V. V. Nikulin
Nagoya Math. J.
Vol. 157 (2000), 73–92

A REMARK ON ALGEBRAIC SURFACES WITH POLYHEDRAL MORI CONE

VIACHESLAV V. NIKULIN[1]

To 75-th Birthday of I.R. Shafarevich

Abstract. We denote by $FPMC$ the class of all non-singular projective algebraic surfaces X over $\mathbb{C}$ with finite polyhedral Mori cone $\mathrm{NE}(X) \subset \mathrm{NS}(X) \otimes \mathbb{R}$. If $\rho(X) = \mathrm{rk}\ \mathrm{NS}(X) \geq 3$, then the set $\mathrm{Exc}(X)$ of all exceptional curves on $X \in FPMC$ is finite and generates $\mathrm{NE}(X)$. Let $\delta_E(X)$ be the maximum of $(-C^2)$ and $p_E(X)$ the maximum of $p_a(C)$ respectively for all $C \in \mathrm{Exc}(X)$. For fixed $\rho \geq 3$, δ_E and p_E we denote by $FPMC_{\rho,\delta_E,p_E}$ the class of all algebraic surfaces $X \in FPMC$ such that $\rho(X) = \rho$, $\delta_E(X) = \delta_E$ and $p_E(X) = p_E$. We prove that the class $FPMC_{\rho,\delta_E,p_E}$ is bounded in the following sense: for any $X \in FPMC_{\rho,\delta_E,p_E}$ there exist an ample effective divisor h and a very ample divisor h' such that $h^2 \leq N(\rho, \delta_E)$ and $h'^2 \leq N'(\rho, \delta_E, p_E)$ where the constants $N(\rho, \delta_E)$ and $N'(\rho, \delta_E, p_E)$ depend only on ρ, δ_E and ρ, δ_E, p_E respectively.

One can consider Theory of surfaces $X \in FPMC$ as Algebraic Geometry analog of the Theory of arithmetic reflection groups in hyperbolic spaces.

§1. Introduction

Let X be a non-singular projective algebraic surface over algebraically closed field with finite polyhedral Mori cone $\mathrm{NE}(X) \subset \mathrm{NS}(X) \otimes \mathbb{R}$ where $\mathrm{NS}(X)$ is the Neron–Severi lattice of X. If $\rho = \mathrm{rk}\ \mathrm{NS}(X) \geq 3$, then the set $\mathrm{Exc}(X)$ of exceptional curves of X is finite and generates the cone $\mathrm{NE}(X)$. Further we assume that $\rho \geq 3$. One can introduce natural invariants of X:

$$\rho = \mathrm{rk}\ \mathrm{NS}(X), \quad \delta_E = \max_{C \in \mathrm{Exc}(X)} (-C^2), \quad p_E = \max_{C \in \mathrm{Exc}(X)} p_a(C).$$

The main result of the paper (Theorem 1.1) is that *the class of surfaces X with finite polyhedral Mori cone and fixed invariants $\rho \geq 3$, δ_E and p_E is bounded:* there exists an effective ample divisor h and a very ample divisor h' on X such that $h^2 \leq N(\rho, \delta_E)$ and $h'^2 \leq N(\rho, \delta_E, p_E)$.

Received November 16, 1998.
[1]Supported by Grant of Russian Fund of Fundamental Research

The key step in the proof the theorem is using the old result of the author [N4] on "narrow parts" of convex finite polyhedra of finite volume in hyperbolic spaces (for the special case of Theorem 1.1 it is formulated in Lemma 1.1). It was used in [N4] to prove some finiteness results on arithmetic reflection groups in hyperbolic spaces. We also use some standard results about symmetric matrices with non-negative coefficients (Perron–Frobenius Theorem) and Reider's Theorem [R] on very ample divisors of surfaces. These considerations permit to write down the h and h' as linear combination of exceptional curves on the X.

In Example 1.2 we show that Theorem 1.1 is not valid if one of the invariants $\rho \geq 3$, $\delta_E \geq 3$ and p_E is not fixed.

Because of Theorem 1.1, one can ask about classification of surfaces X with finite polyhedral Mori cone and small invariants ρ, δ_E and p_E.

In Example 1.3 we consider classification for $\delta_E = 1$. Using results of [N2] and [N3], we then have $\rho \leq 9$. In Example 1.3.1 we additionally to $\delta_E = 1$ suppose that $p_E = 0$. Then one gets non-singular Del Pezzo surfaces whose classification is well-known, e. g. see [Ma].

In Example 1.4 we consider classification for $\delta_E = 2$. Using results of [V1] and [E], we have $\rho \leq 22$. In Example 1.4.1 we additionally to $\delta_E = 2$ suppose that $p_E = 0$. Then one gets (minimal) K3 surfaces, (minimal) Enriques surfaces, minimal resolutions of singularities of Del Pezzo surfaces with Du Val singularities, and rational surfaces with $K^2 = 0$ and nef $-K$. Classification of the last class of surfaces with finite polyhedral Mori cone was not considered in literature, and we give this classification. It uses Ogg–Shafarevich theory of elliptic surfaces and results [Hal], [D] and [CD] about rational elliptic surfaces. It is interesting that not all rational surfaces with $K^2 = 0$, nef $-K$ and finite polyhedral Mori cone are elliptic.

It seems, nobody tried to classify surfaces X with finite polyhedral Mori cone and $p_E \geq 1$.

It Sect. 2 we consider generalization of the main theorem 1.1 above to some surfaces with locally finite polyhedral Mori cone.

I am grateful to I.V. Dolgachev and I.R. Shafarevich for very useful discussion on elliptic surfaces.

§1. Algebraic surfaces with finite polyhedral Mori cone

Let X be a non-singular projective algebraic surface over an algebraically closed field. Let $\mathrm{NS}(X)$ be *Neron–Severi lattice* of X (i. e. the group of divisors on X by numerical equivalence considered together with the intersection pairing). By Hodge Index Theorem, the lattice $\mathrm{NS}(X)$ is hyperbolic: it has signature $(1, \rho - 1)$ where $\rho = \mathrm{rk}\ \mathrm{NS}(X)$. We denote by $\mathrm{NE}(X) \subset \mathrm{NS}(X) \otimes \mathbb{R}$ the *Mori cone* of X generated over $\mathbb{R}_+ = \{t \in \mathbb{R} \,|\, t \geq 0\}$ by all effective curves on X. By definition, a surface X has a *finite polyhedral Mori cone* $\mathrm{NE}(X)$ if $\mathrm{NE}(X)$ is generated by a finite set of rays (we denote by $FPMC$ the class of all these surfaces). The minimal set of these rays is called the *set of extremal rays*. We denote by $V^+(X)$ the half-cone containing a polarization (i. e. an ample divisor) of the light cone $V(X) = \{x \in \mathrm{NS}(X) \otimes \mathbb{R} \ \mid \ x^2 > 0\}$. By Riemann–Roch Theorem, the cone $\mathrm{NE}(X)$ contains the half-cone $V^+(X)$. It follows that for $\rho(X) \geq 3$, the set of extremal rays of $X \in FPMC$ is equal to $\mathbb{R}_+ E$, $E \in \mathrm{Exc}(X)$, where $\mathrm{Exc}(X)$ is the set of all exceptional (i. e. irreducible and having negative square) curves of X. In particular, the set $\mathrm{Exc}(X)$ is finite. Thus, we can introduce *natural invariants* of $X \in FPMC$:

$$\rho(X) = \mathrm{rk}\ \mathrm{NS}(X), \tag{1.1}$$

$$\delta_E(X) = \max_{C \in \mathrm{Exc}(X)} (-C^2), \tag{1.2}$$

$$p_E(X) = \max_{C \in \mathrm{Exc}(X)} p_a(C), \tag{1.3}$$

where $p_a(C) = \frac{C^2 + C \cdot K}{2} + 1$ is the arithmetic genus of a curve C, K is the canonical divisor of X.

Surfaces $X \in FPMC$ are interesting because of the following reasons:

1) Polyhedrality of the Mori cone $\mathrm{NE}(X)$ is very important in Mori Theory (see [Mo]). It is interesting and curious to ask what will be if one requires the only this condition.

2) We consider surfaces $X \in FPMC$ as Algebraic Geometry analog of arithmetic groups generated by reflections in hyperbolic spaces (e. g. see [N4], [N5] and [N8]). We also expect that they are connected with some analog of automorphic products introduced by R. Borcherds (see [B1], [B2], [GN]–[GN7] and [N10]–[N12]).

3) We expect that quantum cohomology related with surfaces $X \in FPMC$ are very interesting: one can consider the set $\mathrm{Exc}(X)$ as analog of a system of simple real roots. Here "related" means that not necessarily the quantum cohomology of X itself, but e. g. quantum cohomology of varieties fibrated by $X \in FPMC$ might be interesting ones. See some examples in [CCL], [HM], [Ka], [Moo] and also [GN3], [GN7].

4) A generalization of results of this paper to 3-folds (e. g. to Calabi-Yau 3-folds) would be very interesting.

EXAMPLE 1.1. (Basic example) We have the following basic example of surfaces $X \in FPMC$. It shows that there are plenty of surfaces $X \in FPMC$. For this example, let Y be a normal projective algebraic surface such that $-K_Y$ is nef and $K_Y^2 > 0$ (in particular, one can take Y to be a numerical Del Pezzo surface with normal singularities). Let X be the minimal resolution of singularities of Y. Then $X \in FPMC$ if $\rho(X) \geq 3$. This follows from the Mori Theory [Mo] applied to the non-singular projective algebraic surface X. E. g. see [N9].

For fixed invariants $\rho \geq 3$, δ_E, p_E, we denote by $FPMC_{\rho,\delta_E,p_E}$ the class of all algebraic surfaces $X \in FPMC$ such that $\rho(X) = \rho$, $\delta_E(X) = \delta_E$ and $p_E(X) = p_E$.

In Theorem 1.1 below we want to show that the class $FPMC_{\rho,\delta_E,p_E}$ is bounded. We remind that any non-singular projective algebraic surface has a linear projection embedding into $\mathbb{P}^5$ (e. g. see [Sh2]). This projection keeps the degree. Surfaces in $\mathbb{P}^5$ of the fixed degree depend on a finite number of Chow coordinates (e. g. see [Sh2]).

THEOREM 1.1. *For $\rho \geq 3$, there are constants $N(\rho,\, \delta_E)$ and $N'(\rho,\, \delta_E,\, p_E)$ depending only on $(\rho,\, \delta_E)$ and $(\rho,\, \delta_E,\, p_E)$ respectively such that for any $X \in FPMC_{\rho,\delta_E,p_E}$ there exists an ample effective divisor h such that $h^2 \leq N(\rho,\, \delta_E)$, and if the ground field is $\mathbb{C}$, there exists a very ample divisor h' such that $h'^2 \leq N'(\rho,\, \delta_E,\, p_E)$.*

Proof. Let $\mathrm{NEF}(X) = \mathrm{NE}(X)^*$ be the dual nef cone. We have $\mathrm{NEF}(X) \subset \overline{V^+(X)} = \overline{V^+(X)}^{\,*} \subset \mathrm{NE}(X)$. Therefore, the nef cone defines a finite polyhedron $\mathcal{M} = \mathrm{NEF}(X)/\mathbb{R}_+$ of finite volume in the hyperbolic space $\mathcal{L}(X) = V^+(X)/\mathbb{R}_{++}$, where $\mathbb{R}_{++} = \{t \in \mathbb{R} \mid t > 0\}$. The set $\mathrm{Exc}(X)$ is the set of orthogonal vectors to faces (of the highest dimension) of $\mathcal{M}$. By [N4], Appendix, Theorem 1 (see also [N14] about much more exact statements), we have the following

LEMMA 1.1. *There exist exceptional curves $E_1, \dots, E_\rho \in \mathrm{Exc}(X)$ such that the conditions* (a), (b) *and* (c) *below are valid:*

(a) *$E_1, \dots, E_\rho$ generate $\mathrm{NS}(X) \otimes \mathbb{Q}$;*

(b) $\frac{2\,(E_i\cdot E_j)}{\sqrt{E_i^2E_j^2}} < 62$;

(c) *The dual graph of* $E_1,\dots,E_\rho$ *is connected, i. e. one cannot divide this set in two non-empty subsets orthogonal to one another.*

Let us consider the $\rho\times\rho$ matrix $\Gamma=(\gamma_{ij})=(E_i\cdot E_j)$ where $1\le i,\, j\le\rho$. By conditions of Theorem 1.1 and by Lemma 1.1, we have $-\delta_E\le\gamma_{ii}<0$ for any $1\le i\le\rho$, and $0\le\gamma_{ij}<31\delta_E$ for any $1\le i,\, j\le\rho$ and $i\neq j$. It follows that the set of possible matrices Γ is finite. Thus, in further considerations we can fix one of the possible matrices Γ.

The matrix Γ has non-negative coefficients except its diagonal. Moreover, it is symmetric and indecomposable (by the condition (c)). Thus, by Perron-Frobenius Theorem, its maximal eigenvalue λ has multiplicity one and has the eigenvector $v=b_1E_1+\cdots+b_\rho E_\rho$ with positive coordinates $b_i>0$. Since the lattice $\mathrm{NS}(X)$ is hyperbolic, the eigenvalue $\lambda>0$. It follows that $\Gamma v=\lambda v$ and $E_j\cdot v=\lambda b_j>0$ for any $1\le j\le\rho$. Moreover, $v^2=\lambda(b_1^2+\cdots+b_\rho^2)>0$.

We can replace real numbers b_i by very closed positive rational numbers b'_i keeping the inequalities $E_j\cdot(b'_1E_1+\cdots+b'_\rho E_\rho)>0$ for any $1\le j\le\rho$, and $(b'_1E_1+\cdots+b'_\rho E_\rho)^2>0$. Multiplying numbers b'_i by an appropriate positive natural number N, finally we find natural numbers $a_i=Nb'_i$ such that $E_j\cdot(a_1E_1+\cdots+a_\rho E_\rho)>0$ for any $1\le j\le\rho$, and $(a_1E_1+\cdots+a_\rho E_\rho)^2>0$. It follows

LEMMA 1.2. *Under the conditions* (a) *and* (c) *of Lemma* 1.1*, there exist* $a_i\in\mathbb{N}$*,* $i=1,\dots,\rho$*, depending only on the matrix* $\Gamma=(E_i\cdot E_j)$*,* $1\le i,\, j\le\rho$*, such that for* $h=a_1E_1+\cdots+a_\rho E_\rho$ *one has*

$$E_j\cdot h>0$$

for any $1\le j\le\rho$*, and* $h^2>0$.

Suppose that C is an irreducible curve on X different from $E_1,\dots,E_\rho$. Then C defines a non-zero element in $\mathrm{NS}(X)$ because $C\cdot H>0$ for a hyperplane section H. It follows that $C\cdot E_i\ge 0$ for any $0\le i\le\rho$, and at least one of these inequalities is strong because $E_1,\dots,E_\rho$ generate $\mathrm{NS}(X)\otimes\mathbb{Q}$. Thus, $C\cdot h>0$. It follows that $C\cdot h>0$ for any effective curve C. Since $h^2>0$, by Nakai–Moishezon criterion, the divisor h is ample. By the construction, h is effective. Since for the fixed $(\rho,\,\delta_E)$ the set of possible matrices Γ is finite, $h^2 \le N(\rho,\,\delta_E)$ for a constant $N(\rho,\,\delta_E)$ depending only on $(\rho,\,\delta_E)$. It proves the first statement of Theorem 1.1.

To prove second statement, let us additionally fix $p_a(E_i)$, $1 \le i \le \rho$. Since $0 \le p_a(E_i) \le p_E$, there exists only a finite number of possibilities. Let $K = d_1E_1 + \cdots + d_\rho E_\rho$ be the canonical class of X where $d_i \in \mathbb{Q}$. One can find $d_1, \dots, d_\rho$ from equations $\frac{E_i^2+(E_i\cdot K)}{2} + 1 = p_a(E_i)$, $1 \le i \le \rho$. Since the lattice $\mathrm{NS}(X)$ is non-degenerate and $E_1, \dots, E_\rho$ give a bases of $\mathrm{NS}(X)\otimes\mathbb{Q}$, these equations define $d_1, \dots, d_\rho$ uniquely.

Suppose that the ground field is $\mathbb{C}$. By Reider's Theorem [R] (see also [L]), we have $h' = K + 4h$ is very ample. It follows that $h'^2 \le N'(\rho,\,\delta_E,\,p_E)$ where $N'(\rho,\,\delta_E,\,p_E)$ is bounded by a constant depending on $(\rho,\,\delta_E,\,p_E)$. (We remark that this is the only place in the proof of Theorem 1.1 where we use that the ground field is $\mathbb{C}$.) This finishes the proof of Theorem 1.1.

Remark 1.1. The same proof shows that surfaces $X \in FPMC_{\rho,\delta_E,p_E}$ belong to a finite set of Hilbert schemes determined by a finite set of Hilbert polynomials (for the fixed invariants ρ, δ_E, p_E). By this argument, one can proof Theorem 1.1 for any characteristic p of the ground field, avoiding using of the Reider's Theorem.

In Theorem 1.1, we fixed invariants ρ, δ_E and p_E. In Example 1.2 below, we show that Theorem 1.1 is not in general true if one does not fix one of these invariants.

EXAMPLE 1.2. Let us consider a non-singular curve C_g of genus g and an invertible sheaf $\mathcal{L}$ on C_g of the degree $-n$ where $n \in \mathbb{N}$. The ruled surface $\pi : Y = \mathbb{P}(\mathcal{O}_{C_g} \oplus \mathcal{L}) \to C_g$ has the exceptional section C_g of genus g such that $(C_g)^2 = -n$ (e. g. see [Har], Ch. V, Example 2.11.3). Let E_0 be a fiber of π and X the blow up of Y in a point of E_0 which does not belong to the section C_g. Let F_0 be the exceptional curve of the blow up. We get three exceptional curves C_g, E_0 and F_0 on X of the genus g, 0 and 0 respectively and with the intersection matrix

$$\begin{pmatrix} -n & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

Using this matrix, it is easy to prove that the cone $\mathbb{R}_+C_g + \mathbb{R}_+E_0 + \mathbb{R}_+F_0$ contains the cone $V^+(X)$. (One needs to show that the Gram matrix of any two of these three curves is either negative or semi-negative definite.) It follows that $\mathrm{NE}(X)=\mathbb{R}_+C_g+\mathbb{R}_+E_0+\mathbb{R}_+F_0$, and $X\in FPMC_{3,n,g}$ where the numbers $n>0$ and $g\ge 0$ can be arbitrary. The surfaces X give the infinite dimensional family of surfaces in $FPMC_{3,\delta_E=n}=\cup_{p_E\ge 0}FPMC_{3,n,p_E}$. This shows that Theorem 1.1 is not true if one does not fix the invariant p_E. The same example shows that Theorem 1.1 is not true if one does not fix the invariant δ_E.

Now let us show that ρ is not bounded for $X\in FPMC_{\delta_E,p_E}=\cup_{\rho\ge 3}FPMC_{\rho,\delta_E,p_E}$ for any fixed $\delta_E\ge 3$ and $p_E\ge 0$. Let us consider the surface X above with $n=\delta_E\ge 3$ and $g=p_E$. It has three exceptional curves C_g, E_0 and F_0. Curves E_0 and F_0 (different from C_g) define a connected tree of curves and have one intersection point of the curves. Consider the blow up X_1 of X in the intersection point. Then X_1 has four exceptional curves C_g, E_0, F_0 and F_1 where F_1 is the exceptional curve of the blow-up. Like for Example 1.1, one can show that $X_1\in FPMC$ and C_g, E_0, F_0 and F_1 are all exceptional curves of X_1. It has $\rho(X_1)=4$. Curves E_0, F_0 and F_1 (different from C_g) also define a tree and have two intersection points. We can repeat this procedure considering blow up X_2 of X_1 in one of these two points. Repeating this procedure, we get an infinite sequence X_k, $k\ge 0$, of surfaces with $\rho(X_k)=3+k$. We have: $X_k\in FPMC$ and X_k has exactly $3+k$ exceptional curves where $2+k$ of them are proper preimages of exceptional curves of X_{k-1} and one is the exceptional curve of the blow-up in an intersection point of two exceptional curves (different from C_g) of X_{k-1}. One can see that this sequence contains an infinite sequence of surfaces $X_k\in FPMC_{\rho=3+k,\delta_E,p_E}$. Actually one can find an infinite sequence X_k such that the curve C_g of X_k has $(C_g)^2=-n=-\delta_E$, and all other exceptional curves E of X_k have $-3\le E^2<0$. It follows, that $X_k\in FPMC_{\delta_E,p_E}$ since we assume that $\delta_E\ge 3$.

Because of Theorem 1.1, one can ask about classification of surfaces $X\in FPMC_{\rho,\delta_E,p_E}$ for small invariants ρ, δ_E and p_E.

Example 1.3. For this example, we suppose that $\delta_E=1$. Then $\rho\le 9$. Really, for $\delta_E=1$ any exceptional curve $E\in \mathrm{Exc}(X)$ has $E^2=-1$ and defines a reflection of $\mathrm{NS}(X)$ which maps $E\to -E$ and is identical on $E^\perp$. It is given by the formula $x\mapsto x+2(E\cdot x)E$, $x\in\mathrm{NS}(X)$. All $E\in \mathrm{Exc}(X)$ generate a reflection group $W\subset O(\mathrm{NS}(X))$ with the fundamental chamber $\mathrm{NEF}(X)/\mathbb{R}_+\subset\mathcal{L}(X)$ of finite volume. It follows that $[O(\mathrm{NS}(X)):$

$W^{(-1)}(\mathrm{NS}(X))] < \infty$ where $W^{(-1)}(\mathrm{NS}(X))$ is generated by reflections in all $\alpha \in \mathrm{NS}(X)$ with $\alpha^2 = -1$. It was shown in [N2], [N3] that rk $S \le 9$ for any hyperbolic lattices S with $[O(S) : W^{-1}(S)] < \infty$, and all hyperbolic lattices S with this property were found; see also [N6], [N8]. From Theorem 1.1, it then follows that the family $FPMC_{\delta_E=1,p_E}$ is bounded and may be described (in principle) for a fixed p_E. Here we denote $FPMC_{\delta_E=k,p_E} = \cup_{\rho\ge 3} FPMC_{\rho,k,p_E}$.

EXAMPLE 1.3.1. Let us additionally (to the condition $\delta_E = 1$) assume that $p_E = 0$. We have: *The family* $X \in FPMC_{\delta_E=1,p_E=0}$ *consists of all non-singular Del Pezzo surfaces* X *and is well-known* (e. g. see [Ma]). Really, we have that $-K \cdot E = 1 > 0$ for any $E \in \mathrm{Exc}(X)$. Since $\mathrm{Exc}(X)$ is finite and generates $\mathrm{NE}(X)$, it then follows that $(-K)^2 > 0$, and $-K$ is ample by Nakai–Moishezon criterion. The opposite statements is a very particular case of Basic Example 1.1. One can get Del Pezzo surfaces as blow up of $\mathbb{P}^2$ in ≤ 8 points in "general" position. It follows that they define a bounded family of algebraic surfaces and illustrates Theorem 1.1 for this very particular case. We even have more: their moduli have finite number of connected components.

It seems, nobody tried to classify $FPMC_{\delta_E=1,\,p_E}$ for $p_E \ge 1$.

EXAMPLE 1.4. For this example, we suppose that $\delta_E = 2$. Thus, $E^2 = -1$ or -2 for any $E \in \mathrm{Exc}(X)$. Then again any $E \in \mathrm{Exc}(X)$ defines reflection of $\mathrm{NS}(X)$. It is given by the formula $x \mapsto x - \left(2(E\cdot x)/E^2\right)E$, $x \in \mathrm{NS}(X)$. The same arguments as above show that $\rho \le 22$; see [V1], [V2] and [E]. We mention that the same result is valid for $X \in FPMC$ if all $E \in \mathrm{Exc}(X)$ define reflections of $\mathrm{NS}(X)$. We also remark that if $E^2 = -2$ for any $E \in \mathrm{Exc}(X)$, then $\rho \le 19$, see [N2], [N3]. By Theorem 1.1, it then follows that the class $FPMC_{\delta_E=2,\,p_E}$ is bounded for a fixed p_E.

EXAMPLE 1.4.1. Let us additionally (to the condition $\delta_E = 2$) assume that $p_E = 0$. Let $X \in FPMC_{\delta_E=2,\,p_E=0}$. Then $E^2 = -1$ or -2 and E is non-singular rational for any $E \in \mathrm{Exc}(X)$. By the formula for genus of curve, $-K \cdot E = 1$ if $E^2 = -1$, and $-K \cdot E = 0$ if $E^2 = -2$. It follows that $-K$ is nef and $K^2 \ge 0$. Vice versa, by the formula for genus of curve, any surface X with nef $-K$ has only non-singular rational exceptional curves E with $E^2 = -1$ or -2. Considering these surfaces $X \in FPMC$, we get one of cases below where we for simplicity suppose that the basic field $k = \mathbb{C}$ (one can consider arbitrary k using results from [CD]).

Case 1. *Suppose that* $E^2 = -2$ *for any* $E \in \mathrm{Exc}(X)$. Then $K \equiv 0$ and X is minimal. By classification of surfaces (e. g. see [Sh1]), we then have $mK = 0$ for some $m \in \mathbb{N}$. We can suppose that m is minimal with this property. Then K defines the cyclic m-sheeted covering $\pi : \widetilde{X} \to X$ where $K_{\widetilde{X}} = 0$ and $\widetilde{X}$ is either $K3$ or Abelian surface. The preimage $\pi^{-1}(E)$ of an exceptional curve E of X contains an exceptional curve of $\widetilde{X}$ because $\pi^{-1}(E)^2 = mE^2 < 0$. Abelian surfaces do not have exceptional curves. Thus, $\widetilde{X}$ is K3 surface and $m = 1$ or 2. It follows that X is either K3 (if $m = 1$), or Enriques (if $m = 2$) surface. Thus *we get K3 or Enriques surfaces* X. All K3 and Enriques surfaces with finite polyhedral Mori cone (equivalently, when their automorphism group is finite [P-ŠŠ] were classified in [N2], [N3], [N6], [N7], [N8] and [Ko]. Their moduli have finite number of connected components. It is interesting that to get this classification, general arguments of Lemma 1.1 were used in some cases (see [N6]).

Case 2. *Suppose that there exists* $E \in \mathrm{Exc}(X)$ *with* $E^2 = -1$. Then $K \cdot E = -1$ and K is not numerically zero. Since $-K$ is nef, it follows that linear systems $|nK|$ are empty for all $n > 0$. By classification of surfaces, X is either ruled or rational. Suppose that X is ruled and $\pi : X \to C$ is a morphism on a curve C. Since $\mathrm{Exc}(X)$ generates $\mathrm{NS}(X) \otimes \mathbb{Q}$, the image of one of exceptional curves E is equal to C. Since E is rational, the curve C is rational. It follows that X *is a rational surface with nef* $-K$.

Case 2a. *Suppose that* X *is rational,* $-K$ *is nef and* $K^2 > 0$. Since $E \cdot K = 0$ for any $E \in \mathrm{Exc}(X)$ with $E^2 = -2$, the intersection matrix of these curves is negative definite. It follows that these curves define several disjoint configurations of the Dynkin type A_n, D_n or E_n. It follows that there exists the contraction map $\pi : X \to Y$ of these curves where Y is a surface with Du Val singularities. We then have $K_X = \pi^* K_Y$. It follows that $(-K_Y)$ is numerically ample, and Y is Del Pezzo surface with Du Val singularities. It is a particular case of Basic Example 1.1. Thus, we get that *the surfaces* X *are minimal resolutions of singularities of Del Pezzo surfaces with Du Val singularities.* Classification of these surfaces as blow-ups of $\mathbb{P}^2$ in ≤ 8 points in "general" position was obtained by Nagata [Na]. See also [Ma]. It follows that these surfaces define a bounded family of algebraic surfaces and illustrates Theorem 1.1 for this very particular case. Their moduli have finite number of connected components. Classification of all possible graphs $\Gamma(\mathrm{Exc}(X))$ of exceptional curves on X was obtained in [AN1] and [AN2].

Case 2b. *Suppose that X is rational, $-K$ is nef and $K^2 = 0$.* By Noether formula, $\rho = \text{rk NS}(X) = 10$. Since $\text{NE}(X)$ is finite polyhedral, the dual cone $\text{NEF}(X) \subset \overline{V^+(X)}$ is also finite polyhedral and the polyhedron $\mathcal{M}(X) = \text{NEF}(X)/\mathbb{R}_+$ is a finite polyhedron in the hyperbolic space $\mathcal{L}(X)$. The polyhedron $\mathcal{M}(X)$ has $\mathbb{R}_{++}(-K)$ as its infinite vertex and should be finite in a neighbourhood of this vertex. It follows that the set of exceptional curves $E \in \text{Exc}(X)$ which are orthogonal to $-K$ is parabolic and has the rank 8. It means that any connected component Γ_i of the dual graph Γ of the set of these curves has semi-negative definite Gram matrix of the rank r_i and the sum $\sum_i r_i = 8$. Since $E^2 = -2$ if $E \in \text{Exc}(X)$ and $E \cdot K = 0$, it follows that the graphs Γ_i are extended Dynkin diagrams of types $\widetilde{A}_{r_i}$, $\widetilde{D}_{r_i}$ or $\widetilde{E}_{r_i}$ where $\sum_i r_i = 8$.

Let us show that the opposite statement is also true. Suppose that X is rational, $-K$ is nef, $K^2 = 0$, any connected component of the set of exceptional curves $E \in \text{Exc}(X)$ with $E^2 = -2$ is extended Dynkin diagram of the rank r_i and $\sum_i r_i = 8$. We claim that then $X \in FPMC$.

Since $-K$ is nef, $K^2 = 0$ and K is not numerically zero, the polyhedron $\mathcal{M}(X) = \text{NEF}(X)/\mathbb{R}_+ \subset \mathcal{L}(X)$ is parabolic relative to $\mathbb{R}_{++}(-K)$. E. g. see [N9] (this follows from the Mori Theory applied to non-singular surfaces). This means that $\mathcal{M}(X)$ is finite in any angle of the hyperbolic space $\mathcal{L}(X)$ with the vertex $\mathbb{R}_{++}(-K)$. (This is example of surfaces with almost finite polyhedral Mori cone which we shall consider in Sect. 2). Let Q be the set of all exceptional curves $E \in \text{Exc}(X)$ with $E^2 = -2$ and $\mathcal{H}_E^+ = \{0 \neq x \in \text{NS}(X) \otimes \mathbb{R} \mid x \cdot E \geq 0\}/\mathbb{R}_{++}$. We have $\mathcal{M}(X) \subset \bigcap_{E \in Q} \mathcal{H}_E^+$. Curves $E \in Q$ are all orthogonal to $-K$, and by the condition on the set of these curves, the set $\bigcap_{E \in Q} \mathcal{H}_E^+$ is a finite polyhedral angle in $\mathcal{L}(X)$ with the vertex $\mathbb{R}_{++}(-K)$. It follows that $\mathcal{M}(X)$ is finite polyhedral in a neighbourhood of $\mathbb{R}_{++}(-K)$ in $\mathcal{L}(X)$. It follows that $\mathcal{M}(X)$ and $\text{NEF}(X)$ are finite. Thus, the dual cone $\text{NE}(X)$ is also finite. This proves the statement.

$(*)$ *Thus, the surfaces X are rational surfaces with nef $-K$, $K^2 = 0$, and such that any connected component of the set of exceptional curves $E \in$ $\text{Exc}(X)$ with $E^2 = -2$ is an extended Dynkin diagram Γ_i of the rank r_i and* $\sum_i r_i = 8$.

Classification of surfaces from (*) was not considered in literature. So, we are forced to give this classification below.

Suppose that X satisfies (*). Let E_j, $j = 1, \dots, r_i + 1$ are exceptional curves with $E_j^2 = -2$ which give an extended Dynkin diagram Γ_i. There are natural a_j, $j = 1, \ldots, r_i + 1$ such that $D_i = \sum_j a_j E_j$ has $D_i^2 = 0$ and one of coefficients a_j is equal to one. Obviously, $D_i \in |-m_i K|$ where m_i is the invariant of the divisor D_i and of the connected component Γ_i of the set of exceptional curves with square -2 of X. By Riemann–Roch Theorem, the linear system $|-K|$ is not empty and contains a divisor D. If D is different from one of divisors D_i, the linear system $|-m_i K|$ has positive dimension. It follows that for some natural m the linear system $|-mK|$ is a pencil. By Bertini Theorem, this pencil contains a non-singular curve which is an elliptic curve. Thus, $|-mK|$ is elliptic pencil, and the surface X is then rational minimal elliptic surface (see [D] and [CD] about theory of rational elliptic surfaces). We shall consider classification of elliptic surfaces from (*) in more details below. Our consideration also shows that X might be not elliptic only if there exists exactly one extended Dynkin diagrams Γ_1, it has rank 8 and the divisor $D_1 \in |-K|$.

Thus, the set of surfaces from (*) and its classification is divided in two parts: Elliptic surfaces from (*), and

$(***)$ *rational surfaces X with nef $-K$ and $K^2 = 0$ such that all exceptional curves F_j with $F_j^2 = -2$ of X define a connected extended Dynkin diagram of the rank 8. There exist natural a_j such that one of a_j is equal to one and $D = \sum_j a_j F_j \in |-K|$.*

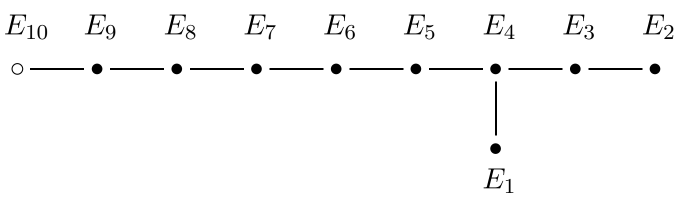
First we consider classification of surfaces (***). The graph Γ of $F_1, \ldots, F_9$ has type $\widetilde{E_8}$, $\widetilde{D_8}$ or $\widetilde{A_8}$. Since X is not relatively minimal, there exists an exceptional curve E of the first kind. We have $E \cdot D = 1$. It follows that $E \cdot F_{j_0} = 1$ for one of curves F_{j_0} having $a_{j_0} = 1$, and $E \cdot F_j = 0$ if $j \neq j_0$.

Suppose that the graph $\Gamma = \widetilde{E_8}$. Then the exceptional curves F_j and E have the dual graph $H\widetilde{E_8}$ below where black vertices correspond to the curves F_j with $F_j^2 = -2$. An edge of the graph means transversal intersection of curves in one point. Analyzing the graph $H\widetilde{E_8}$, one can see that the dual cone $\{x \in \mathrm{NS}(X) \otimes \mathbb{R} \mid x \cdot E_i \geq 0\}$ of curves $E_1, \ldots, E_{10}$ is contained in $\overline{V^+(X)}$. It follows that $X \in FPMC$, and the curves $E_1, \ldots, E_{10}$ are all exceptional curves of X. Thus, $H\widetilde{E_8}$ is the graph of all exceptional curves on X if $\Gamma = \widetilde{E_8}$.

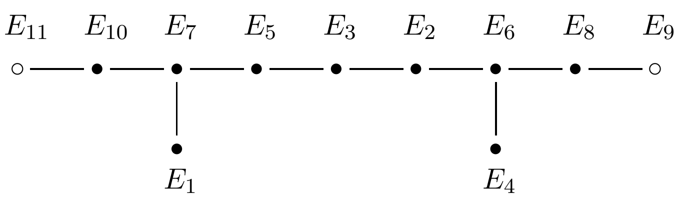
If $\Gamma = \widetilde{D_8}$, considerations are a little bit more complicated. Curves F_j and E define a subgraph of the graph $H\widetilde{D_8}$ below with vertices $E_1, \ldots, E_{10}$. Curves E_2, E_4, E_6, E_8 and E_9 of the subgraph define an extended Dynkin diagram of the type $\widetilde{B_4}$. The divisor $C = E_2 + E_4 + 2E_6 + 2E_8 + 2E_9$ is nef and has $C^2 = 0$. Curves E_{10}, E_1, E_7 and E_5 of the subgraph are orthogonal to C and define Dynkin diagram of type D_4. The nef cone $NEF(X)$ should be finite in a neighbourhood of $\mathbb{R}_+C$. It follows that there exists another exceptional curve E' of the first kind such that E' together with the curves E_{10}, E_1, E_7 and E_5 defines an extended Dynkin diagram of the type $\widetilde{B_4}$. It follows that X has exceptional curves with the graph $H\widetilde{D_8}$. Analyzing this graph, one can see that $X \in FPMC$, and the graph $H\widetilde{D_8}$ gives all exceptional curves of X. Here one should use some elementary facts about polyhedra with acute angles in hyperbolic spaces (e. g. see [V2]).

Similarly one can prove that all exceptional curves of X have graph $H\widetilde{A_8}$ below if curves with square -2 of X from (***) have the extended Dynkin diagram $\widetilde{A_8}$.

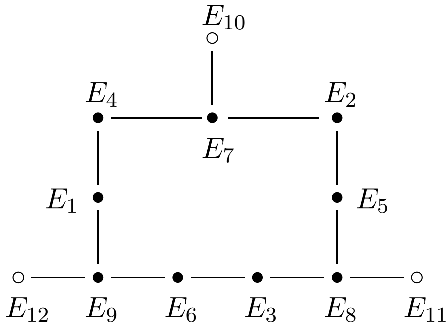
This gives: *Classification of surfaces* $(***)$*: The set of all exceptional curves of the surfaces X has one of dual graphs $H\widetilde{E_8}$, $H\widetilde{D_8}$ or $H\widetilde{A_8}$ given below. Here a black vertex E_i has $E_i^2 = -2$, and a white vertex E_i has $E_i^2 = -1$. An edge E_i, E_j of the graph means that $E_i \cdot E_j = 1$. General surfaces X of these type are not elliptic* (we shall show this below).



Graph $H\widetilde{E_8}$



Graph $H\widetilde{D_8}$



Graph $H\widetilde{A_8}$

Considering a sequence of contractions of exceptional curves of the first kind of the surfaces X (their preimages are numerated in descending order by vertices E_i of the graphs $H\widetilde{E_8}$, $H\widetilde{D_8}$ and $H\widetilde{A_8}$), one obtains existence and description of parameters of the surfaces X. Surfaces X with graphs $H\widetilde{E_8}$, $H\widetilde{D_8}$ or $H\widetilde{A_8}$ are obtained from a line E_1, two different lines E_1, E_2 and three non-collinear lines E_1, E_2, E_3 respectively on a plane $\mathbb{P}^2$ by blow up of appropriate sequences of 9 points. They correspond to vertices of the graphs $H\widetilde{E_8}$, $H\widetilde{D_8}$ and $H\widetilde{A_8}$ respectively in increasing order.

Let us show that general surfaces from (***) are not elliptic: for any natural m the linear system $|-mK|$ contains only the divisor $-mD$ and is zero-dimensional. First, let us consider $m = 1$. Suppose that for a surface X from (***) the linear system $|-K_X|$ is 1-dimensional. To be concrete, assume that X has the graph $H\widetilde{A_8}$ of exceptional curves. Let X_1 be the surface obtained by contraction of the curve E_{10}. Then $|-K_{X_1}|$ contains a nef divisor $D_1 = E_1 + E_4 + E_7 + E_2 + E_5 + E_8 + E_3 + E_6 + E_9$ with $D_1^2 = 1$. By Kawamata-Viehweg vanishing [Kaw], [Vie] and Riemann-Roch theorem, one than gets $\dim |-K_{X_1}| = 1$. Since $|-K_{X_1}|$ contains the image of $|-K_X|$ with $\dim |-K_X| = 1$, we obtain that the linear system $|-K_{X_1}| = |-K_X|$ is one-dimensional with the base point on the curve E_7. This base point is equal to the image of E_{10}. Let X' be a surface which is blow up of X_1 in a point of the curve E_7 which is different from the base point of $|-K_{X_1}|$, and from points $E_7 \cap E_4$ and $E_7 \cap E_2$. The surface X' belongs to (***), has the same graph $H\widetilde{A_8}$ of exceptional curves, and $\dim|-K_{X'}| = 0$. This shows that surfaces (***) have one additional parameter to elliptic surfaces X from (***) with not zero-dimensional $|-K_X|$. Now suppose that $|-mK|$ is not zero-dimensional, but $|-(m-1)K|$ is zero-dimensional for some $m > 0$. Then $|-mK|$ is elliptic pencil with the fiber mD of multiplicity m where D has the type $\widetilde{A_8}$. Below we shall see that for any fixed $m \geq 1$ these elliptic surfaces have the same number of parameters as for $m = 1$. Thus, our consideration above for $m = 1$ shows that general surfaces X from (***) are not elliptic: they have one additional parameter to elliptic surfaces from (***).

Now let us consider classification of elliptic surfaces X from (*). We denote this class of surfaces by (**). It consists of:

(**)*Rational surfaces X with $K^2 = 0$ having an elliptic pencil $|-mK|$ for some $m > 0$ (equivalently, X is rational minimal elliptic surface) and such that the sum $\sum_i r_i = 8$ for ranks r_i of reducible fibers of the pencil (these rational minimal elliptic surfaces are called maximal). The invariant m is called index (it was first observed in* [Hal]). See [D] and [CD], Sect. 5.6 about these surfaces.

Classification of surfaces from (**) can be obtained using general Ogg–Shafarevich theory of elliptic surfaces [O], [Sh3] , applied to the special case of rational elliptic surfaces. This was done in [D] and [CD]. In fact, below, we just review these results.

Suppose that the index $m = 1$. Then the elliptic pencil $|-K_X|$ is Jacobian, it has a section which is defined by an exceptional curve E of the first kind. For this case, there exists a sequence of contractions of 9 exceptional curves of the first kind $\pi : X \to \mathbb{P}^2$ such that image of the pencil $|-K_X|$ is a pencil of plane cubics. Condition $\sum_i r_i = 8$ is then equivalent to finiteness of the Mordell–Weil group defined by sections (i. e. exceptional curves of the first kind of X). Using this description, one can classify these surfaces X according to Dynkin diagrams Γ_i of reducible fibers. It is known [CD] that there are the following and the only following possibilities for Γ_i with $\sum_i r_i = 8$, and Mordell–Weyl groups, see [CD], Corollary 5.6.7:

Types of fibers	**Mordell–Weil groups**
$\widetilde{E}_8$	(1)
$\widetilde{D}_8$	$\mathbb{Z}/2\mathbb{Z}$
$\widetilde{A}_8$	$\mathbb{Z}/3\mathbb{Z}$
$\widetilde{E}_7+\widetilde{A}_1(\widetilde{A}_1^*)$	$\mathbb{Z}/2\mathbb{Z}$
$\widetilde{A}_7+\widetilde{A}_1(\widetilde{A}_1^*)$	$(\mathbb{Z}/2\mathbb{Z})^2$
$\widetilde{E}_6+\widetilde{A}_2(\widetilde{A}_2^*)$	$\mathbb{Z}/3\mathbb{Z}$
$\widetilde{D}_5+\widetilde{A}_3$	$\mathbb{Z}/4\mathbb{Z}$
$2\widetilde{D}_4$	$(\mathbb{Z}/2\mathbb{Z})^2$
$2\widetilde{A}_4$,	$\mathbb{Z}/5\mathbb{Z}$
$\widetilde{D}_6+2\widetilde{A}_1(\widetilde{A}_1^*)$	$(\mathbb{Z}/2\mathbb{Z})^2$
$\widetilde{A}_5+\widetilde{A}_1(\widetilde{A}_1^*)+\widetilde{A}_2(\widetilde{A}_2^*)$	$(\mathbb{Z}/3\mathbb{Z})\oplus(\mathbb{Z}/2\mathbb{Z})$
$2\widetilde{A}_3+2\widetilde{A}_1(\widetilde{A}_1^*)$	$(\mathbb{Z}/4\mathbb{Z})\oplus(\mathbb{Z}/2\mathbb{Z})$
$4\widetilde{A}_2(\widetilde{A}_2^*)$	$(\mathbb{Z}/3\mathbb{Z})^2$.

It is not difficult to extend these possible diagrams of exceptional curves with square -2 adding exceptional curves of the first kind. Their number is equal to the order of the Mordell–Weil group. E. g. for diagrams $\widetilde{E}_8$, $\widetilde{D}_8$ and $\widetilde{A}_8$ one gets graphs $H\widetilde{E}_8$, $H\widetilde{D}_8$ and $H\widetilde{A}_8$ respectively given above.

Now assume that the index $m>1$. Then the elliptic pencil $|-mK_X|$ has a unique multiple fiber mD where $D\in|-K_X|$. The Jacobian fibration J of X is also a rational minimal elliptic surface (see [CD], Proposition 5.6.1) with the same base and the same fibers as X. We have considered rational Jacobian fibrations J above. The multiple fiber mD of $|-mK_X|$ fixes a point $x_D\in D$ of order m of the corresponding fiber D of J. By Ogg–Shafarevich theory [O] and [Sh3], the triplet $(x_D\in D\subset J)$ defines the elliptic pencil $|-mK_X|$ uniquely, any triplet $(x_D\in D\subset J)$ (where J is a Jacobian rational minimal elliptic surface, D its fiber and $x_D\in D$ its point of order m) is possible and defines a rational elliptic surface X with elliptic pencil $|-mK_X|$ of index m, multiple fiber mD and the Jacobian fibration J. See [D] and [CD], Ch. 5 for details. In particular, considering of the X with a multiple fiber mD where D has the type $\widetilde{A_8}$ (this case has been considered above), is equivalent to considering of triplets $(x_D\in D\subset J)$ with the reducible fiber D of the type $\widetilde{A_8}$ and the point $x_D\in D$ of order m. For the fixed J, the set of possible $x_D\in D$ is obviously finite, and the number of parameters of the X is equal to the number of parameters of the J. We have used this above.

We emphasize that the index m of a surface X from (**) can be arbitrary $m \in \mathbb{N}$. It follows that the number of connected components of the moduli space of surfaces X in (**) is infinite. It follows that *number of connected components of moduli space of surfaces $X \in FPMC_{\delta_E=2,p_E=0}$ of the case* 2b *is infinite: it has three connected components with non-elliptic general X and infinite number of connected components with elliptic X corresponding to infinite number of possible indexes $m \in \mathbb{N}$ of X.*

It seems, nobody tried to classify surfaces $X \in FPMC_{\delta_E=2,\,p_E}$ for $p_E \geq 1$.

For a surface $X \in FPMC_{\rho\geq 3}$, the very important invariant is the *dual graph* $\Gamma(\mathrm{Exc}(X))$ *of the set* $\mathrm{Exc}(X)$ *of exceptional curves of* X. Here we mark vertices $E \in \mathrm{Exc}(X)$ of this graph by the pair $(E^2,\, p_a(E))$, and edges (E_i, E_j) of the graph by $E_i \cdot E_j$ if $E_i \cdot E_j > 0$.

Using considerations in the proof of Theorem 1.1, we can prove:

Theorem 1.2. *For fixed invariants $\rho \geq 3$, δ_E and p_E, the set of possible graphs $\Gamma(\mathrm{Exc}(X))$ of $X \in FPMC_{\rho,\delta_E,p_E}$ is finite if $K_X^2 > 0$.*

Proof. We argue as in the proof of Theorem 1.1. For the fixed matrix Γ (one from a finite set) we have that the hyperbolic lattice $\mathrm{NS}(X)$ is an intermediate lattice $[E_1, \dots, E_\rho] \subset \mathrm{NS}(X) \subset [E_1, \dots, E_\rho]^*$ where the lattice $[E_1, \dots, E_\rho]$ is fixed by Γ. Thus there exists only a finite number of possibilities for the overlattice $[E_1, \dots, E_\rho] \subset \mathrm{NS}(X)$. We fix one of them. We also fix one of finite possibilities for the canonical class $K = K_X \in \mathrm{NS}(X)$. If $K^2 > 0$, there exists only a finite set of elements $e \in \mathrm{NS}(X)$ such that $-\delta_E \leq e^2 < 0$ and $0 \leq \frac{e^2+e\cdot K}{2} + 1 \leq p_E$ because the lattice $\mathrm{NS}(X)$ is non-degenerate and hyperbolic. It shows that number of possible graphs $\Gamma(\mathrm{Exc}(X))$ is finite. This finishes the proof.

The condition $K_X^2 > 0$ of Theorem 1.2 is necessary. Surfaces X of the case 2b of Example 1.4.1 have $K_X^2 = 0$, have infinite number of connected components of the moduli space and infinite number of possible graphs $\Gamma(\mathrm{Exc}(X))$.

§2. Algebraic surfaces with some locally polyhedral Mori cone

Here we want to outline some generalization of results of Sect. 1 for more general class of surfaces (we hope to give details in forthcoming publications). They are Algebraic Geometry analog of reflection groups of hyperbolic lattices of elliptic, parabolic or hyperbolic type (see [N8], [N9], [N11], [N13], [N14]).

For surfaces $X \in FPMC_{\rho\geq 3}$ the nef cone defines an elliptic (i. e. finite and of finite volume) polyhedron $\mathrm{NEF}(X)/\mathbb{R}_+ \subset \mathcal{L}(X)$. The key Lemma 1.1 which we used in the proof of Theorem 1.1, can be generalized (with some bigger absolute constant instead of 62) for locally finite polyhedra of restricted parabolic or restricted hyperbolic type in hyperbolic spaces. See [N11], [N13], [N14]. Thus, we shall have Theorem 1.1 for surfaces with $\mathrm{NEF}(X)/\mathbb{R}_+$ of these types. Below we introduce surfaces for which this is true.

Like in Sect. 1, we consider only non-singular projective algebraic surfaces X over algebraically closed field.

Definition 2.1. Definition 2.1 Let $\rho = \rho(X) \geq 3$. We say that X has *almost finite polyhedral Mori cone* $\overline{NE}(X)$ if (1), (2) and (3) below hold:

(1) There exist finite maximums:

$$\delta_E(X) = \max_{C\in \mathrm{Exc}(X)} -C^2 \quad \text{and} \quad p_E(X) = \max_{C\in \mathrm{Exc}(X)} p_a(C).$$

(2) There exists a non-zero $r \in \mathrm{NS}(X)$ such that any extremal ray of $\overline{\mathrm{NE}}(X)$ is either generated by an exceptional curve $E \in \mathrm{Exc}(X)$ or by $c \in \mathrm{NS}(X) \otimes \mathbb{R}$ such that $c^2 = 0$ and $c \cdot r = 0$.

(3) The set $\mathrm{Exc}(X)\cdot r$ is bounded: $-R \leq \mathrm{Exc}(X)\cdot r \leq R$ for some finite $R > 0$.

We remark that if $K \not\equiv 0$, one can always put $r = K$. Then (3) follows from (1).

There are plenty of surfaces X with almost finite polyhedral Mori cone. Let Y be a normal projective algebraic surface with nef anticanonical class $-K_Y$ and with at least one non Du Val singularity if $-K_Y \equiv 0$. Then the minimal resolution X of singularities of Y has almost finite polyhedral Mori cone (e. g. see [N9]).

One can show that surfaces X with almost finite polyhedral Mori cone have the polyhedron $\mathrm{NEF}(X)/\mathbb{R}_+ \subset \mathcal{L}(X)$ either of elliptic or restricted parabolic or restricted hyperbolic type. For polyhedra $\mathcal{M}$ of this type one can prove Lemma 1.1 with the constant 62 replaced by some other absolute constant. See [N11], [N13], [N14]. From this Lemma we get (like in proof of Theorem 1.1)

Theorem 2.1. *For $\rho \geq 3$, there are constants $N(\rho, \delta_E)$ and $N'(\rho, \delta_E, p_E)$ depending only on (ρ, δ_E) and (ρ, δ_E, p_E) respectively such that for any X with almost finite polyhedral Mori cone and $\rho(X) = \rho$, $\delta_E(X) = \delta_E$ and $p_E(X) = p_E$, there exists an ample effective divisor h such that $h^2 \leq N(\rho, \delta_E)$, and if the ground field is $\mathbb{C}$, there exists a very ample divisor h' such that ${h'}^2 \leq N'(\rho, \delta_E, p_E)$.*

References

[AN1] V.A. Alexeev and V.V. Nikulin, *The classification of Del Pezzo surfaces with log terminal singularities of the index ≤ 2, involutions of $K3$ surfaces and reflection groups in Lobachevsky spaces* (Russian), Doklady po matematike i prilogeniyam, MIAN, **2** (1988), no. 2, 51–150.

[AN2] ———, *The classification of Del Pezzo surfaces with log terminal singularities of the index ≤ 2 and involutions of $K3$ surfaces*, Dokl. AN SSSR, **306** (1989), no. 3, 525–528.

[B1] R. Borcherds, *Automorphic forms on $O_{s+2,2}$ and infinite products*, Invent. Math., **120** (1995), 161–213.

[B2] ———, *The moduli space of Enriques surfaces and the fake monster Lie superalgebra*, Topology, **35** (1996), no. 3, 699–710.

[CCL] G.L. Cardoso, G. Curio and D. Lüst, *Perturbative coupling and modular forms in $N = 2$ string models with a Wilson line*, Nucl. Phys., **B491** (1997), 147–183; hep-th/9608154.

[CD] F.R. Cossec and I.V. Dolgachev, Enriques surfaces I, Birkhäuser, Progress in Mathematics, Vol. 76, 1989, pp. 397.

[D] I.V. Dolgachev, *On rational surfaces with elliptic pencil* (Russian), Izv. AN SSSR, Ser. matem., **30** (1966), 1073–1100.

[E] F. Esselmann, *Über die maximale Dimension von Lorentz-Gittern mit coendlicher Spiegelungsgruppe*, J. Number Theory, **61** (1996), 103–144.

[GN] V.A. Gritsenko and V.V. Nikulin, *Siegel automorphic form correction of some Lorentzian Kac–Moody Lie algebras*, Amer. J. Math., **119** (1997), no. 1, 181–224; alg-geom/9504006.

[GN2] ———, *Siegel automorphic form correction of a Lorentzian Kac–Moody algebra*, C. R. Acad. Sci. Paris Sér. A–B, **321** (1995), 1151–1156.

[GN3] ———, *$K3$ surfaces, Lorentzian Kac–Moody algebras and mirror symmetry*, Math. Res. Lett., **3** (1996), no. 2, 211–229; alg-geom/9510008.

[GN4] ———, *The Igusa modular forms and "the simplest" Lorentzian Kac–Moody algebras*, Matem. Sbornik, **187** (1996), no. 11, 27–66.

[GN5] ———, *Automorphic forms and Lorentzian Kac-Moody algebras. Part I*, Intern. J. Math., **9** (1998), no. 2, 153–199; alg-geom/9610022.

[GN6] ———, *Automorphic forms and Lorentzian Kac-Moody algebras. Part II*, Intern. J. Math., **9** (1998), no. 2, 201–275; alg-geom/9611028.

[GN7] ———, *The arithmetic mirror symmetry and Calabi–Yau manifolds*, Commun. Math., to appear; alg-geom/9612002.

[Hal] Halphen, *Sur les courbes planes du sixiéme degré à neuf points doubles*, Bull. Soc. math. France, **10** (1881), 162–172.

[Har] R. Hartshorne, Algebraic Geometry, Springer, 1977, pp. 496.

[HM] J. Harvey and G. Moore, *Algebras, BPS-states, and strings*, Nucl. Physics., **B463** (1996), 315–368; hep-th/9510182.

[Ka] T. Kawai, *String duality and modular forms*, Phys. Lett., **B397** (1997), 51–62; hep-th/9607078.

[Kaw] Y. Kawamata, *A generalization of Kodaira–Ramanujam's vanishing theorem*, Math. Ann., **261** (1982), 43–46.

[Ko] S. Kondo, *Enriques surfaces with finite automorphism groups*, Japanese J. Math., **12** (1986), no. 2, 191–282.

[L] R. Lazarsfeld, *Lectures on linear systems*, Complex Algebraic Geometry (J. Kollar, eds.), Amer. Math. Soc. (IAS/Park City, Math. Series, Vol. 3) (1997).

[Ma] Yu.I. Manin, Cubic forms: Algebra, Geometry, Arithmetic, Nauka, 1972.

[Mo] S. Mori, *Threefolds whose canonical bundles are not numerically effective*, Ann. Math., **116** (1982), no. 1, 133–176.

[Moo] G. Moore, *String duality, automorphic forms, and generalized Kac–Moody algebras*, Preprint (1997; hep-th/9710198).

[Na] M. Nagata, *On rational surfaces I, II*, Mem. of College of Sci. Univ. of Kyoto, Ser. A, **32** (1960), no, 3 351–370.

[N1] V.V. Nikulin, *Integral symmetric bilinear forms and some of their geometric applications*, Izv. Akad. Nauk SSSR Ser. Mat., **43** (1979), 111–177.

[N2] ———, *On factor groups of the automorphism groups of hyperbolic forms modulo subgroups generated by 2-reflections*, Dokl. Akad. Nauk SSSR, **248** (1979), 1307–1309.

[N3] ———, *On the quotient groups of the automorphism groups of hyperbolic forms by the subgroups generated by 2-reflections, Algebraic-geometric applications*, Current Problems in Math. Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, **18** (1981), 3–114.

[N4] ———, *On arithmetic groups generated by reflections in Lobachevsky spaces*, Math. USSR Izv., **16** (1981), 637–669.

[N5] ———, *On the classification of arithmetic groups generated by reflections in Lobachevsky spaces*, Math. USSR Izv., **18** (1982), **45** (1981), no. 1, 113–142.

[N6] ———, *Surfaces of type K3 with finite automorphism group and Picard group of rank three*, Trudy Inst. Steklov, **165** (1984), 113–142.

[N7] ———, *On a description of the automorphism groups of Enriques surfaces*, Dokl. AN SSSR, **277** (1984), 1324–1327.

[N8] ———, *Discrete reflection groups in Lobachevsky spaces and algebraic surfaces*, Proc. Int. Congr. Math. Berkeley 1986, 1, pp. 654–669.

[N9] ———, *Basis of the diagram method for generalized reflection groups in Lobachevsky spaces and algebraic surfaces with nef anticanonical class*, Intern. J. of Math.., **7** (1996), no. 1, 71–108.

[N10] ———, *A lecture on Kac–Moody Lie algebras of the arithmetic type*, Preprint Queen's University, Canada, **#1994-16,** (1994).

[N11] ———, *Reflection groups in Lobachevsky spaces and the denominator identity for Lorentzian Kac–Moody algebras*, Izv. Akad. Nauk of Russia. Ser. Mat., **60** (1996), no. 2, 73–106.

[N12] ———, *The remark on discriminants of $K3$ surfaces moduli as sets of zeros of automorphic forms*, J. of Mathematical Sciences,, **81** (1996), no. 3, Plenum Publishing; alg-geom/9512018, 2738–2743.

[N13] ———, *$K3$ surfaces with interesting groups of automorphisms*, J. of Math. Sciences, to appear; alg-geom/ 9701011.

[N14] ———, *On the classification of hyperbolic root systems of the rank three.*, Preprint Max-Planck-Institut für Mathematik,, **MPI-99** (1999).

[O] A. Ogg, *Cohomology of abelian varieties over function fields*, Ann. Math., **76** (1962), 185–212.

[P-ŠŠ] I.I. Pjatetckiĭ-Šapiro and I.R. Šafarevich, *A Torelli theorem for algebraic surfaces of type $K3$*, Izv. Akad. Nauk SSSR Ser. Mat., **35** (1971), 530–572.

[R] I. Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann. Math., **127** (1988), 309–316.

[Sh1] I.R. Šafarevič (ed.), Algebraic surfaces, Proc. Steklov Math. Instit., 1965.

[Sh2] ———, Basic Algebraic Geometry, Nauka, 1988.

[Sh3] ———, *Principal homogeneous spaces over function fields*, Proc. Steklov Inst. Math., **64** (1961), 316–346.

[Vie] E. Viehweg, *Vanishing theorems*, J. reine und angew. Math., **335** (1982), 1–8.

[V1] É.B. Vinberg, *The absence of crystallographic reflection groups in Lobachevsky spaces of large dimension*, Trudy Moscow. Mat. Obshch., **47** (1984), 67–102.

[V2] ———, *Hyperbolic reflection groups*, Uspekhi Mat. Nauk, **40** (1985), 29–66.

Steklov Mathematical Institute
ul. Gubkina 8, Moscow 117966, GSP-1, Russia

Dept. of Pure Math. of the University of Liverpool
Liverpool L69 3BX, England
slava@nikulin.mian.su
V.Nikulin@liverpool.ac.uk