S. Kebekus
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ON THE CLASSIFICATION OF 3-DIMENSIONAL $SL_2(\mathbb{C})$-VARIETIES

STEFAN KEBEKUS

Abstract. In the present work we describe 3-dimensional complex SL_2-varieties where the generic SL_2-orbit is a surface. We apply this result to classify the minimal 3-dimensional projective varieties with Picard-number 1 where a semisimple group acts such that the generic orbits are 2-dimensional.

This is an ingredient of the classification [Keb99] of the 3-dimensional relatively minimal quasihomogeneous varieties where the automorphism group is not solvable.

§1. Introduction

In [Keb99] we give a classification of the 3-dimensional relatively minimal quasihomogeneous projective varieties where the automorphism group is linear algebraic and not solvable. By "relatively minimal" we mean varieties having at most $\mathbb{Q}$-factorial terminal singularities and allowing an extremal contraction of fiber type. These varieties always occur at the end of the minimal model program if one starts with a projective rational quasihomogeneous manifold whose automorphism group is not solvable.

Certain aspects of this project utilize results on non-transitive $SL_2(\mathbb{C})$-actions which in our opinion are of separate interest. We have chosen to present these here as opposed to including them in the midst of the classification work, where the methods are essentially different.

The aim of the first part of this paper is to describe 3-dimensional complex SL_2-varieties where the generic SL_2-orbit is a surface. More precisely, we give elementary criteria for the fibers of the categorical quotient to be irreducible or normal and describe neighborhoods of reduced fibers (see Proposition 3.1). We reduce to this case by using concretely constructed equivariant Galois coverings which are étale in codimension one. Under certain restrictions on the isotropy group, a stronger classification is known —

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see [Arz98].

In the main part of the paper we apply these results to yield the following ingredient of the classification in [Keb99].

Theorem 1.1. *Let X be a $\mathbb{Q}$-factorial projective 3-dimensional variety with Picard-number $\rho(X) = 1$ having at most terminal singularities. Assume that a semisimple linear algebraic group S acts algebraically on X such that generic S-orbits are 2-dimensional. Then X is isomorphic to to the smooth 3-dimensional quadric or to one of the (weighted) projective spaces $\mathbb{P}_3$, $\mathbb{P}_{(1,1,1,2)}$ or $\mathbb{P}_{(1,1,2,3)}$.*

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§2. On the normality of fibers of the categorical quotient

Recall that for an affine variety the quotient is defined as the spectrum of the ring of invariant functions. The following are the results of this section:

Proposition 2.1. *Let X be an irreducible complex affine 3-dimensional normal SL_2-variety. Then all fibers of the categorical quotient map $q : X \to Y$ are irreducible. If X is additionally Cohen-Macaulay, then a q-fiber is normal if it is reduced.*

Under additional assumptions on the singularities, the claim is true for non-reduced fibers as well.

Proposition 2.2. *In the setting of Proposition 2.1 assume additionally that X has at most canonical singularities. Then every fiber of the categorical quotient is normal with its reduced structure.*

Before proceeding with the proofs we recall two elementary facts: First, the only normal affine complex SL_2-surfaces with non-trivial action are

the smooth affine quadric $\mathbb{Q}_2^a$: this space is SL_2-homogeneous. The isotropy group of a point is a torus.

$\mathbb{P}_2$ minus a quadric curve: this is a quotient of $\mathbb{Q}_2^a$ by $\mathbb{Z}_2$. We denote it by $\mathbb{Q}_2^a/\mathbb{Z}_2$. The isotropy group is the normalizer of a torus.

the affine cone over a rational normal curve: we denote this by $\mathbb{S}_n^a$, where n is the degree of the curve. The isotropy group is generated by a unipotent part and a cyclic group, isomorphic to $\mathbb{Z}_n$. This space contains an open SL_2-orbit and an SL_2-fixed point.

See [Huc86] for a more detailed description.

Second, if X is a 3-dimensional SL_2-variety with non-trivial action and $D_1 \subset X$ is a divisor, then SL_2 acts non-trivially on D_1. This follows directly from a linearization argument; see [HO80, I. 1.5] for matters concerning linearization. In particular, if X is affine and D_2 is another divisor, then $D_1 \cap D_2$ must be a single point.

Proof of Proposition 2.1. Assume without loss of generality that $\dim Y = 1$, for the proposition is trivial otherwise. Since all q-fibers are connected, as a first step we rule out the possibility that there is a point $y \in Y$ such that $q^{-1}(y)$ is connected and not irreducible. If this was the case, then the irreducible components of $q^{-1}(y)$ can only meet in the unique SL_2-fixed point in $q^{-1}(y)$, i.e., $q^{-1}(y)$ is not connected in codimension one. On the other hand, Hartshorne's connectedness theorem states that X is connected in dimension 2 (see [Eis95, Thm. 18.12 and the preceding discussion]). Now Y is normal, hence smooth, so that $q^{-1}(y)$ is Cartier. In this situation Grothendieck's connectedness theorem shows that $q^{-1}(y)$ must be connected in dimension 1 (see [Gro62, Exp. XIII]), a contradiction.

If X is Cohen-Macaulay, then every q-fiber automatically satisfies Serre's condition S_2 (see [Rei87]). If it is reduced, its singular set is either the unique SL_2-fixed point or empty. The normality follows directly from Serre's criterion. ∎

Proof of Proposition 2.2. Again it is sufficient to consider the case that $\dim Y = 1$. If $y \in Y$ is a point such that $q^{-1}(y)$ has multiplicity $m > 1$, let $\pi : \tilde{X} = \operatorname{Spec} R \to X$ be the associated cyclic cover: set $D := (q^{-1}(y))_{red}$ and $R := \bigoplus_{i=0}^{m-1} \mathcal{O}_X(-iD)$. The map π is an SL_2-equivariant cyclic cover with group $\mathbb{Z}_m$, branched only over the unique SL_2-fixed point in D, if at all. This has two consequences: first, [Rei80, Prop. 1.7] applies, showing that $\tilde{X}$ has canonical singularities, so that $\tilde{X}$ is Cohen-Macaulay and $\tilde{D} = (\pi^{-1}(D))_{red}$ is normal. Secondly, we claim that the induced map $\pi : \tilde{D} \to X$ is just the quotient by the action of $\mathbb{Z}_m$. If this claim holds true, D must also be normal and we are done.

In order to show the claim, recall that by construction $\tilde{D}$ is defined by a function f such that if $\xi \in \mathbb{Z}_m$ is a primitive element, then $f \circ \xi = \xi f$. Because of the universal property of the categorical quotient we have a map $\pi' : \tilde{D}/\mathbb{Z}_m = \operatorname{Spec}(R/f)^{\mathbb{Z}_m} \to X = \operatorname{Spec} R^{\mathbb{Z}_m}$. We show that π' is a closed immersion, i.e., that every element in $(R/f)^{\mathbb{Z}_m}$ can be written as $r' + fR$, where $r' \in R^{\mathbb{Z}_m}$. For this, note that if $r + fR \in (R/f)^{\mathbb{Z}_m}$, then $r \circ \xi - r \in fR$. In particular

$$r + fR = \underbrace{\frac{1}{m} \sum_{\xi \in \mathbb{Z}_m} r \circ \xi}_{=:r'} + fR.$$

□

There exists a preprint of I. V. Arzhantsev where, using the techniques of [LV83], a proof of Proposition 2.2 is indicated for arbitrary normal singularities.

§3. Neighborhoods of fibers

Now we consider the neighborhood of reduced fibers.

Proposition 3.1. *In the setting of Proposition* 2.1*, if Y is a curve and $y \in Y$ is a point such that $q^{-1}(y)$ is reduced, then there exists a Zariski-open neighborhood Δ of y such that $q^{-1}(\Delta)$ is equivariantly isomorphic to one of the following*:

- *a product $\mathbb{S}_n^a \times \Delta$ where SL_2 acts on $\mathbb{S}_n^a$ only*
- $\{((x, y, z), \delta) \in \mathbb{C}^3 \times \Delta \mid 4xz - y^2 = P(\delta)\}$*, where $P \in \mathcal{O}(\Delta)$, having zeros only at y and SL_2 acts on $\mathbb{C}^3$ via the* 3*-dimensional irreducible representation.*
- *a quotient of the latter by $\mathbb{Z}_2$, acting with weights $(1, 1, 1)$ on $\mathbb{C}^3$ and trivially on Δ.*

The proof follows from two technical considerations. Recall from [Kra85, II. 2.4] that there is an equivariant embedding $\iota : X \to \bigoplus V_{k_i}$, where the V_{k_i} are irreducible SL_2-representation spaces.

Lemma 3.2. *There exists a $j \in \mathbb{N}$ such that the projection $\pi : \bigoplus V_{k_i} \to V_{k_j}$ is a closed embedding if restricted to $q^{-1}(y)$.*

Proof. We consider the possibilities for the central fiber separately:

if $q^{-1}(y) \cong \mathbb{Q}_2^a/\mathbb{Z}_2$: then every non-trivial equivariant map is a closed embedding because the isotropy of $\mathbb{Q}_2^a/\mathbb{Z}_2$ is maximal.

if $q^{-1}(y) \cong \mathbb{Q}_2^a$: the only possible images of an SL_2-equivariant morphism which is not an embedding are $\mathbb{Q}_2^a/\mathbb{Z}_2$ and $\{0\}$. Both have normalizers of tori in their isotropy groups, but $\mathbb{Q}_2^a$ has not. Thus, there must be a projection with image $\mathbb{Q}_2^a$. This must be an embedding.

if $q^{-1}(y) \cong \mathbb{S}_n^a$: one has to rule out that all projections map $q^{-1}(y)$ to $\{0\}$ or to $\mathbb{S}_{kn}^a$, $k > 1$, this being the only possible images. Assume to the contrary and let $U < SL_2$ be a unipotent subgroup. Its fixed point set is a line C, isomorphic to $\mathbb{C}$, and all projections map C to $\{0\}$ or are branched covers, ramified at zero. Thus, the rank of the Jacobian of $\iota|_C$ drops at zero — a contradiction to ι being an embedding.

∎

Having embedded the central fibers, we show that the restriction to a neighboring q-fiber is injective as well.

LEMMA 3.3. *There exists a Zariski-open neighborhood U of y such that for all $\eta \in U$ the restriction of π to the q-fiber $X_\eta = q^{-1}(\eta)$ is a closed embedding.*

Proof. Choose U to be a maximal neighborhood of $y \in Y$ such that $\pi(X_\eta) \neq 0$ for all $\eta \in Y$ and such that all q-fibers over $U \backslash \{y\}$ are isomorphic. By the remarks above, this is always possible. Again we perform a case-by-case check:

$q^{-1}(y) \cong \mathbb{S}_1^a$: in this case V_{k_j} must be $\mathbb{C}^2$. As $\pi(X_\eta) \neq \{0\}$, we have $\pi(X_\eta) \cong \mathbb{S}_1^a$ and X_η must be isomorphic to $\mathbb{S}_1^a$ itself, there being no SL_2-equivariant cover.

$q^{-1}(y) \cong \mathbb{S}_2^a$: here V_{k_j} is the irreducible 3-dimensional representation space. The only SL_2-invariant divisors in here are $\mathbb{S}_2^a$ and smooth quadrics. Arguing as above, one must show that the generic q-fiber X_η is not isomorphic to a cover of $\mathbb{S}_2^a$ or $\mathbb{Q}_2^a$, i.e., $X_\eta \ncong \mathbb{S}_1^a$. If this was the case, then linearize the center Z of SL_2 at a smooth point of $q^{-1}(y)$. This gives an analytic curve germ $C \subset X$, invariant under Z and intersecting $q^{-1}(y)$ transversally in a single point. As Z is not contained in the isotropy group of any point in X_η other than 0, C must intersect the neighboring fiber twice. This is a contradiction to $q^{-1}(y)$ being reduced.

$q^{-1}(y) \cong \mathbb{S}^a_n$ where $n = 3$ or $n > 4$: a similar linearization argument as above, using a $\mathbb{Z}_n$ from the isotropy group of a generic point in $q^{-1}(y)$, shows that the generic X_η must contain a $\mathbb{Z}_n$-fixed curve. Classification yields that $X_\eta \cong \mathbb{S}^a_{kn}$ for one $k \in \mathbb{N}$. But k must be 1: every X_η contains a curve which is $\mathbb{Z}_{kn}$-fixed and $q^{-1}(y)$ must, too.

$q^{-1}(y) \cong \mathbb{S}^a_4$: here V_{k_j} is the irreducible 5-dimensional representation space where the only SL_2-invariant surfaces are $\mathbb{S}^a_4$ or are isomorphic to $\mathbb{Q}^a_2/\mathbb{Z}_2$. A linearization argument similar to the one used above rules out that $X_\eta \cong \mathbb{Q}^a_2$ or a cover of $\mathbb{S}^a_4$: assume to the contrary and choose a smooth point $x \in q^{-1}(y)$. Recall that the isotropy group $(SL_2)_x$ contains a group $Z \cong \mathbb{Z}_4$. Linearize the Z-action at x and follow the argumentation of the case where $q^{-1} \cong \mathbb{S}^a_2$ verbatim.

$q^{-1}(y) \cong \mathbb{Q}^a_2$: here V_{k_j} contains two types of 2-dimensional SL_2-orbits: $\mathbb{Q}^a_2$ and $\mathbb{S}^a_{k_j}$. We know that $\pi(X_\eta) \cong \mathbb{Q}^a_2$, as otherwise $\pi(X_\eta)$ must contain a U-pointwise fixed curve and $\pi(q^{-1}(y))$ must, too. A contradiction. Again $\pi|_{X_\eta}$ must be injective as there is no SL_2-equivariant cover of $\mathbb{Q}^a_2$.

$q^{-1}(y) \cong \mathbb{Q}^a_2/\mathbb{Z}_2$: apply the linearization argument involving a generic isotropy group, i.e., the normalizer of a torus to see that the neighboring q-fibers cannot be isomorphic to $\mathbb{Q}_2$. Now argue as in the last case.

□

With this information we start the

Proof of Proposition 3.1. Choose $\Delta \subset Y$ as in Lemma 3.3. Then the map $(\pi \circ \iota) \times q : X \to V_{k_j} \times Y$ is injective if restricted to $q^{-1}(\Delta)$.

Recall the classical fact that every irreducible representation space of SL_2 contains a unique SL_2-orbit whose closure is isomorphic to $\mathbb{S}^a_n$ — see e.g. [MU83, Lem. 1.4] for a modern proof. Thus the claim of Proposition 3.1 holds if all q-fibers are isomorphic to $\mathbb{S}^a_n$.

If $q^{-1}(y) \cong \mathbb{S}^a_2$, and the generic fiber is a smooth quadric, then $q^{-1}(\Delta)$ can be equivariantly embedded into $V_2 \times \Delta$. Fix a torus $T < SL_2$ and note that its fixed point set V_2^T is a 1-dimensional linear subspace. If y

is a linear coordinate on V_2^T, then the intersection $X^T := X \cap (V_2^T \times \Delta)$ is given by $\{-y^2 = P(\delta)\}$ where $P \in \mathcal{O}(\Delta)$. This is because X^T is 2:1 over Δ and invariant under multiplication of V_2 with -1. By choice of Δ, P has no zero on $\Delta \setminus \{0\}$. Recall that we can find two additional linear coordinates (x, z) on V_2 such that (x, y, z) is a coordinate system where every SL_2-invariant surface is given as $\{(x, y, z) \in \mathbb{C}^3 \mid 4xz - y^2 = \text{const.}\}$. Since X is uniquely determined by X^T as $X = \overline{SL_2.X^T}$ shows that X is given by $\{((x, y, z), \delta) \in \mathbb{C}^3 \times \Delta \mid 4xz - y^2 = P(\delta)\}$. Thus, the claim is shown as well.

If all fibers are isomorphic to $\mathbb{Q}_2^a$ and $V_{k_j} \not\cong V_2$, then argue similarly: $V_{k_j}^T$ is 1-dimensional and $X^T = X \cap (V_{k_j}^T \times \Delta)$ is given by $\{-y^2 = P(\delta)\}$ where $P \in \mathcal{O}^*(\Delta)$. We show that X is isomorphic to $\{((x, y, z), \delta) \in \mathbb{C}^3 \times \Delta \mid 4xz - y^2 = P(\delta)\} =: X_2 \subset V_2 \times \Delta$. A linear identification of V_2^T and $V_{k_j}^T$ yields an isomorphism between X^T and X_2^T. Let $\Gamma^T \subset (V_2^T \times \Delta) \times (V_{k_j}^T \times \Delta) \subset (V_2 \times \Delta) \times (V_{k_j} \times \Delta)$ be the graph and set $\Gamma := SL_2.\Gamma^T \subset (V_2 \times \Delta) \times (V_{k_j} \times \Delta)$. Now X^T and X_2^T both having isotropy group T at any point implies that Γ is the graph of a bijective morphism, i.e., an isomorphism between the (normal) varieties X and X_2.

If $q^{-1}(y) \cong \mathbb{S}_4^a$ or all q-fibers are isomorphic to $\mathbb{Q}_2^a/\mathbb{Z}_2$ one uses the same line of argumentation with the only difference that the SL_2-invariant surfaces in the 5-dimensional representation space are given by the ideal generated by

$$\begin{aligned} &3d^2 - 8ce + 4\delta e, && cd - 6be + \delta d, \\ &3bd - 48ae + 2\delta c + 2\delta^2, && c^2 - 36ae + 2\delta c + \delta^2, \\ &bc - 6ad + \delta b, && 3b^2 - 8ac + 4\delta a. \end{aligned}$$

This variety is a quotient of $\{4xz - y^2 = \delta\}$ by $\mathbb{Z}_2$, where the SL_2-equivariant quotient map is given by $(x, y, z) \mapsto (x^2, 2xy, 2xz + y^2, 2yz, z^2)$. □

The next lemma covers a special case which we will need to consider later.

PROPOSITION 3.4. *In the setting of Proposition* 3.1, *if* $Y \cong \mathbb{C}$ *and all fibers over* $Y \setminus \{0\}$ *are isomorphic, then* X *is equivariantly isomorphic to*

- $\mathbb{S}_n^a \times \mathbb{C}$ *where* SL_2 *acts on* $\mathbb{S}_n^a$ *only*
- $X_k := \{((x, y, z), \delta) \in \mathbb{C}^3 \times \mathbb{C} \mid 4xz - y^2 = \delta^k\}$ *where* $k \in \mathbb{N}$ *and* SL_2 *acts on* $\mathbb{C}^3$ *via the* 3*-dimensional irreducible representation.*

- *a quotient of the latter by $\mathbb{Z}_2$, acting with weights $(1,1,1)$ on $\mathbb{C}^3$ and trivially on the base.*

Proof. If all q-fibers are isomorphic to $\mathbb{S}_n^a$, then Proposition 3.1 shows the local triviality. Note that the only automorphisms of $\mathbb{S}_n^a$ commuting with the SL_2-action are in $\mathbb{C}^*$. But $H^1(\mathbb{C}, \mathcal{O}^*)$ is trivial so that the local trivializations glue together to give a global one.

If the generic fiber is $\mathbb{Q}_2^a$, then employ the same methods as in the proof of Proposition 3.1: embed $X \to \bigoplus V_{k_i} \times \mathbb{C}$ and assume that $(\pi_0 \times \mathrm{Id}) : \bigoplus V_{k_i} \times \mathbb{C} \to V_{k_0} \times \mathbb{C}$ is an embedding, if restricted to q^{-1} of a neighborhood of y. Choose a torus T and let $X^T \subset V_{k_0}^T$ be the T-fixed point set. Assuming without loss of generality that $y = 0$, $(\pi_0 \times \mathrm{Id})(X^T)$ is given as $\{-y^2 = c \cdot \delta^k \cdot \prod (y_j - \delta)^{m_j}\}$ where $y_j \neq 0$, δ is to coordinate on $Y \cong \mathbb{C}$ and $c \neq 0$ is a constant. We know that X^T is locally (analytically) reducible over each of the y_j. Its $(\pi_0 \times \mathrm{Id})$-image is, as well. Thus, the m_j are even.

Let $U_0 \subset \mathbb{C}$ be the maximal set such that all fibers are isomorphic to $\mathbb{Q}_2^a$. To construct an isomorphism $X_k \to X$ over U_0, it is necessary to find an isomorphism between $X^T \cap q^{-1}(U_0)$ and $X_k^T := \{-y^2 = \delta^k \mid \delta \in U_0\}$ and then apply the construction from the proof of Proposition 3.1, involving the graph Γ. Note that $X^T \cap q^{-1}(U_0)$ and X_k^T are both smooth and have a birational morphism onto $(\pi_0 \times \mathrm{Id})(X^T)$, the latter being given by

$$\begin{aligned} X_k^T &\longrightarrow (\pi_0 \times \mathrm{Id})(X^T) \\ (y, \delta) &\longmapsto \left(y \prod (y_j - \delta)^{m_j/2} \sqrt{c}, \delta \right). \end{aligned}$$

Thus, they must be isomorphic. Now the construction gives an isomorphism over U_0.

If $U_0 \subset \mathbb{C}$ is not the whole of $\mathbb{C}$, then set $U_1 := \mathbb{C} \setminus \{y_1, \dots, y_k\}$. Recall that V_{k_0} is necessary 3-dimensional, equip it with coordinates x, y and z and set

$$\begin{aligned} \left\{4xz - y^2 = \delta^k\right\} &\longrightarrow \left\{4xz - y^2 = \delta^k \cdot \prod (y_j - \lambda)^{m_j}\right\} \\ ((x,y,z),\delta) &\longmapsto \left(\prod (y_j - \delta)^{m_j/2} (x,y,z), \delta \right) \end{aligned}$$

where $((x,y,z),\delta)$ are coordinates on $\mathbb{C}^3 \times \mathbb{C}$.

We have to show that the two local isomorphisms over U_0 and U_1 agree. Note that the only automorphisms of $\mathbb{Q}_2^a$ commuting with the SL_2-action are in $\mathbb{Z}_2$. Now $H^1(\mathbb{C}, \mathbb{Z}_2)$ being trivial shows hat after multiplying one of the local isomorphisms with (-1), if necessary, we can always glue.

Again the analogous construction works if the generic fiber is isomorphic to $\mathbb{Q}_2^a/\mathbb{Z}_2$. □

Remark 3.5. Propositions 3.1 and 3.4 could also be proved using elementary deformation theory, see [Pin74].

Now we describe certain quasi-projective varieties which will play an important role in the next chapter. For this, a categorical quotient of a quasi-projective variety is an invariant affine surjective morphism $q : X \to Y$ which is a categorical quotient on an affine cover of the base.

PROPOSITION 3.6. *Let X be a quasi-projective normal complex SL_2-variety with at most terminal singularities and categorical quotient $q : X \to \mathbb{P}_1$. If every q-fiber over $\mathbb{C}^* \subset \mathbb{P}_1$ is isomorphic to $\mathbb{C}^2$, then the singularities of X are of type $\frac{1}{n}(1,1,-1)$ and $\frac{1}{m}(1,1,-1)$ (i.e., are locally isomorphic to $\mathbb{C}^3/\mathbb{Z}_n$, where $\mathbb{Z}_n$ acts with weights $(1,1,-1)$) and X is toric. Here n and m are the multiplicities of the exceptional q-fibers.*

Proof. Set $X^0 := q^{-1}(\mathbb{C})$, let n be the multiplicity of $q^{-1}(0)$ and consider the n-th root fibration:

$$\begin{array}{ccc} \tilde{X}^0 & \xrightarrow{\gamma} & X^0 \\ {\scriptstyle \tilde{q}}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbb{C} & \xrightarrow{n:1} & \mathbb{C} \end{array}$$

By construction, X^0 is a cyclic quotient of $\tilde{X}^0$ by the group $\mathbb{Z}_n$ and there is a natural SL_2-action on $\tilde{X}^0$ such that γ is equivariant. Using Proposition 3.4, identify $\tilde{X}^0$ with $\mathbb{C}^2 \times \mathbb{C}$, where SL_2 acts only on the first factor and take coordinates $((\tilde{x}, \tilde{y}), \tilde{\lambda})$. Then, if $\xi \in \mathbb{Z}_n$ is a primitive root, we may write without loss of generality $\xi((\tilde{x}, \tilde{y}), \tilde{\lambda}) = (a(\xi, \tilde{\lambda})(\tilde{x}, \tilde{y}),\ \xi \cdot \tilde{\lambda})$, where $a(\xi, \tilde{\lambda}) \in \mathrm{Aut}(\mathbb{C}^2)$. Note that, since the SL_2-action preserves γ-fibers, ξ must commute with SL_2. Thus, $a(\xi, \tilde{\lambda}) \in \mathrm{Aut}(\mathbb{C}^2)$ is a homothety. Furthermore, since there is no non-constant algebraic morphism from $\mathbb{C}$ to $\mathbb{C}^*$, $a(\xi, \tilde{\lambda})$ does not depend on $\tilde{\lambda}$. We may thus view $\mathbb{Z}_n$ as acting on $\mathbb{C}^2 \times \mathbb{C}$ with weights $(a, a, 1)$. In order to show that X has singularities of type

$\frac{1}{n}(1,1,-1) = \frac{1}{n}(-1,-1,1)$, i.e., that $a = 1$. Note that a and n must be coprime; this is because γ is étale in codimension one. In this setting, the classification theory of terminal singularities asserts that only $a = -1$ is possible if the quotient is supposed to be terminal, see [Rei87, Sect. 5.1]. The same argument applies to $X^\infty := q^{-1}(\mathbb{P}_1 \setminus \{0\})$.

Now we will show that X is toric. Let $(\mathbb{C}^*)^3$ act on $\tilde{X}^0$ as

$$\begin{array}{ccc} (\mathbb{C}^*)^3 \times \tilde{X}^0 & \longrightarrow & \tilde{X}^0 \\ (r,s,t)((\tilde{x},\tilde{y}),\tilde{\lambda}) & \longmapsto & ((rt^{-m}\tilde{x}, st^{-m}\tilde{y}), t^m\tilde{\lambda}) \end{array}$$

Note that this action commutes with the action of the Galois group $\mathbb{Z}_n$ so that we have an induced action on X^0. Over $\mathbb{C}^* \subset \mathbb{C}$ this action has a particularly simple form: set $\tilde{X}^* := \tilde{q}^{-1}(\mathbb{C}^*)$, $X^* := q^{-1}(\mathbb{C}^*) \cong \mathbb{C}^2 \times \mathbb{C}^*$ and note that the map γ is given as

$$\begin{array}{cccc} \gamma : & \tilde{X}^* & \longrightarrow & X^* \\ & ((\tilde{x},\tilde{y}),\tilde{\lambda}) & \longmapsto & ((\tilde{x}\tilde{\lambda}, \tilde{y}\tilde{\lambda}), \tilde{\lambda}^n) = ((x,y),\lambda) \end{array}$$

Thus, the induced $(\mathbb{C}^*)^3$-action on X^* can be written as

$$\begin{array}{ccc} (\mathbb{C}^*)^3 \times X^* & \longrightarrow & X^* \\ (r,s,t)((x,y),\lambda) & \longmapsto & ((rx, sy), t^{nm}\lambda) \end{array}$$

A similar argument shows that we can find an action of $(\mathbb{C}^*)^3$ on X^∞ which agrees on X^* with the one defined above. Thus, X is toric. ∎

§4. An application

Our main goal here is to prove Theorem 1.1, but first we consider the example which is actually realized.

Example 4.1. Let X be the weighted projective space $\mathbb{P}_{(1,1,2,3)}$. By [Ful93, p. 35], identify X with the toric variety X whose fan is constructed from the vectors e_1, e_2, e_3 and $v = (-1,-2,-3) \in \mathbb{Z}^3$, i.e., whose cones are generated by any three of these vectors. We claim that X has $\mathbb{Q}$-factorial terminal singularities and Picard-number $\rho(X) = 1$. Furthermore, SL_2 acts on X such that the generic orbit is 2-dimensional.

***X* has $\mathbb{Q}$-factorial terminal singularities:** The cones generated by (e_1, e_2, e_3) and (e_2, e_3, v) describe smooth varieties. The cone generated by (e_1, e_2, v) can be brought into the form $(e_1, e_2, (-1, -2, 3))$

using the matrix $\mathrm{Diag}(1,1,-1) \in GL(2,\mathbb{Z})$. The latter cone is known (see [Ful93, p. 35]) to describe the singularity $\mathbb{C}^3/\mathbb{Z}_3$, where $\mathbb{Z}_3$ acts with weights $(1,1,-1)$. Analogously,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix} (e_1, e_3, v) = (e_1, e_3, (-1,2,-1)).$$

This describes the singularity $\mathbb{C}^3/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts with weights $(1,1,1)$. Both singularities are terminal — compare [Rei87, Thm. on p. 379].

X has Picard-number one: By [Ful93, p. 64f], $\mathrm{Pic}(X) = \mathbb{Z}$.

SL_2-action: If $[x:y:z:w]$ are weighted homogeneous coordinates associated with the weights $(1,1,2,3)$, let SL_2 act on x and y via the 2-dimensional representation.

Since the proof of Theorem 1.1 is rather long, we subdivide it into a number of lemmata. For the rest of this paper, we use the notation of Theorem 1.1 without mentioning it further.

LEMMA 4.2. *In the situation of Theorem* 1.1, $S \cong SL_2$.

Proof. If it was not, then the generic S-orbit is a homogeneous divisor. Two of them don't intersect, a contradiction to $\rho(X) = 1$. ∎

LEMMA 4.3. *Every irreducible S-invariant divisor $D_\alpha \subset X$ is S-quasi-homogeneous and its normalization is isomorphic to either*

1. *the projective cone over the rational normal curve of degree n, $\mathbb{S}_n$, where S acts with a fixed point (this includes $\mathbb{P}_2$),*
2. *$\mathbb{P}_2$, where S acts via the* 3*-dimensional irreducible SL_2-representation, or*
3. *the Hirzebruch surface Σ_0, where S acts diagonally*

There exists a curve $C \subset X$, $C \cong \mathbb{P}_1$ such that (set-theoretically) $C = D_\alpha \cap D_\beta$ for all S-invariant divisors D_α and D_β. Furthermore, every irreducible S-invariant divisor is locally (analytically) irreducible at any point of C.

Proof. We have remarked at the very beginning that D_α cannot be pointwise S-fixed. Suppose that the generic S-orbit in D_α was 1-dimensional. Then D_β intersects D_α in finitely many orbits and therefore has empty intersection with a generic S-orbit in D_α, a contradiction to $\rho(X) = 1$. Consequently, D_α is S-quasihomogeneous and if $\eta : \tilde{D}_\alpha \to D_\alpha$ is the normalization, then classification (see [Huc86]) shows that $\tilde{D}_\alpha$ must be isomorphic a variety in the list, or to Σ_n, $n > 0$ with S stabilizing two sections.

Due to $\rho(X) = 1$ there is a number $k \in \mathbb{N}$ such that kD_β is CARTIER and ample. In particular, $\eta^*(kD_\beta)$ is an effective ample divisor with S-invariant support. This is possible in all cases save Σ_n, $n > 0$. In the allowed cases, there is a unique irreducible S-invariant curve $C \subset \tilde{D}_\alpha$ which yields the assertion. □

One of the key points in the proof of Theorem 1.1 is the following local description of the D_α in the neighborhood of C.

LEMMA 4.4. *Let $x \in C$ and $T < S$ be a torus fixing x. Then there exists a 2-dimensional T-representation space E with positive weights, a neighborhood V of $0 \in E$, a neighborhood Δ of $0 \in \mathbb{C}$ and an immersion $\phi : V \times \Delta \to U \subset X$ with the following properties*:

1. $\phi^{-1}(C) = \{0\} \times \Delta$
2. *$D_\alpha \subset X$ is an S-invariant divisor if and only if there exists a T-invariant curve $N \subset E$ such that $\phi^{-1}(D_\alpha) = N \times \Delta$.*

Proof. Let $U \subset T_xX$ be a sufficiently small neighborhood of 0 and let $\lambda : U \to X$ be a linearization of the T-action. As C is smooth, the tangent space $T_xC \subset T_xX$ coincides with one of the weight spaces. Take two weight vectors v_1 and $v_2 \in T_xX$ which, together with T_xC span the whole space T_xX. Let E be the space spanned by v_1 and v_2.

As a first step, we claim that after replacing T be T^{-1}, if necessary, all weights of the T-action on T_xX are positive. In order to show this it is sufficient to show that for generic $y \in U$ the T-orbit Ty contains x in the closure. This, however, is true because $\lambda(y)$ is contained in an S-invariant divisor and $x \in \overline{Ty}$ by the classification of Lemma 4.3.

In order to construct the map ϕ, note that $\lambda|_V$ is immersive. The image of the tangential map $T(\lambda|_V)$ is transversal to T_xC. Let $H < S$ be a unipotent one-parameter group not fixing x. The associated vector field, evaluated at x is contained in T_xC so that the map

$$\begin{array}{rccc} \phi: & V\times H & \longrightarrow & X \\ & (v,h) & \longmapsto & h\cdot\lambda(v) \end{array}$$

has maximal rank at $(0,0)$. Thus, ϕ is invertible in a small neighborhood. Property (1) holds by construction.

In order to show property (2) it is sufficient to consider irreducible D_α. Claim that D_α is S-invariant if and only if there exists a point $y\in D_\alpha\cap U$ such that $D_\alpha=\overline{H.\overline{T.y}}$. Indeed, if D_α is S-invariant, then it contains C, and therefore also a point $y\in D_\alpha\cap\lambda(V)\setminus C$ and $\overline{T.y}$ is necessarily a curve containing x in the closure. Therefore $\overline{T.y}$ is *not* H-invariant and $\overline{H.\overline{T.y}}$ is an irreducible component of D_α, hence equal to D_α. This shows already that if we set $N:=\overline{T.\phi^{-1}(y)}$, then $\phi(N\times\Delta)$ is contained in D_α. If $\phi^{-1}(D_\alpha)\neq N\times\Delta$, then it contains another irreducible component, a contradiction to the local irreducibility of D_α. □

We utilize the local description to draw conclusions concerning the global configuration of the divisors D_α.

Corollary 4.5. *There are at least two different S-invariant divisors D_0 and D_∞ in X which are smooth along C. Unless X is isomorphic to the smooth 3-dimensional quadric $\mathbb{Q}_3$, to $\mathbb{P}_3$ or to $\mathbb{P}_{(1,1,1,2)}$, the normalizations are isomorphic to $\tilde{D}_0\cong\mathbb{S}_n$ and $\tilde{D}_\infty\cong\mathbb{S}_m$ with $n,\,m>1$. If $\tilde{D}_\alpha$ is the normalization of a generic S-invariant divisor, then either*

1. *m and n are coprime and $\tilde{D}_\alpha\cong\mathbb{P}_2$ where S acts with a fixed point, or*
2. *m and n are even, $m/2$, $n/2$ are coprime and $\tilde{D}_\alpha\cong\Sigma_0$, or*
3. *m and n are divisible by four, $m/4$, $n/4$ are coprime and $\tilde{D}_\alpha\cong\mathbb{P}_2$ where S acts via the 3-dimensional irreducible representation.*

Proof. Using the notation from the preceding Lemma 4.4, let n and m denote the weights of the T-action on E. Taking N to be one of the weight spaces immediately yields two divisors D_0 and D_∞ which are smooth along C.

Note that by Lemma 4.3 two S-invariant divisors intersect in C only so that the generic S-invariant divisor D_α does not meet the singular set of X. Use the standard argument linearizing the S-action at a fixed point to exclude the possibility that $D_\alpha \cong \mathbb{S}_k$ where $k > 1$. Thus, D_α is smooth away from C.

Results of Bădescu state that X is isomorphic to a cone or to $\mathbb{Q}_3$ if there is a Cartier divisor in X which is isomorphic to $\mathbb{P}_2$ or Σ_0; see [Bǎd82, Thms. 1 and 5] for the cases that X is smooth or that $D_\alpha \cong \mathbb{P}_2$ and [Bǎd84, Thm. 3] for the remaining case. Remark that if X is a cone then the classification from [Mor82, Thm. 3.3 and Cor. 3.4] yields that $X \cong \mathbb{P}_3$ or $\mathbb{P}_{(1,1,1,2)}$ if X is assumed to have $\mathbb{Q}$-factorial and terminal singularities; note that a cone over Σ_0 is never $\mathbb{Q}$-factorial as there are Weyl-divisors intersecting in a single point.

Thus, excluding the cases discussed in the preceding paragraph, we can assume the following:

First, the divisors D_α must be singular along C. By the local description of Lemma 4.4, the generic T-stable curve in E must be singular. Equivalently, say that n is not divisible by m and vice versa. Furthermore, if a number k divides n and m, then D_α contains a curve which is pointwise fixed under to action of the group $\mathbb{Z}_k \subset T$. However, the classification of Lemma 4.3 and the knowledge of the SL_2-actions show that, depending on the isomorphism class of D_α, only $k = 2$ and $k = 4$ are possible. This gives conditions (1)–(3).

Secondly, the normalizations of D_0 and D_∞ are isomorphic to $\mathbb{S}_\bullet$, since otherwise $D_\bullet \setminus C$ would be homogeneous, would not intersect the (finite) singular set of X and would thus be Cartier. Again the fact the D_0 and D_∞ contain $\mathbb{Z}_n$ (resp. $\mathbb{Z}_m$)-pointwise fixed curves and the knowledge of the SL_2-action yields that $D_0 \cong \mathbb{S}_n$ and $D_\infty \cong \mathbb{S}_m$. ∎

Note that the set of semi-stable points with respect to the unique lifting of the SL_2-action to $\mathcal{O}(D_\alpha)$ is $X \setminus C$. Let $q : X \setminus C \to Y$ denote the resulting quotient in the sense of geometric invariant theory.

Corollary 4.6. *We have* $Y \cong \mathbb{P}_1$ *and either* $X \cong \mathbb{Q}_3$, $\mathbb{P}_3$ *or* $\mathbb{P}_{(1,1,1,2)}$ *or and there are points* $0, \infty \in \mathbb{P}_1$ *such that* $q^{-1}(0) = n'D_0$, $q^{-1}(\infty) = m'D_\infty$ *and all other* q*-fibers are reduced. Here* $n' = n$, $n/2$ *or* $n/4$, *according to the cases of Corollary* 4.5; m' *similarly.*

Proof. The description of Lemma 4.4 guarantees that the quotient map extends to a rational map $X \dashrightarrow Y$ which becomes regular if we perform a weighted blow-up of C with weights n and m. Since the exceptional set of this blow-up is rational, Y is, as well. Thus $Y \cong \mathbb{P}_1$. In particular, all q-fibers are linearly equivalent, and the D_α are linearly equivalent up to positive multiplicities.

In order to see that all the D_α have multiplicity 1 as q-fibers, it is sufficient to see that the divisors D_α are linearly equivalent. By Lemma 4.4, D_α is locally given by a curve N having the equation $x^{(n')} = y^{(m')}$, $D_0 = \{x = 0\}$ and $D_\infty = \{y = 0\}$. Thus,

$$D_0.D_\alpha = m'C, \quad D_\infty.D_\alpha = n'C, \quad D_\alpha.D_\alpha = n'm'C.$$

Consequently

$$D_0 \sim \frac{1}{n'} D_\alpha \quad \text{and} \quad D_\infty \sim \frac{1}{m'} D_\alpha$$

as $\mathbb{Q}$-divisors. This finishes the proof. ∎

4.1. Proof of Theorem 1.1

With these preparations we start the proof of the Main Theorem 1.1. If $X \cong \mathbb{Q}_3$, $\mathbb{P}_3$ or $\mathbb{P}_{(1,1,1,2)}$, we can stop here. Otherwise, we are in one of the cases (1)–(3) of Corollary 4.5. We treat these cases separately.

Proof of 1.1 *in case* (1) *of Corollary* 4.5. By Proposition 3.1, all q-fibers over $\mathbb{P}_1 \setminus \{0, \infty\}$ are isomorphic to $\mathbb{C}^2$. By Proposition 3.6, X has two singularities of type $\frac{1}{n}(1, 1, -1)$ and $\frac{1}{m}(1, 1, -1)$, and $X \setminus C$ is toric. Since X is smooth along C, the associated vector fields extend to X, showing that X is toric, too.

Consequence: X can be given as a fan in $\mathbb{Z}^3$. Let σ_1 and $\sigma_2 \subset \mathbb{Z}^3$ be the cones describing the smooth $(\mathbb{C}^*)^3$-fixed points on C, σ_3 describe the point $\frac{1}{n}(1, 1, -1)$ and σ_4 be associated with $\frac{1}{m}(1, 1, -1)$. There can be no further fixed points.

Choose coordinates such that σ_1 is spanned by the unit vectors $(e_1, e_2, e_3) \in \mathbb{Z}^3$. Because every cone is spanned by four rays, there must be a vector $v = (a, b, c) \in \mathbb{Z}^3$ such that σ_2, σ_3 and σ_4 are spanned by 2 unit vectors and v each. After renaming the e_i, if necessary, assume that $\sigma_2 = (e_1, e_2, v)$, $\sigma_3 = (e_1, e_3, v)$ and $\sigma_4 = (e_2, e_3, v)$. We will find out the possibilities for v. First, note that two cones must not intersect in anything but a face. Thus, a, b and c must be negative. We use the local description of the singularities:

σ_2 is smooth: consequently, (e_1, e_2, v) must be a basis of $\mathbb{Z}^3$ and $c = -1$.

σ_3 is $\frac{1}{n}(1, 1, -1)$: It is known (see [Ful93, p. 35]) that the cone generated by $(e_1, e_3, -(n-1)e_1 + ne_2 - e_3)$ corresponds to a singularity of type $\frac{1}{n}(1, 1, -1)$. Thus, there exists a $g \in GL(3, \mathbb{Z})$ such that $g(e_1, e_3, -(n-1)e_1 + ne_2 - e_3) = (e_1, e_3, (a, b, -1))$. Calculating the product

$$\begin{pmatrix} 1 & \alpha & 0 \\ 0 & \beta & 0 \\ 0 & \gamma & 1 \end{pmatrix} \begin{pmatrix} -(n-1) \\ n \\ -1 \end{pmatrix} = \begin{pmatrix} -n+1+\alpha n \\ \beta n \\ \gamma n - 1 \end{pmatrix} \overset{!}{=} \begin{pmatrix} a \\ b \\ -1 \end{pmatrix}$$

yields $\gamma = 0$, and $a \in \mathbb{Z}n + 1$. Since $\det g \in \{\pm 1\}$, $\beta \in \{\pm 1\}$. The inequality $b < 0$ gives $b = -n$.

σ_4 is $\frac{1}{m}(1, 1, -1)$: Similar to the above there is a $g \in GL(3, \mathbb{Z})$ such that $g(e_2, e_3, me_1 - (m-1)e_2 - e_3) = (e_1, e_3, (a, -n, -1))$. The same calculation shows $a = -m$ and $b \in \mathbb{Z}m + 1$.

Summarizing the above, we need to find all n and m such that there are numbers μ, $\nu \in \mathbb{Z}$ with

$$m = \mu n - 1 \tag{1}$$

$$n = \nu m - 1 \tag{2}$$

By assumption, n and m are coprime so that we can always assume without loss of generality that $n > m > 1$. Then equation 1 holds iff $\mu = 1$ and $m = n - 1$. Inserting this into equation 2 gives $m(\nu - 1) = 2$ which in turn implies $m = 2$. Now compare v to the description of Example 4.1. ∎

Proof of 1.1 *in case* (2) *of Corollary* 4.5. Let us begin by giving a detailed description of this case over a trivialization. Set $X^0 := X \setminus D_\infty$. By Corollary 4.6, $q^{-1}(0)$ has support on D_0 and multiplicity n'; this is the only q-fiber with non-trivial multiplicity over $\mathbb{C}$. Let $\tilde{q} : \tilde{X}^0 \to \mathbb{C}$ be the n'-th root fibration associated with $q : X^0 \to \mathbb{C}$. Now $\tilde{X}^0$ is a quotient of X_k by the cyclic group $\mathbb{Z}_{n'}$ acting freely in codimension 1. Choose an analytic disk $\Delta \subset \mathbb{C}$ around 0 which is $\mathbb{Z}_{n'}$-invariant. Then, after proper choice of coordinates, $\tilde{q}^{-1}(\Delta) \cong \{4xz - y^2 = \delta^k\}$ as ensured by Proposition 3.1.

It is elementary to see that every automorphism of $\tilde{q}^{-1}(\Delta)$ over Δ commuting with SL_2 is given by $((x, y, z), \delta) \to (\pm t^{k/2}(x, y, z), t\delta)$ for some $t \in \mathbb{C}^*$. Thus, the action of $\mathbb{Z}_{n'}$ on X_k extends to $\mathbb{C}^3 \times \Delta$ where $\tilde{q}^{-1}(\Delta)$ is SL_2-equivariantly embedded. Since the action of $\mathbb{Z}_{n'}$ must commute with SL_2, the weights of the $\mathbb{Z}_{n'}$-action on $\mathbb{C}^3 \times \{0\}$ must be equal.

An analogous construction can be given at ∞. We now show that n' and m' are not coprime. This contradicts the assumption.

Consider the following cases:

k = 0: In this case there is no SL_2-fixed point in $\tilde{q}^{-1}(\Delta)$, and consequently none on D_0. Thus, X must be a cone or $\mathbb{Q}_3$; see the proof of Corollary 4.5 for this. A contradiction to the assumption.

***k* = 1:** In this case (x, y, z) are coordinates for $\tilde{X}$. The quotient is terminal iff the weights are of the form $(a, -a, 1)$ (see [Rei87, Sect. 5.1]). Thus, $n' = 2$.

***k* > 1:** Note that there is no $\mathbb{Z}_{n'}$-fixed subspace in $\mathbb{C}^3 \times \Delta$. In this situation [Mor85, Thm. 12] shows that n' must be 4.

Now apply the same argumentation to D_∞ and realize that the coprimeness assertion of Corollary 4.5 is necessarily violated. This yields the claim. ∎

Proof of 1.1 *in case* (3) *of Corollary* 4.5. As before, set $X^0 := X \setminus D_\infty$ and consider the divisor $L := K_{X^0} - D_0$. By adjunction formula, $L|_{D_0} = K_{D_0}$ which has index $n/2$. Thus, the index of L in X^0 is in $\frac{n}{2}\mathbb{N}$. Now perform the cyclic cover associated with L (see [Rei80, Cor. 1.9] for details): $\gamma : \tilde{X}^0 \to X^0$. Stein-factorization gives a diagram

$$\begin{array}{ccc} \tilde{X}^0 & \xrightarrow{\gamma} & X^0 \\ \tilde{q}\downarrow & & \downarrow q \\ \tilde{Y} & \longrightarrow & Y \end{array}$$

where we can choose $\tilde{Y}$ to be normal, hence smooth. We are interested in the preimage of D_0. First, note that every vector field on $X^0 \setminus \mathrm{Sing}(X^0)$ can be lifted to $\tilde{X} \setminus q^{-1}(\mathrm{Sing}(X^0))$. Since $\tilde{X}$ is normal, we obtain an action of the associated 1-parameter group on $\tilde{X}$. In particular, since S is simply connected, S acts on $\tilde{X}^0$ in a way that γ is equivariant.

As a next step we need to show that $\tilde{q}^{-1}(0)$ is reduced. By Corollary 4.6, $q^{-1}(0)$ has multiplicity $n' = n/4$. On the other hand, generic $(\gamma \circ q)$-fibers have at least $n/4$ components. This is due to the fact that $D_\alpha \cong \mathbb{P}_2$, where SL_2 acts without fixed point, admits only Σ_0 as a connected S-equivariant cover. Consequence: $\tilde{q}^{-1}(0) = \gamma^{-1}(D_0)$ is reduced and isomorphic to $\mathbb{S}_2^a$. Now apply the argumentation from the proof in case (2). ∎

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Mathematisches Institut der Universität Bayreuth
95440 Bayreuth
Germany
FAX: +49 (0)921/55-2785
stefan.kebekus@uni-bayreuth.de