u|\bigg)&\leqq\delta\bigg(\frac{J\zeta h_i}{\big(1+|Eu|^2\big)^{3/2}},\ (D_i D_k u)^2\bigg)\\
&\quad+C(\delta)\big(\zeta+\zeta^{-1}|D\zeta|^2,\ \sqrt{1+|Eu|^2}\big).
\end{aligned}
$$

Accordingly we obtain

$$
\begin{aligned}
\text{(4.16)}\ \bigg(D_i\Big(\frac{J h_i D_i u}{\sqrt{1+|Eu|^2}}\Big),\ D_k(\zeta D_k v)\bigg)&\geqq\frac{1}{4}\bigg(\frac{J\zeta h_i}{\big(1+|Eu|^2\big)^{3/2}},\ (D_i D_k u)^2\bigg)\\
&-C(M)\big[1+\big(\zeta+|D\zeta|+\zeta^{-1}|D\zeta|^2,\ |Du|\big)\big].
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
|(JH,\ D_k(\zeta D_k v))|&=|(D_k(JH),\ \zeta D_k v)|\\
&\leqq C\|H\|_{1,\infty}(\zeta,\ |Dv|)\\
&\leqq C\|H\|_{1,\infty}[(\zeta,\ |Du|)+M],
\end{aligned}
$$

where $\| \;\|_{1,\infty}$ is the $C^{0,1}(\overline{\Omega})$-norm. Combining the above, (4.6), (4.16) and (4.2) with (3.1), we finally conclude that

$$\varepsilon_j\big(\zeta,\, |DD_k u|^2\big) + \left(\frac{\zeta}{\big(1+|Eu|^2\big)^{3/2}},\, |DD_k u|^2\right)$$

is uniformly bounded with respect to j. This completes the proof. ∎

If we eliminate the assumption $k \neq n$ in Proposition 4.1, we have

Proposition 4.2. *Let ρ be the positive number in Proposition* 4.1. *Let $k = 1, \ldots, n$. Then there is a positive constant C independent of j such that*

$$\int_{B_\rho(O)\cap\{\xi_n \geqq 0\}} \frac{|D_\xi D_{\xi_k} u_j|^2}{\big(1+|D_\xi u_j|^2\big)^{7/2}}\, d\xi \leqq C.$$

Proof. The equation (4.1) can be written with

$$Jh_i\left(\varepsilon_j D_i^2 u + D_i\Big(\frac{D_i u}{\sqrt{1+|Eu|^2}}\Big)\right) = F_1 + nJH,$$

where

$$|F_1| \leqq C(1 + \varepsilon_j |Du|).$$

Hence

$$\begin{aligned} &Jh_n\left(\varepsilon_j D_n^2 u + D_n\Big(\frac{D_n u}{\sqrt{1+|Eu|^2}}\Big)\right) \\ &\quad = -\sum_{i\neq n} Jh_i\left(\varepsilon_j D_i^2 u + D_i\Big(\frac{D_i u}{\sqrt{1+|Eu|^2}}\Big)\right) + F_1 + nJH. \end{aligned}$$

This becomes

$$\varepsilon_j D_n^2 u + \frac{1}{\sqrt{1+|Eu|^2}}\left(1 - \frac{h_n (D_n u)^2}{1+|Eu|^2}\right) D_n^2 u = F_2 + n h_n^{-1} H,$$

where

$$|F_2| \leqq C\left[1 + \varepsilon_j |Du| + \left(\varepsilon_j + \frac{1}{\sqrt{1+|Eu|^2}}\right) \sum_{(i,k)\neq(n,n)} |D_i D_k u|\right].$$

Accordingly

$$\left(\varepsilon_j + \frac{1}{\left(1+|Eu|^2\right)^{3/2}}\right)|D_n^2 u| \leqq |F_2| + C|H|,$$

which means that

$$\begin{aligned}&\left(\frac{\zeta}{\left(1+|Eu|^2\right)^{7/2}},\, (D_n^2 u)^2\right)\\ &\quad\leqq C\Bigg[1+\varepsilon_j^2(\zeta,\,|Du|)+\varepsilon_j^2\sum_{(i,k)\neq(n,n)}\left(\frac{\zeta}{\left(1+|Eu|^2\right)^{1/2}},\,(D_iD_ku)^2\right)\\ &\qquad+\sum_{(i,k)\neq(n,n)}\left(\frac{\zeta}{\left(1+|Eu|^2\right)^{3/2}},\,(D_iD_ku)^2\right)\Bigg].\end{aligned}$$

Therefore from Proposition 4.1 and (3.1) we have obtained the required. ∎

In Proposition 4.1, the estimation contains the second derivatives $D_\xi D_{\xi_k} u_j$, but we have assumed that $k \neq n$. If $k = n$, we have the following

PROPOSITION 4.3. *Suppose that $\partial u_j/\partial \mathbf{n} \geqq 0$ on $\Gamma_1 \cap \partial\Omega_j$ for each j. Then there is a positive constant d_0 depending only on Γ_1 such that if $H \geqq d_0$ on Γ_1, it holds that*

$$\int_{B_\rho(O)\cap\{\xi_n\geqq 0\}}\left(\varepsilon_j + \frac{1}{\left(1+|D_\xi u_j|^2\right)^{3/2}}\right)|D_\xi D_{\xi_n} u_j|^2\, d\xi \leqq C, \quad j=1,2,\dots,$$

where ρ and C are two positive constants independent of j.

Proof. As in the proof of Proposition 4.1, we denote $u_j(\phi_j)$ with $u(\phi)$, respectively. And $v = u - \phi$ $(= u_j - \phi_j)$. Our assumption means that $D_n u \geqq 0$ on $\{\xi_n = 0\}$. We write by $\langle\ ,\ \rangle$ the $L^2(\{\xi_n = 0\})$-inner product.

We multiply (4.1) with $D_n(\zeta D_n v)$, where ζ is the function in the proof of Proposition 4.1. We define

$$I_i = \begin{cases} -\langle D_i(Jh_iD_iu),\, \zeta D_n v\rangle & (i\neq n),\\ 0 & (i=n).\end{cases}$$

Then it holds that

$$(4.17)\qquad (D_i(Jh_iD_iu),\, D_n(\zeta D_nv) = (D_n(Jh_iD_iu),\, D_i(\zeta D_nv)) + I_i,$$

In fact, it is trivial for $i = n$. If $i \neq n$, by integration by parts

$$\begin{aligned}
&(D_i(Jh_iD_iu),\, D_n(\zeta D_nv)) \\
&\quad = -\langle D_i(Jh_iD_iu),\, \zeta D_nv\rangle - (D_iD_n(Jh_iD_iu), \zeta D_nv) \\
&\quad = -\langle D_i(Jh_iD_iu),\, \zeta D_nv\rangle + (D_n(Jh_iD_iu),\, D_i(\zeta D_nv)).
\end{aligned}$$

Hence (4.17) is correct.

Similarly we obtain

$$(4.18)\qquad \begin{aligned}
&\Big(D_i\Big(\frac{Jh_iD_iu}{\sqrt{1+|Eu|^2}}\Big),\, D_n(\zeta D_nv)\Big) \\
&\quad = \Big(D_n\Big(\frac{Jh_iD_iu}{\sqrt{1+|Eu|^2}}\Big),\, D_i(\zeta D_nv)\Big) + J_i,
\end{aligned}$$

where

$$J_i = \begin{cases} -\Big\langle D_i\Big(\dfrac{Jh_iD_iu}{\sqrt{1+|Eu|^2}}\Big),\, \zeta D_nv\Big\rangle & (i \neq n), \\ 0 & (i = n). \end{cases}$$

We estimate I_i and J_i, respectively. Let M be the constant in the proof of Proposition 4.1. First we see that

$$\begin{aligned}
|I_i| &\leqq CM\langle\zeta,\, |D_nv|\rangle \\
&\leqq CM\big[\delta\langle\zeta,\, (D_nv)^2\rangle + C(\delta)\big], \quad \delta > 0,
\end{aligned}$$

and

$$\begin{aligned}
\big\langle\zeta,\, (D_nv)^2\big\rangle &= -\big(1,\, D_n(\zeta(D_nv)^2)\big) \\
&\leqq \big(\zeta + |D\zeta|,\, |Dv|^2\big) + \big(\zeta,\, (D_n^2v)^2\big).
\end{aligned}$$

Hence

$$(4.19)\qquad |I_i| \leqq CM\big[\delta\big(\zeta,\, (D_n^2v)^2\big) + \big(\zeta + |D\zeta|,\, |Dv|^2\big) + C(\delta)\big]$$

Next we estimate J_i for $i \neq n$. By integration by parts

$$\begin{aligned}
J_i &= \Big\langle Jh_iD_iu,\, \frac{D_i(\zeta D_nv)}{\sqrt{1+|Eu|^2}}\Big\rangle \\
&= \Big\langle Jh_iD_iu,\, \frac{D_i(\zeta D_nu)}{\sqrt{1+|Eu|^2}}\Big\rangle - \Big\langle Jh_iD_iu,\, \frac{D_i(\zeta D_n\phi)}{\sqrt{1+|Eu|^2}}\Big\rangle.
\end{aligned}$$

Thus we can write

$$J_i = A_1 + A_2, \tag{4.20}$$

where

$$A_1 = \left\langle J h_i D_i \phi,\, \zeta \frac{D_i D_n u}{\sqrt{1+|Eu|^2}} \right\rangle, \quad |A_2| \leqq CM.$$

In general, let $f(t)$, $g(t)$ and $h(t)$ be three given functions such that g, $h > 0$. It is easily seen that

$$\begin{aligned}
&\big(\log(\sqrt{h} f + \sqrt{g + hf^2})\big)' \\
&\quad = \frac{1}{\sqrt{h} f + \sqrt{g+hf^2}} \\
&\qquad \times \left[\sqrt{h} f' + (\sqrt{h})' f + \frac{1}{2}(g + hf^2)^{-1/2}(2hff' + h'f^2 + g') \right] \\
&\quad = \frac{\sqrt{h} f'}{\sqrt{g+hf^2}} + \frac{(\sqrt{h})' f}{\sqrt{h} f + \sqrt{g+hf^2}} + \frac{h'f^2 + g'}{2\sqrt{g+hf^2}\big(\sqrt{h} f + \sqrt{g+hf^2}\big)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{f'}{\sqrt{g+hf^2}} = \frac{1}{\sqrt{h}} \big(\log(\sqrt{h} f + \sqrt{g+hf^2})\big)' - \frac{h'}{2h} \frac{f}{\sqrt{h} f + \sqrt{g+hf^2}} \\
- \frac{1}{2\sqrt{h}} \frac{h'f^2 + g'}{\sqrt{g+hf^2}(\sqrt{h} f + \sqrt{g+hf^2})}.
\end{aligned}$$

We set

$$f = D_n u,\ \psi = \sum_{k \neq n} h_k (D_k u)^2,\ g = 1 + \psi \text{ and } h = h_n.$$

From the above, it follows that

$$\begin{aligned}
\frac{D_i D_n u}{\sqrt{1+|Eu|^2}} = \frac{1}{\sqrt{h_n}} D_i \big(\log(\sqrt{h_n} D_n u + \sqrt{1+|Eu|^2})\big) \\
- \frac{1}{2h_n} \frac{D_i h_n \cdot D_n u}{\sqrt{h_n} D_n u + \sqrt{1+|Eu|^2}} \\
- \frac{1}{2\sqrt{h_n}} \frac{D_i h_n \cdot (D_n u)^2}{\sqrt{1+|Eu|^2}\big(\sqrt{h_n} D_n u + \sqrt{1+|Eu|^2}\big)} \\
- \frac{1}{2\sqrt{h_n}} \frac{D_i \psi}{\sqrt{1+|Eu|^2}\big(\sqrt{h_n} D_n u + \sqrt{1+|Eu|^2}\big)}.
\end{aligned}$$

Thus we have

$$
\begin{aligned}
(4.21)\ A_1 = &\Big\langle Jh_i \frac{\zeta}{\sqrt{h_n}} D_i\phi,\ D_i\big(\log(\sqrt{h_n}D_n u + \sqrt{1+|Eu|^2})\big)\Big\rangle \\
&-\frac{1}{2}\Big\langle Jh_i \frac{D_i h_n}{h_n}\zeta D_i\phi,\ \frac{D_n u}{\sqrt{h_n}D_n u + \sqrt{1+|Eu|^2}}\Big\rangle \\
&-\frac{1}{2}\Big\langle Jh_i \frac{D_i h_n}{\sqrt{h_n}}\zeta D_i\phi,\ \frac{(D_n u)^2}{\sqrt{1+|Eu|^2}\big(\sqrt{h_n}D_n u + \sqrt{1+|Eu|^2}\big)}\Big\rangle \\
&-\frac{1}{2}\Big\langle Jh_i \frac{\zeta}{\sqrt{h_n}} D_i\phi,\ \frac{D_i\psi}{\sqrt{1+|Eu|^2}\big(\sqrt{h_n}D_n u + \sqrt{1+|Eu|^2}\big)}\Big\rangle \\
\equiv &\sum_{i=1}^{4} A_{1i}, \quad \text{say.}
\end{aligned}
$$

Since $D_n u \geqq 0$ on $\{\xi_n = 0\}$, we have

$$1 \leqq \sqrt{h_n}D_n u + \sqrt{1+|Eu|^2} \leqq 2\sqrt{1+|Eu|^2},$$

from which

$$0 \leqq \log\big(\sqrt{h_n}D_n u + \sqrt{1+|Eu|^2}\big) \leqq \log 2 + \frac{1}{2}\log\big(1+|Eu|^2\big) \quad \text{on } \{\xi_n = 0\}.$$

Now we estimate A_{11}. By integration by parts

$$A_{11} = -\Big\langle D_i\Big(Jh_i\frac{\zeta}{\sqrt{h_n}}D_i\phi\Big),\ \log\big(\sqrt{h_n}D_n u + \sqrt{1+|Eu|^2}\big)\Big\rangle.$$

Hence

$$|A_{11}| \leqq CM\big[1 + \langle \zeta + |D\zeta|,\ \log(1+|Eu|^2)\rangle\big].$$

Since

$$\big\langle \zeta + |D\zeta|,\ \log(1+|Eu|^2)\big\rangle = -\big(1,\ D_n((\zeta+|D\zeta|)\log(1+|Eu|^2))\big),$$

and

$$\big|D_n \log(1+|Eu|^2)\big| \leqq C\Big(1 + \frac{|DD_n u|}{\sqrt{1+|Eu|^2}}\Big),$$

we obtain

$$|A_{11}| \leqq CM\Big[1 + \Big(\zeta + |D\zeta|,\ \frac{|DD_n u|}{\sqrt{1+|Eu|^2}}\Big) + \big(|D\zeta| + |D|D\zeta||,\ \sqrt{1+|Eu|^2}\big)\Big].$$

Obviously $|A_{12}|$, $|A_{13}| \leqq CM$, and

$$|D_i\psi| \leqq C\Big[\sum_{k\neq n}(D_k\phi)^2 + \sum_{i,k\neq n}|D_i\phi||D_iD_k\phi|\Big] \quad \text{on } \{\xi_n = 0\}.$$

Hence

$$|A_{14}| \leqq CM.$$

From the above and (4.21) it follows that

$$|A_1| \leqq CM\Big[1 + \Big(\zeta + |D\zeta|,\, \frac{|DD_nu|}{\sqrt{1+|Eu|^2}}\Big) + \big(|D\zeta| + |D(|D\zeta|)|, \sqrt{1+|Eu|^2}\big)\Big]. \tag{4.22}$$

Therefore we obtain by (4.20)

$$|J_i| \leqq \text{the right-hand side of (4.22)}.$$

By Cauchy's inequality it follows that

$$|J_i| \leqq CM\Big[1 + \delta\Big(\zeta,\, \frac{|DD_nu|^2}{(1+|Eu|^2)^{3/2}}\Big) + C(\delta)\big(\zeta + |D\zeta| + \zeta^{-1}|D\zeta|^2 + |D(|D\zeta|)|,\, \sqrt{1+|Eu|^2}\big)\Big]. \tag{4.23}$$

Lastly we have

$$(JH,\, D_n(\zeta D_nv)) = (JH,\, D_n(\zeta D_nu)) - (JH,\, D_n(\zeta D_n\phi))$$

and

$$(JH,\, D_n(\zeta D_nu)) = -(D_n(JH),\, \zeta D_nu) - \langle JH,\, \zeta D_nu\rangle.$$

Hence combining (4.17) and (4.18) with (4.1), we obtain

$$\begin{aligned} &\varepsilon_j(D_n(Jh_iD_iu),\, D_i(\zeta D_nv)) \\ &\quad + \Big(D_n\Big(\frac{Jh_iD_iu}{\sqrt{1+|Eu|^2}}\Big),\, D_i(\zeta D_nv)\Big) + n\langle JH,\, \zeta D_nu\rangle \\ &= -n(D_n(JH),\, \zeta D_nu) - n(JH,\, D_n(\zeta D_n\phi)) - \varepsilon_j\sum_i I_i - \sum_i J_i. \end{aligned} \tag{4.24}$$

Here we use (4.19), (4.23) and (3.1). Then we can put

$$\text{the right hand side of (4.24)} = G, \tag{4.25}$$

where

$$|G| \leqq C(M+1)\Big[\varepsilon_j \delta\big(\zeta,\, (D_n^2 u)^2\big) + \varepsilon_j M^2 + \delta\Big(\zeta,\, \frac{|DD_n u|^2}{(1+|Eu|^2)^{3/2}}\Big) + C(\delta)\Big].$$

Hereafter we proceed in parallel with the proof of Proposition 4.1, by replacing D_k with D_n. The situation is different only for the two terms I_{32} and K_1 in the proof of Proposition 4.1. By integration by parts

$$I_{32} = -\frac{1}{2}\Big(D_n\Big(\frac{\zeta D_n(Jh_l)}{\sqrt{1+|Eu|^2}}\Big),\, (D_l u)^2\Big) - \frac{1}{2}\Big\langle \frac{\zeta D_n(Jh_l)}{\sqrt{1+|Eu|^2}},\, (D_l u)^2\Big\rangle$$

and

$$K_1 = -\frac{1}{4}\Big(D_n\Big(\frac{\zeta D_n J\cdot h_l}{\sqrt{1+|Eu|^2}}\Big),\, (D_l u)^2\Big) - \frac{1}{4}\Big\langle \zeta\frac{D_n J\cdot h_l}{\sqrt{1+|Eu|^2}},\, (D_l u)^2\Big\rangle.$$

On the last equality we set $K_1 = \tilde{K}_1 + L$. Let K_2 be the same in the proof of Proposition 4.1, where k is replaced with n. Let A_i $(i = 1, 2, \ldots)$ be the terms, which will be decided later such that

$$|A_i| \leqq C\Big[\Big(\zeta,\, \frac{|DD_n u|}{\sqrt{1+|Eu|^2}}\Big) + (\zeta + |D\zeta|,\, |Du|)\Big].$$

Then we can write

$$K_2 = \frac{1}{2}(\tilde{K}_1 + L) - A_1, \quad J_2 + I_{33} = \tilde{K}_1 + L + A_2$$

and

$$\tilde{K}_1 = K_2 + A_3.$$

Here we use the equality:

$$\begin{aligned}\frac{1}{2}\tilde{K}_1 &= (\tilde{K}_1 - K_2) + \Big(K_2 - \frac{1}{2}\tilde{K}_1\Big)\\ &= \frac{1}{2}L + A_3 - A_1.\end{aligned}$$

Then

$$J_2 + I_{33} = 2L + (2A_3 - 2A_1 + A_2).$$

Thus (4.13) holds for $k = n$. It needs to estimate the above two boundary integrals. The remained part of the proof is similar to that of Proposition 4.1. If $l \neq n$, $D_l u = D_l \phi$ on $\{\xi_n = 0\}$ and

$$\left|\left\langle \frac{\zeta D_n(Jh_l)}{\sqrt{1+|Eu|^2}},\, (D_l\phi)^2 \right\rangle\right|, \quad \left|\left\langle \frac{\zeta D_n J \cdot h_l}{\sqrt{1+|Eu|^2}},\, (D_l\phi)^2 \right\rangle\right| \leqq CM.$$

Hence, from (4.24) it is enough to prove

$$n\langle JH,\, \zeta D_n u\rangle - \frac{1}{2}\left\langle \frac{\zeta D_n(Jh_n)}{\sqrt{1+|Eu|^2}},\, (D_n u)^2 \right\rangle - \frac{1}{2}\left\langle \frac{\zeta D_n J \cdot h_n}{\sqrt{1+|Eu|^2}},\, (D_n u)^2 \right\rangle \geqq 0. \tag{4.26}$$

Since $D_n u \geqq 0$ on $\{\xi_n = 0\}$, we have

$$D_n u \geqq \sqrt{h_n}\, \frac{(D_n u)^2}{\sqrt{1+|Eu|^2}},$$

from which (4.26) holds if

$$nJ\sqrt{h_n}H - \frac{1}{2}D_n(Jh_n) - \frac{1}{2}D_n J \cdot h_n \geqq 0 \quad \text{on } \{\xi_n = 0\}.$$

Thus we can take d_0 as follows:

$$Jd_0 \geqq \frac{1}{2n}\Big(\frac{1}{\sqrt{h_n}}D_n(Jh_n) + D_n J \cdot \sqrt{h_n}\Big). \tag{4.27}$$

We have finished the proof. □

We give an example of the constant d_0 in (4.27).

EXAMPLE 3. Let us denote by C_R the circumference with its center $(0, -R)$ and with its radius R. Let Γ_1 be an open arc on C_R such that Γ_1 contains the origin (see Figure 3). Let Ω be a bounded domain in the exterior of C_R such that $\partial\Omega \supset \Gamma_1$. We transform Γ_1 into a flat and calculate J and h_2. For this sake we use the polar coordinates transformation $x = \xi_2 \cos \xi_1$, $y = -\xi_2 \sin \xi_1 - R$. Then

$$J = \frac{D(x,y)}{D(\xi_1,\xi_2)} = \xi_2, \quad h_2 = (D_x\xi_2)^2 + (D_y\xi_2)^2 = 1.$$

Hence

$$\text{the right-hand side of (4.27)} = \frac{1}{2},$$

namely we can take $d_0 = 1/2R$.

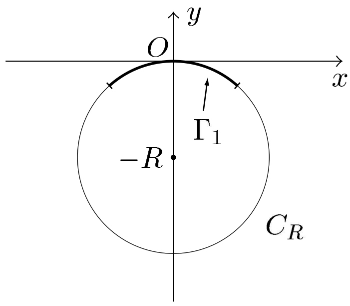

Figure 3.

§5. Proof of our Theorems

Let $\{u_j\}$ be the sequence of solutions in (2.1) and u be the function in (2.2). We recall that (3.5) holds. We denote by the same $\{\nu\}$ any subsequence of $\{j\}$. Before proving our Theorems we prepare the following

Proposition 5.1. *There is a positive sequence $\{\alpha_\nu\}$ with $\alpha_\nu \to 0$ $(\nu \to \infty)$ such that for $1 \leqq i \leqq n$*

$$\frac{D_i u_\nu}{\left(1+|Du_\nu|^2\right)^{\alpha_\nu}} \to D_i u \quad \textit{in } L^1(\Omega') \textit{ as } \nu \to \infty,$$

where Ω' is any subdomain of Ω such that $\partial\Omega' \cap \partial\Omega \subset \Gamma_1$ and $\overline{\Omega}' \cap \Gamma_2 = \phi$.

Proof. By the convergence theorem

$$\frac{D_i u}{\left(1+|Du_\nu|^2\right)^{\alpha_\nu}} \to D_i u \quad \text{in } L^1(\Omega')\ (\nu \to \infty).$$

Hence it is sufficient to prove that

$$\frac{D_i(u_\nu - u)}{\left(1+|Du_\nu|^2\right)^{\alpha_\nu}} \to 0 \quad \text{in } L^1(\Omega')\ (\nu \to \infty). \tag{5.1}$$

From (2.2) we can take a sequence $\{G_k\}$, subdomains of Ω such that $\overline{G}_k \subset \Omega$, $G_k \uparrow \Omega$ $(k \to \infty)$ and

$$D_i u_\nu \rightrightarrows D_i u \quad \text{in } G_k\ (\nu \to \infty).$$

We denote with the same notation any subsequence of $\{u_\nu\}$. Then we may assume that

$$|D_i u_\nu - D_i u| < \frac{1}{\nu} \quad \text{in } G_\nu.$$

Let us take a positive sequence $\{\alpha_\nu\}$ such that $\alpha_\nu \to 0$ $(\nu \to \infty)$ and

$$|\Omega - G_\nu|^{2\alpha_\nu} \to 0 \quad (\nu \to \infty). \tag{5.2}$$

We set

$$\int_{\Omega'} \frac{|D_i(u_\nu - u)|}{\left(1 + |Du_\nu|^2\right)^{\alpha_\nu}}\, dx = \int_{G_\nu} + \int_{\Omega' - G_\nu} \equiv I_\nu + J_\nu, \quad \text{say.}$$

Easily, $I_\nu \to 0$ $(\nu \to \infty)$. And

$$|J_\nu| \leqq \int_{\Omega' - G_\nu} |D_i u_\nu|^{1-2\alpha_\nu}\, dx + \int_{\Omega' - G_\nu} |D_i u|\, dx.$$

Since $u \in W^{1,1}(\Omega)$,

$$\int_{\Omega - G_\nu} |D_i u|\, dx \to 0 \quad (\nu \to \infty).$$

By Hölder's inequality

$$\int_{\Omega' - G_\nu} |D_i u_\nu|^{1-2\alpha_\nu}\, dx \leqq |\Omega - G_\nu|^{2\alpha_\nu} \left(\int_{\Omega'} |Du_\nu|\, dx \right)^{1-2\alpha_\nu}.$$

Hence $J_\nu \to 0$ $(\nu \to \infty)$ from (5.2) and (3.1). Thus we obtain (5.1). This completes the proof. □

Now we prove Theorem 1.

Proof of Theorem 1. Let P be any fixed point on Γ_1 and δ be a sufficiently small positive number. Let ζ be any fixed function in $C_0^\infty(B_\delta(P))$. Since from (3.3)

$$\left| D_i \left(\frac{u_\nu - \phi_\nu}{\left(1 + |Du_\nu|^2\right)^{5/4}} \right) \right| \leqq C \left(1 + \frac{|DD_i u_\nu|}{\left(1 + |Du_\nu|^2\right)^{7/4}} \right) \quad \text{in } B_\delta(P),$$

it follows from Proposition 4.2 that $\left\{\zeta(u_\nu - \phi_\nu)/(1+|Du_\nu|^2)^{5/4}\right\}$ is uniformly bounded in $W_0^{1,2}(\Omega)$. Hence a subsequence of $\left\{\zeta(u_\nu - \phi_\nu)/(1+|Du_\nu|^2)^{5/4}\right\}$

converges weakly to a function w in $W_0^{1,2}(\Omega)$. On the other hand, from (2.2) it converges to $\{\zeta(u-\phi)/(1+|Du|^2)^{5/4}\}$ pointwise in Ω. Therefore by the usual argument

$$w = \frac{\zeta(u-\phi)}{\left(1+|Du|^2\right)^{5/4}} \in W_0^{1,2}(\Omega),$$

which implies that $(u-\phi)/(1+|Du|^2)^{5/4} \in W_0^{1,2}(\Omega;\Gamma_1)$. In Section 2 we have already stated that the boundary condition is satisfied on Γ_2. Thus Theorem 1 holds. □

Proof of Theorem 2. We take the sequence $\{\alpha_\nu\}$ in Proposition 5.1. Let $1 \leqq i \leqq n$. We have

$$D_i\biggl(\frac{u_\nu-\phi_\nu}{\left(1+|Du_\nu|^2\right)^{\alpha_\nu}}\biggr) - \frac{D_i(u_\nu-\phi_\nu)}{\left(1+|Du_\nu|^2\right)^{\alpha_\nu}} = -2\alpha_\nu(u_\nu-\phi_\nu)\frac{Du_\nu\cdot DD_iu_\nu}{\left(1+|Du_\nu|^2\right)^{\alpha_\nu+1}}.$$

Hence from (3.3)

$$\left|D_i\biggl(\frac{u_\nu-\phi_\nu}{\left(1+|Du_\nu|^2\right)^{\alpha_\nu}}\biggr) - \frac{D_i(u_\nu-\phi_\nu)}{\left(1+|Du_\nu|^2\right)^{\alpha_\nu}}\right| \leqq C\alpha_\nu\frac{|DD_iu_\nu|}{\left(1+|Du_\nu|^2\right)^{(2\alpha_\nu+1)/2}}.$$

Let P be any fixed point in Γ_1 and $\delta>0$ be small. Then from Proposition 4.3

$$\int_{B_\delta(P)\cap\Omega_\nu}\frac{|DD_iu_\nu|^2}{\left(1+|Du_\nu|^2\right)^{3/2}}\,dx \leqq C.$$

Accordingly by Schwarz inequality

$$\begin{aligned}&\int_{B_\delta(P)\cap\Omega_\nu}\frac{|DD_iu_\nu|}{\left(1+|Du_\nu|^2\right)^{(2\alpha_\nu+1)/2}}\,dx\\ &\quad\leqq\biggl(\int_{\Omega_\nu}\sqrt{1+|Du_\nu|^2}\,dx\biggr)^{1/2}\biggl(\int_{B_\delta(P)\cap\Omega_\nu}\frac{|DD_iu_\nu|^2}{\left(1+|Du_\nu|^2\right)^{3/2}}\,dx\biggr)^{1/2}\\ &\quad\leqq C\quad\text{(from (3.1))}.\end{aligned}$$

We take any function $\psi\in C_0^\infty(B_\delta(P))$ and denote by (,) the $L^2(\Omega)$-inner product. Then from the above it follows that

$$\text{(5.3)}\quad \biggl(D_i\Bigl(\frac{u_\nu-\phi_\nu}{\left(1+|Du_\nu|^2\right)^{\alpha_\nu}}\Bigr),\,\psi\biggr)-\biggl(\frac{D_i(u_\nu-\phi_\nu)}{\left(1+|Du_\nu|^2\right)^{\alpha_\nu}},\,\psi\biggr)\to 0\quad(\nu\to\infty).$$

On the other hand

$$\Big(D_i\Big(\frac{u_\nu-\phi_\nu}{\big(1+|Du_\nu|^2\big)^{\alpha_\nu}}\Big),\,\psi\Big)=-\Big(\frac{u_\nu-\phi_\nu}{\big(1+|Du_\nu|^2\big)^{\alpha_\nu}},\,D_i\psi\Big)$$

and

$$\frac{D_i\phi_\nu}{\big(1+|Du_\nu|^2\big)^{\alpha_\nu}}\to D_i\phi\quad\text{in } L^1(\Omega\cap\{\operatorname{supp}\psi\})\ (\nu\to\infty).$$

Hence from Proposition 5.1 we have

$$\Big(\frac{D_i(u_\nu-\phi_\nu)}{\big(1+|Du_\nu|^2\big)^{\alpha_\nu}},\,\psi\Big)\to(D_i(u-\phi),\,\psi)\quad(\nu\to\infty).$$

Further it follows from (2.2), (3.3) and the convergence theorem that

$$\frac{u_\nu-\phi_\nu}{\big(1+|Du_\nu|^2\big)^{\alpha_\nu}}\to u-\phi\quad\text{in } L^1(\Omega\cap\{\operatorname{supp}\psi\})\ (\nu\to\infty).$$

Combining the above with (5.3), we obtain

$$(u-\phi,\,D_i\psi)=-(D_i(u-\phi),\,\psi),$$

which means that $u-\phi\in W_0^{1,1}(\Omega;\Gamma_1)$.

Next it is known that $u\in C^0(\Omega\cup\Gamma_2)$ and $u=\phi$ on Γ_2, by the usual method of barrier functions (see e.g., [18]).

Lastly we have

$$\Big|D_i\Big(\frac{u_\nu-\phi_\nu}{\big(1+|Du_\nu|^2\big)^{1/4}}\Big)\Big|\leqq C\Big(\big(1+|Du_\nu|^2\big)^{1/4}+\frac{|DD_iu_\nu|}{\big(1+|Du_\nu|^2\big)^{3/4}}\Big)\ \text{ in } B_\delta(P).$$

Using Proposition 4.3, we proceed in parallel with the proof of Theorem 1. Then the final statement in Theorem 2 is obtained. We complete the proof. □

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