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Nagoya Math. J.
Vol. 157 (2000), 47–57

# SOME REMARKS ON COMPLEX LIE GROUPS

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**Abstract.** First we show that any complex Lie group is complete Kähler. Moreover we obtain a plurisubharmonic exhaustion function on a complex Lie group as follows. Let $\mathfrak{k}$ the real Lie algebra of a maximal compact real Lie subgroup $K$ of a complex Lie group $G$. Put $q := \dim_{\mathbb{C}} \mathfrak{k} \cap \sqrt{-1}\mathfrak{k}$. Then we obtain that there exists a plurisubharmonic, strongly $(q+1)$-pseudoconvex in the sense of Andreotti-Grauert and $K$-invariant exhaustion function on $G$.

## §1. Introduction

To get our aim of this paper, we may assume that every complex Lie group is always connected throughout this paper.

Let $G$ be a complex Lie group of complex dimension $n$ and $\mathfrak{k}$ the real Lie algebra of a maximal compact real Lie subgroup $K$ of $G$. Put $q := \dim_{\mathbb{C}} \mathfrak{k} \cap \sqrt{-1}\mathfrak{k}$. Since all maximal compact subgroups are conjugate each other([2]), the integer $q$ is independent of a choice of a maximal compact subgroup. Concerning the pseudoconvexity of complex Lie groups, we have the following theorem which was proved in [10] and, partially, in [3, 4].

Theorem 1. *There exists a $C^\infty$ plurisubharmonic function $\varphi$ on $G$ satisfying* (1) *and* (2).

(1) *The Levi form of $\varphi$ is positive semidefinite and has $n-q$ positive eigenvalues at every point of $G$, in other words, $\varphi$ is plurisubharmonic and strongly $(q+1)$-pseudoconvex on $G$ in the sense of* [1].

(2) *$\varphi$ is an exhaustion function on $G$, i.e., for any $c \in \mathbb{R}$.*

$$\{x \in G | \varphi(x) < c\} \subset\subset G.$$

On the Stein group $GL(n, \mathbb{C})$ there exists a natural strongly plurisubharmonic exhaustion function

Received September 21, 1998.

$$\varPhi(a) := \mathrm{trace}(a\ a^*) + \frac{1}{\det(a\ a^*)}$$

$(a \in GL(n, \mathbb{C}))$. This function $\varPhi$ is invariant with respect to the left and right actions of the unitary subgroup $U(n) := \{a \mid a \in GL(n, \mathbb{C}),\ a^{-1} = a^*\}$ which is a maximal compact real Lie subgroup of $GL(n, \mathbb{C})$, i.e., for $a \in GL(n, \mathbb{C})$ and $x, y \in U(n)$,

$$\varPhi(xay) = \varPhi(a).$$

The purpose of this paper is to consider whether or not there exists a plurisubharmonic and strongly $(q+1)$-pseudoconvex exhaustion function in the sense of [1] on $G$ which is invariant on a given maximal compact real Lie subgroup $K$ and to show, as it's application, the existence of complete Kähler metric for every complex Lie group.

We wish to thank Prof. Hideki Kosaki and Prof. Mitsuo Morimoto who gave important advice for us to prove Lemma 3 and Theorem 3, respectively.

## §2. Linear groups and abelian Lie groups

A closed complex Lie subgroup of the complex general linear group $GL(k, \mathbb{C})$ for some positive integer $k$ is said to be a closed complex linear group.

LEMMA 1. *Let $G$ be a closed complex linear group and $K$ a maximal compact subgroup of $G$. Then there exists a $C^\infty$ function*

$$\varphi : G \longrightarrow \mathbb{R}$$

*such that*

(1) *$\varphi$ is strongly plurisubharmonic,*

(2) *$\varphi$ is an exhaustion function on $G$, i.e., for any $c \in \mathbb{R}$.*

$$\{x \in G | \varphi(x) < c\} \subset\subset G,$$

(3) *$\varphi$ is $K$-invariant, that is,*

$$\varphi(x) = \varphi(yxz)$$

*for any $x \in G, y, z \in K$.*

*Proof.* Let $G$ be a closed complex Lie subgroup of the complex general linear group $GL(k,\mathbb{C})$. There exists a maximal compact subgroup $K_1$ of $GL(k,\mathbb{C})$ such that

$$K \subset K_1.$$

Since all maximal compact subgroups are conjugate each other([2]), one can find $a \in GL(k,\mathbb{C})$ so that

$$a\,K_1\,a^{-1} \;=\; U(k),$$

where $U(k)$ is the unitary subgroup of $GL(k,\mathbb{C})$. Taking a function

$$\varphi(x) := \Phi(axa^{-1}),$$

where $\Phi(x) = \mathrm{trace}(x\;x^*) + \dfrac{1}{\det(x\;x^*)}$ , we get the assertion of this lemma.

In the case of complex abelian Lie groups we obtain a similar result as Lemma 1.

Lemma 2. *Let $G$ be a complex abelian Lie group of complex dimension $n$ and $K$ a maximal compact subgroup of $G$. Then there exists a $K$-invariant $C^\infty$ function*

$$\varphi : G \longrightarrow \mathbb{R}$$

*satisfying the same statements* (1) *and* (2) *in Theorem* 1.

*Proof.* Let $e$ be the unit element of the complex abelian Lie group $G$. Put

$$G^0 := \{x \,|\, f(x) = f(e) \text{ for every holomorphic function } f \text{ on } G\}$$

From the result of [7] $G^0$ is a complex Lie subgroup of $G$ which is a toroidal group, that is, a connected complex Lie group without non-constant holomorphic functions and there exists a lattice $\Gamma$ of $\mathbb{C}^m$ such that $G^0 \cong \mathbb{C}^m/\Gamma$, where $m$ is the complex dimension of $G^0$. We may assume $\Gamma = \mathbb{Z}\{e_1, e_2, \ldots, e_m, v_1, v_2, \ldots, v_{q_0}\}$, where $e_i$ is the $i$-th standard unit vector of $\mathbb{C}^m$ and $e_1, e_2, \ldots, e_m, v_1, v_2, \ldots, v_{q_0}$ are $\mathbb{R}$-linearly independent, and $K^0 \cong \mathbb{R}\{e_1, e_2, \ldots, e_m, v_1, v_2, \ldots, v_{q_0}\}/\Gamma$. Let $v_i = (v_{i1}, v_{i2}, \ldots, v_{im})$ and $v_i = \mathrm{Re}\, v_i + \sqrt{-1}\ \mathrm{Im}\, v_i$ ($\mathrm{Re}\, v_i, \mathrm{Im}\, v_i \in \mathbb{R}^m$). Since $e_1, e_2, \ldots, e_m, v_1, v_2, \ldots, v_{q_0}$ are $\mathbb{R}$ -linearly independent, $\mathrm{Im}\, v_1, \ldots, \mathrm{Im}\, v_{q_0}$ are $\mathbb{R}$ -linearly independent. Without loss of generality we may assume the $q_0 \times q_0$ real matrix

$\Big(v_{ij}\ ;\ 1 \leq i,j \leq q_0\Big)$ is invertible. We put $v_{q_0+1} := \sqrt{-1}e_{q_0+1}, v_{q_0+2} := \sqrt{-1}e_{q_0+2}, \dots, v_m := \sqrt{-1}e_m$. Then $e_1, e_2, \dots, e_m,\ v_1, v_2, \dots, v_{q_0}, v_{q_0+1}, \dots, v_m$ are $\mathbb{R}$-linearly independent vectors of $\mathbb{C}^m$. For any $z = (z_1, z_2, \dots, z_m)$ there exists a unique vector $(t_1, t_2, \dots, t_{2m}) \in \mathbb{R}^{2m}$ such that

$$z = \sum_{i=1}^{m} t_i e_i + \sum_{i=1}^{q_0} t_{m+i}\sqrt{-1}\ \operatorname{Im} v_i + \sum_{i=q_0+1}^{m} t_{m+i} v_i.$$

We define the exhaustion function

$$\psi : \mathbb{C}^m/\Gamma \ni z + \Gamma \longmapsto \sum_{i=q+1}^{m} t_{m+i}^2 \in \mathbb{R}$$

on $\mathbb{C}^m/\Gamma$. We denote by $A = \Big(a_{ij}\Big)$ the inverse matrix of

$$\begin{pmatrix} \operatorname{Im} v_1 \\ \cdot \\ \cdot \\ \cdot \\ \operatorname{Im} v_{q_0} \\ e_{q_0+1} \\ \cdot \\ \cdot \\ \cdot \\ e_m \end{pmatrix}.$$

Let $x_i := \operatorname{Re} z_i$ and $y_i := \operatorname{Im} z_i$. Then we have $t_{m+i} = \sum_{k=1}^m y_k\ a_{ki}$ $(i \geq q_0 + 1)$. The Levi form of $\psi$ is given by

$$\Big(\frac{\partial^2 \psi(z+\Gamma)}{\partial z_i \partial \bar{z}_j}\Big) = \frac{1}{4}\Big(\frac{\partial^2 \psi}{\partial y_i \partial y_j}\Big) = \frac{1}{2} B\ B^t,$$

where $B$ is the matrix

$$\begin{pmatrix} 0 & \dots & 0 & a_{1\ q_0+1} & \dots & a_{1\ m} \\ 0 & & 0 & a_{2\ q_0+1} & \dots & a_{2\ m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & a_{m\ q_0+1} & \dots & a_{m\ m} \end{pmatrix}.$$

Since $B$ is a real $(m,m)$-matrix of rank $m-q_0$, $\frac{1}{2}B\ B^t$ is positive semi-definite with $m-q_0$ positive eigenvalues. By the definition of $\psi$ we can see that $\psi(z+z^*+\Gamma)=\psi(z+\Gamma)$ for any $z\in\mathbb{C}^m$ and $z^*\in K^0$. This means $\psi$ is $K^0$-invariant. By the result of [8] $G\cong G^0\times\mathbb{C}^{*p}\times\mathbb{C}^r$ for some non-negative integers $p$ and $r$ with $m+p+r=n$. Then we may assume $G=G^0\times\mathbb{C}^{*p}\times\mathbb{C}^r$. We take the function

$$\varphi(z+\Gamma,\xi,\eta):=\psi(z+\Gamma)+\sum_{i=1}^{p}(\frac{1}{|\xi_i|^2}+|\xi_i|^2)+\sum_{j=1}^{r}|\eta_j|^2$$

for $(z+\Gamma,\xi,\eta)\in G^0\times\mathbb{C}^{*p}\times\mathbb{C}^r$ ,where $\xi=(\xi_1,\ldots,\xi_p)\in\mathbb{C}^{*p}$ and $\eta=(\eta_1,\ldots,\eta_r)\in\mathbb{C}^r$. Since $G$ is abelian, $G$ has the unique maximal compact subgroup $K=K^0\times\{\xi;|\xi_i|=1\}\times 0\subset G^0\times\mathbb{C}^{*p}\times\mathbb{C}^r$. It is easy to show that $\varphi$ is $K$-invariant and the Levi form of $\varphi$ has $n-q_0$ positive eigenvalues at every point of $G$. Let $\mathfrak{k}$ be the Lie algebra of $K$. Then

$$\mathfrak{k}\ =\ \mathbb{R}\{e_1,e_2,\ldots,e_m,v_1,v_2,\ldots,v_{q_0}\}\oplus\mathbb{R}^q\oplus 0$$

that is a real Lie subalgebra of the Lie algebra $\mathbb{C}^m\times\mathbb{C}^p\times\mathbb{C}^r$ of $G$.

Hence $\mathfrak{k}\cap\sqrt{-1}\mathfrak{k}=\mathbb{R}\{e_1,e_2,\ldots,e_m,v_1,v_2,\ldots,v_{q_0}\}\cap$ $\mathbb{R}\{\sqrt{-1}e_1,\sqrt{-1}e_2,\ldots,\sqrt{-1}e_m,\sqrt{-1}v_1,\sqrt{-1}v_2,\ldots,\sqrt{-1}v_{q_0}\}=$ $\mathbb{C}\{\operatorname{Im}v_1,\ldots,\operatorname{Im}v_{q_0}\}$. Then we have

$$q\ =\ \dim_{\mathbb{C}}\mathfrak{k}\cap\sqrt{-1}\mathfrak{k}\ =\ q_0.$$

## §3. Complete Kähler metric on a complex Lie group

Prof. Hiroshi Yamaguchi proposed the following question.

Question. *Is any complex Lie group a Kähler manifold?*

In this section we will give the affirmative answer to this question, that is, we will obtain the following theorem.

Theorem 2. *Any complex Lie group is complete Kähler.*

*Proof.* Let $\mathfrak{g}$ be the Lie algebra of $G$ and $K_C$ the complex Lie subgroup with the Lie subalgebra $\mathfrak{k}_C:=\mathfrak{k}+\sqrt{-1}\mathfrak{k}$. Then $K_C$ is a closed complex Lie subgroup of $G$([5]). Put $a:=\dim_C G/K_C$. Then $G$ is biholomorphic onto $K_C\times\mathbb{C}^a$ ([5]). From this fact we may assume $G=K_C$. By the result of [5] there exist closed connected complex Lie subgroups $S$ and $Z$ such that

(1) $S$ is semi-simple,

(2) $Z$ is the connected center of $G$,

(3)
$$\rho : Z \times S \ni (x, y) \longmapsto xy \in G$$
is a finite covering homomorphism.

$Z$ is a connected complex abelian Lie group and then is isomorphic onto $\mathbb{C}^p/\Gamma$ for some discrete subgroup $\Gamma$ of $\mathbb{C}^p$. Let $(z_1, ..., z_p)$ be the canonical coordinate system of $\mathbb{C}^p$. The complete Kähler metric

$$\sum_{i=1}^{p} dz_i d\overline{z_i}$$

induces a complete Kähler metric on $Z$ that is $\Gamma$-invariant. Since $\operatorname{Ker}\rho$ is a finite subgroup of $Z \times S$, there exist maximal compact subgroups $K_Z$ and $K_S$ of $Z$ and $S$, respectively such that $\operatorname{Ker}\rho \subset K_Z \times K_S$. Since $S$ is semi-simple, $S$ is isomorphic onto a complex linear group and further we may assume $S$ is a closed complex Lie subgroup of $GL(k, \mathbb{C})$ for some positive integer $k$ ([6]). By Lemma 1 there exists a $C^\infty$ strongly plurisubharmonic exhaustion function

$$\varphi : S \longrightarrow \mathbb{R}$$

that is $K_S$-invariant. The form

$$\sum_{i=1}^{p} dz_i d\overline{z_i} + \sum_{j,\ell=1}^{s} \frac{\partial^2 \varphi}{\partial w_j \partial \overline{w_\ell}} dw_j d\overline{w_\ell}$$

is $\operatorname{Ker}\rho$-invariant and then induces a Kähler metric on $G \cong Z \times S/\operatorname{Ker}\rho$, where $s := \dim_{\mathbb{C}} S$ and $(w_1, ..., w_s)$ is a local coordinate system of $S$. From the technique of Nakano (Proposition 1, [9]), we can find a strictly convex increasing $C^\infty$ function

$$\chi : (0, \infty) \longrightarrow (0, \infty)$$

such that the Kähler metric

$$\sum_{i=1}^{p} dz_i d\overline{z_i} + \sum_{j,\ell=1}^{s} \frac{\partial^2 \chi(\varphi)}{\partial w_j \partial \overline{w_\ell}} dw_j d\overline{w_\ell}$$

is complete.

## §4. Invariant plurisubharmonic exhaustion functions

LEMMA 3. *Let $K$ be a compact topological space with a positive measure $m$ and $a_{ij}(x) : K \longrightarrow \mathbb{C}$ be continuous functions for $1 \leq i, j \leq n$. If $n \times n$ matrix*

$$A(x) := \Big( a_{ij}(x) \Big)$$

*is positive semidefinite Hermitian and has $n-q$ positive eigenvalues at any point $x$ of $K$, then*

$$B := \Big( \int_K a_{ij}(x) dm \Big)$$

*is positive semidefinite and has at least $n-q$ positive eigen values.*

*Proof.* We put $w := {}^t(w_1, ..., w_n)$ for $(w_1, ..., w_n) \in \mathbb{C}^n$. Since

$$(B\ w, w) := \sum_{i,j=1}^{n} w_j \int_K a_{ij}(x) dm\ \overline{w_i} = \int_K \sum_{i,j=1}^{n} w_j\ a_{ij}(x)\ \overline{w_i} dm \geq 0,$$

the matrix $B$ is positive semidefinite Hermitian. Then there exists a unitary matrix $U$ such that $B = {}^t\overline{U}\ \Lambda\ U$, where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \lambda_n \end{pmatrix}$$

and $\lambda_i$ are non-negative eigenvalues of $B$. We get a positive semidefinite Hermitian matrix

$$\sqrt{B} :=^t \overline{U} \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & \sqrt{\lambda_2} & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \sqrt{\lambda_n} \end{pmatrix} U.$$

If $(B\ w, w) = 0$, then $((\sqrt{B})^2\ w, w) = (\sqrt{B}\ w, \sqrt{B}w) = 0$ and hence $\sqrt{B}w = 0$. $\sqrt{B}w = 0$ implies $B\ w = (\sqrt{B})\ (\sqrt{B})\ w = 0$. This shows that

$$B\ w = 0$$

if and only if

$$(B\ w, w) = \sum_{i,j=1}^{n} w_j \int_K a_{ij}(x) dm\ \overline{w_i} = 0.$$

We consider the eigen space

$$E_0 := \{\, w \mid B\ w\ =\ 0 \,\} = \{\, w \mid (B\ w, w)\ =\ 0 \,\}$$

of the eigenvalue 0 of $B$. Since

$$(B\ w, w) = \int_K \sum_{i,j=1}^{n} w_i\ a_{ij}(x)\ \overline{w_j}\ d\ m = 0,$$

we have $\sum_{i,j=1}^{n} w_j\ a_{ij}(x)\ \overline{w_i} = 0$ for any $x \in K$. Then we obtain

$$E_0\ \subset\ \cap_{x\in K}\{\, w \mid A(x)\ w = 0 \,\}\ \subset \{\, w \mid A(x_0)\ w = 0 \,\}$$

for any point $x_0 \in K$. From the assumption it follows that

$$\dim_C\ \{\, w \mid A(x_0)\ w = 0 \,\}\ \leq\ q$$

and then $\dim_C\ E_0\ \leq\ q$.

The following Theorem 3 implies Theorem 1. Moreover, using a different method from those given by [3], [4] and S. Takeuchi [10], we prove Theorem 3 here.

Theorem 3. *Let $G$ be a complex Lie group, $K$ a maximal compact subgroup of $G$. Put $q := \dim_{\mathbb{C}} \mathfrak{k} \cap \sqrt{-1}\mathfrak{k}$. Then there exists a $C^\infty$ function*

$$\varphi : G \longrightarrow \mathbb{R}$$

*such that*

(1) *$\varphi$ is plurisubharmonic and strongly $(q+1)$-pseudoconvex on $G$ in the sense of* [1] ,

(2) *$\varphi$ is an exhaustion function on $G$, i.e., for any $c \in \mathbb{R}$.*

$$\{z \in G \mid \varphi(z) < c\,\} \subset\subset G,$$

(3) *$\varphi$ is $K$-invariant, that is,*

$$\varphi(z) = \varphi(a\ z\ b)$$

*for any $z \in G, a, b \in K$.*

*Proof.* We obtain closed connected Lie subgroups $K_C$, $S$, $Z$, $K_S$, $K_Z$ and a finite covering homomorphism

$$\rho : Z \times S \ni (x, y) \longmapsto x\ y \in K_C$$

and $G$ is biholomorphic onto $K_C \times \mathbb{C}^a$ as in the proof of Theorem 2. Put $n := \dim\ G$, $n_1 := \dim\ Z$ and $n_2 := \dim\ S$ $(\,n = n_1 + n_2 + a\,)$. Since $Z$ is abelian, we have isomorphisms

$$Z \;\cong\; G^0 \times \mathbb{C}^{*p} \times \mathbb{C}^r$$

and $G^0 \;\cong\; \mathbb{C}^m/\Gamma$ $(\,n_1 = m + p + r\,)$, where we put $G^0 := \{x \,|\, f(x) = f(e)$ for every holomorphic function $f$ on $Z\,\}$ and $\Gamma = \mathbb{Z}\{e_1, e_2, \ldots, e_m, v_1, v_2, \ldots, v_{q_0}\}$, using the notations in the proof of Lemma 2. By Lemmas 1 and 2 we obtain a strongly plurisubharmonic exhaustion function $\varphi_S : S \longrightarrow \mathbb{R}$ and a plurisubharmonic exhaustion function $\varphi_Z : Z \longrightarrow \mathbb{R}$, where the Levi form of $\varphi_Z$ has $n_1 - q_0$ positive eigen values at every point. Since $\varphi_S$ and $\varphi_Z$ are $K_S$- and $K_Z$-invariant, respectively and $\operatorname{Ker}\rho \subset K_Z \times K_S$, they induce a plurisubharmonic exhaustion function

$$\varphi_{K_C} := \varphi_Z + \varphi_S : K_C \longrightarrow \mathbb{R}.$$

The Levi form of $\varphi_{K_C}$ has $n - a - q_0$ positive eigenvalues at every point. We take a strongly plurisubharmonic exhaustion function

$$\varphi_a : \mathbb{C}^a \ni (w_1, ..., w_a) \mapsto \sum_{i=1}^{a} |w_i|^2 \in \mathbb{R}.$$

We put

$$\varphi_G := \varphi_{K_C} + \varphi_a : G \longrightarrow \mathbb{R}.$$

Then $\varphi_G$ is a plurisubharmonic exhaustion function whose Levi form has $n - q_0$ positive eigenvalues at every points. Since the maximal compact subgroup $K$ is a Lie group, we can obtain positive left (or right) invariant Haar measure $\mu_\ell$ (or $\mu_r$) on $K$, respectively. Finally we obtain a function

$$\varphi( z ) \ := \int_{x\in K}\int_{y\in K} \varphi_G( y\ z\ x )\ d\ \mu_\ell(x)\ d\ \mu_r(y).$$

Since it's Levi form is

$$\Big(\frac{\partial^2\varphi( z )}{\partial z_i\partial\overline{z_j}}\Big) \ = \ \Big(\int_{x\in K}\int_{y\in K} \frac{\partial^2\varphi_G( y\ z\ x )}{\partial z_i\partial\overline{z_j}}\ d\ \mu_\ell(x)\ d\ mu_r(y)\Big),$$

by Lemma 3 this is positive semi-definite and has at least $n - q_0$ positive eigenvalues at every point of $G$. Furthermore, for $a, b \in K$

$$\begin{aligned}\varphi( a\ z\ b ) \ &= \int_{x\in K}\int_{y\in K} \varphi_G( (y\ a)\ z\ (b\ x) )\ d\ \mu_\ell(x)\ d\ \mu_r(y)\\ &= \int_{x\in K}\int_{y\in K} \varphi_G( y\ z\ x )\ d\ \mu_\ell(x)\ d\ \mu_r(y)\\ &= \varphi( z ).\end{aligned}$$

Let $\mathfrak{k}_Z$ and $\mathfrak{k}_S$ be the Lie algebras of $K_Z$ and $K_S$, respectively. There exists $a \in G$ such that $K_S \subset aU(k)a^{-1}$. The isomorphism $\alpha : G \ni x \mapsto axa^{-1} \in G$ induces the isomorphism $\alpha : \mathfrak{g} \mapsto \mathfrak{g}$ of complex Lie algebras. Since the Lie algebra $\mathfrak{u} = \{\, x \ : \ \overline{{}^t x} + x \ = \ 0 \,\}$ of all skew-Hermitian matrices is the Lie algebra of the unitary group $U(k)$ and $\mathfrak{k}_S \cap \sqrt{-1}\mathfrak{k}_S \subset \alpha(\mathfrak{u} \cap \sqrt{-1}\mathfrak{u})$, we have $\mathfrak{u} \cap \sqrt{-1}\mathfrak{u} \ = \ 0$ and then

$$q = \dim_{\mathbb{C}} \mathfrak{k} \cap \sqrt{-1}\mathfrak{k} = \dim_{\mathbb{C}} \mathfrak{k}_Z \cap \sqrt{-1}\mathfrak{k}_Z = q_0.$$

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