H. Umemura
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ON THE TRANSFORMATION GROUP OF THE SECOND PAINLEVÉ EQUATION

HIROSHI UMEMURA

Abstract. We show that for the second Painlevé equation $y'' = 2y^3 + ty + \alpha$, the Bäcklund transformation group G, which is isomorphic to the extended affine Weyl group of type $\hat{A}_1$, operates regularly on the natural projectification $\mathcal{X}(c)/\mathbb{C}(c,t)$ of the space of initial conditions, where $c = \alpha - 1/2$. $\mathcal{X}(c)/\mathbb{C}(c,t)$ has a natural model $\mathcal{X}[c]/\mathbb{C}(t)[c]$. The group G does not operate, however, regularly on $\mathcal{X}[c]/\mathbb{C}(t)[c]$. To have a family of projective surfaces over $\mathbb{C}(t)[c]$ on which G operates regularly, we have to blow up the model $\mathcal{X}[c]$ along the projective lines corresponding to the Riccati type solutions.

§1. Introduction

As is well known, the (extended) affine Weyl group of type $\tilde{A}_1$ appears as a transformation group of solutions of the second Painlevé equation

$$P_{II}(\alpha) : y'' = 2y^3 + ty + \alpha,$$

where t is the independent variable, $y'' = d^2y/dt^2$ and $\alpha \in \mathbb{C}$ is a parameter. If y is a solution of the second Painlevé equation $P_{II}(\alpha)$, then

$$T_+(y) = -y - \frac{\alpha + \frac{1}{2}}{y' + y^2 + \frac{t}{2}}$$

is a solution of $P_{II}(\alpha + 1)$,

$$T_-(y) = -y + \frac{\alpha - \frac{1}{2}}{y' - y^2 - \frac{t}{2}}$$

is a solution of $P_{II}(\alpha - 1)$ and $I(y) = -y$ is a solution of $P_{II}(-\alpha)$. Let G be the subgroup of the affine transformation group of the affine line $\mathbb{A}^1$ generated by the translations

$$t_+(\alpha) = \alpha + 1, \quad t_-(\alpha) = \alpha - 1$$

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and the reflection

$$i(\alpha) = -\alpha$$

at 0 for $\alpha \in \mathbb{C}$. So G is the affine Weyl group of type $\tilde{A}_1$. We consider the affine space $\mathbb{A}^4$ with coordinate system (y, y', t, α) as well as the affine plane with coordinate system (t, α). We have a vector field

$$\delta(\alpha) = \frac{\partial}{\partial t} + y' \frac{\partial}{\partial y} + (2y^3 + ty + \alpha)\frac{\partial}{\partial y'}$$

on $\mathbb{A}^4$ and a natural fibration $\pi : \mathbb{A}^4 \to \mathbb{A}^2$ by projection $(y, y', t, \alpha) \mapsto (t, \alpha)$. The affine Weyl group G operates on the affine plane $\mathbb{A}^2$ through the second coordinate. The transformations T_+, T_-, I define a birational operation of the affine Weyl group G on $\mathbb{A}^4$ compatible with the derivation $\delta(c)$ such that the fibration $\pi : \mathbb{A}^4 \to \mathbb{A}^2$ is G-equivariant. We construct in this note a projective model of the fibration $\pi : \mathbb{A}^4 \to \mathbb{A}^2$ on which the Weyl group G operates regularly. In fact we construct a projective model $\mathcal{X}$ of the generic fiber of the fibration $\pi : \mathbb{A}^4 \to \mathbb{A}^2$ such that the affine Weyl group G operates regularly on $\mathcal{X}$ (Theorem 2.11). The model $\mathcal{X}$ is the projective surface studied by Okamoto [O1]. More precisely his space of initial conditions is our projective surface $\mathcal{X}$ minus 8 non-singular rational curves with self-intersection number -2 whose dual graph is the extended Dynkin diagram of type $\widetilde{E}_7$. We recall the construction of $\mathcal{X}$ in §2. We first projectify the affine plane $\mathbb{A}^2_{\mathbb{C}(\alpha,t)}$ to a ruled surface isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and then blow up it 8 times to get $\mathcal{X}$. The construction of such a model $\mathcal{Y}$ over $\mathbb{A}^2$ is more subtle. We construct a model $\mathcal{Y}$ that is a complex manifold but is not an algebraic variety (Theorem 4.10). In this paper the ground field is an arbitrary field K of characteristic 0. So $K = \mathbb{Q}$ is the most natural but the readers who are interested in analysis may assume $K = \mathbb{C}$.

§2. Construction of the model

We know that the Painlevé equation

$$P_{II}(\alpha) : y'' = 2y^3 + ty + \alpha$$

is equivalent to both

$$S_{II}(\alpha) : \begin{cases} \dfrac{dq}{dt} = p - q^2 - \dfrac{t}{2}, \\ \dfrac{dp}{dt} = 2pq + \alpha + \dfrac{1}{2}, \end{cases}$$

and

$$S_2(c) : \begin{cases} \dfrac{dq}{dt} = q^2 + p + \dfrac{t}{2}, \\ \dfrac{dp}{dt} = -2qp + c, \end{cases}$$

where $c = \alpha - 1/2$ that is denoted by ε by Okamoto (cf. [O1, p. 50]).

Remark. We consider that the parameter α or c belongs to an extension field or more generally to an over ring of the base field K consisting of constants.

We used $S_{II}(\alpha)$ in [UW]. In this paper we adopt $S_2(\alpha)$. Let us denote by $Sol_2(c)$ the set of solutions of the system $S_2(c)$. We have transformations

$$\begin{aligned} T_+(c, c+1) : Sol_2(c) &\longrightarrow Sol_2(c+1), \\ T_-(c, c-1) : Sol_2(c) &\longrightarrow Sol_2(c-1), \\ I(c, -c) : Sol_2(c) &\longrightarrow Sol_2(-c). \end{aligned}$$

The definition of these transformations is as follows. Let $(q, p) \in Sol_2(c)$.
(i) If $2q^2 + q + 1 \neq 0$, then

$$T_+(c, c+1)(q, p) = \left(-q - \frac{c+1}{2q^2 + p + t}, -2q^2 - p - t\right). \tag{2.1}$$

If $2q^2 + p + t = 0$, then $c = -1$ and

$$T_+(-1, 0)(q, p) = (-q, -2q^2 - p - t).$$

(ii) If $p \neq 0$, then

$$T_-(c, c-1)(q, p) = \left(-q + \frac{c}{p}, -p - 2\left(q - \frac{c}{p}\right)^2 - t\right). \tag{2.2}$$

If $p = 0$, then $c = 0$ and

$$T_-(0, -1)(q, p) = (-q, -2q^2 - p - t).$$

(iii) If $p \neq 0$,

$$I(c, -c)(q, p) = \left(q - \frac{c}{p}, p\right).$$

If $p = 0$, then $c = 0$ and $I(c,-c)(q,p) = (q,p)$. It is easy to check that

$$I(-c,c) \circ I(c,-c) = \mathrm{Id}_{Sol_2(c)}, \tag{2.3}$$

$$\begin{cases} T_-(c+1,c) \circ T_+(c,c+1) = \mathrm{Id}_{Sol_2(c)}, \\ T_+(c-1,c) \circ T_-(c,c-1) = \mathrm{Id}_{Sol_2(c)}, \end{cases} \tag{2.4}$$

and

$$I(-c+1,c-1) \circ T_+(-c,-c+1) \circ I(c,-c) = T_-(c,c-1)$$

for every c. From now on, we assume that we assume that the parameter c is a variable over $K(t)$ and it is convenient to set $L = K(c,t)$. Let now q_c, p_c be variables over L and we consider the polynomial ring $R(c) := L[q_c, p_c]$. If we consider the derivation

$$D(c) = \frac{\partial}{\partial t} + \left(q_c^2 + p_c + \frac{t}{2}\right)\frac{\partial}{\partial q_c} + (-2q_cp_c + c)\frac{\partial}{\partial p_c} : R(c) \longrightarrow R(c),$$

$R(c)$ is a differential algebra and we have

$$\begin{cases} D(c)(q_c) = q_c^2 + q_c + \dfrac{t}{2}, \\ D(c)(p_c) = -2q_cp_c + c, \end{cases} \tag{2.5}$$

i.e., (q_c, p_c) is a solution of the system $S_2(c)$ and for every solution (q,p) of $S_2(c)$ we have a differential L-morphism

$$L[q_c, p_c] \longrightarrow L[q,p] \tag{2.6}$$

of differential algebras sending q_a, p_a respectively to q, p. In fact let Q, P be differential variables over L so that $L\{Q,P\} = L[Q,P,Q',P',\ldots]$ is a differential polynomial ring and we have a differential L-morphism

$$\Phi : L\{Q,P\} \longrightarrow L[q,p]$$

of differential L-algebras sending Q, P respectively to q, p. Since the differential ideal $I(c)$ of the differential algebra $L\{q,p\}$ that is differentially generated by

$$\delta Q - Q^2 - P - \frac{t}{2}, \quad \text{and} \quad \delta P + 2QP - c$$

of the differential polynomial ring $L\{Q, P\}$ is in $\operatorname{Ker}\Phi$, the morphism Φ factors through the residue class map

$$L\{Q, P\} \longrightarrow L\{Q, P\}/I(c) = R(c)$$

and induces a differential L-morphism (2.6). We denote the quotient field of $R(c)$ by $Q(c)$, which is a differential field. We consider $X(c) := \operatorname{Spec} L[q_c, p_c]$ that is nothing but the affine plane $\mathbb{A}^2_L$ over L with the coordinate system (q_c, p_c) endowed with the derivation $\delta(c)$. Since (q_c, p_c) is a solution of $S_2(c)$, it follows from (2.1)

$$T_+(c, c+1)(q_c, p_c) = \left(-q_c - \frac{c+1}{2q_c^2 + p_c + t}, -2q_c^2 - p_c - t\right)$$

is a solution of $S_2(c+1)$. So by (2.6) we have an L-differential morphism

$$R(c+1) \longrightarrow Q(c) \tag{2.7}$$

sending

$$q_{c+1} \text{ to } -q_c - \frac{c+1}{2q_c^2 + p_c + t} \quad \text{and} \quad p_{c+1} \text{ to } -2q_c^2 - p_c - t.$$

Now by (2.4) the morphism (2.7) is birational or it induces an isomorphism

$$Q(c+1) \longrightarrow Q(c)$$

of the differential quotient fields. In other words we have an L-birational map

$$X(c) = \mathbb{A}^2_L = \operatorname{Spec} R(c) \cdots\rightarrow X(c+1) = \mathbb{A}^2_L = \operatorname{Spec} R(c+1) \tag{2.8}$$

compatible with the derivations $\delta(c+1)$ and $\delta(c)$. On the other hand the $K(t)$-isomorphism

$$L = K(c, t) \longrightarrow L = K(c, t), \quad c \longmapsto c+1 \tag{2.9}$$

induces a differential $K(t)$-isomorphism

$$R(c) = L[q_c, p_c] \longrightarrow R(c+1) = L[q_{c+1}, p_{c+1}]$$

sending

$$q_c \longmapsto q_{c+1}, \quad p_c \longmapsto p_{c+1} \quad \text{and} \quad c \longmapsto c+1. \tag{2.10}$$

Composing the isomorphism (2.9) and the L-birational map (2.8), we get a $K(t)$-birational map

$$T_+ : X(c) = \operatorname{Spec} L[q_c, p_c] \longrightarrow X(c+1) = \operatorname{Spec} L[q_{c+1}, p_{c+1}] \cdots \rightarrow X(c) = \operatorname{Spec} L[q_c, p_c]$$

compatible with the derivations such that the diagram

$$\begin{array}{ccc} X(c) & \xrightarrow{T_+} & X(c) \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\ \operatorname{Spec} L & \xrightarrow{t_+} & \operatorname{Spec} L \end{array}$$

is commutative, where the vertical arrow p is the projection and the lower horizontal arrow is the morphism of schemes induced by the isomorphism (2.9). Similarly we have a differential birational map

$$T_- : X(c) \cdots \rightarrow X(c)$$

such that the diagram

$$\begin{array}{ccc} X(c) & \xrightarrow{T_-} & X(c) \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\ \operatorname{Spec} L & \xrightarrow{t_-} & \operatorname{Spec} L \end{array}$$

is commutative, where the lower horizontal arrow is the morphism of schemes induced by the $K(c)$-morphism $L \to L$ of differential fields sending c to $c+1$. We also have a differential birational map

$$I : X(c) \cdots \rightarrow X(c)$$

such that the diagram

$$\begin{array}{ccc} X(c) & \xrightarrow{I} & X(c) \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\ \operatorname{Spec} L & \xrightarrow{i} & \operatorname{Spec} L \end{array}$$

is commutative, where the lower horizontal map is induced by the $K(t)$-isomorphism $L \to L$ of differential fields sending c to $-c$. Let

$$t_+^*, \quad t_-^*, \quad i^*$$

be respectively K-automorphisms of $K(c)$ such that

$$t_+^*(c) = c + 1, \quad t_-^*(c) = c - 1, \quad i^*(c) = -c.$$

They define K-automorphisms t_+, t_-, i of the scheme $\operatorname{Spec} K(c)$. Let G be the subgroup of the automorphism of the scheme $\operatorname{Spec} K(t)$ generated by the automorphisms t_+, t_-, i so that G is the affine Weyl group of type $\tilde{A}_1$. We have

$$t_+^2 = t_-^2 = t_+ \circ t_- = t_- \circ t_+ = i^2 = \mathrm{Id}, \quad i \circ t_+ \circ i = t_-$$

and $G = \langle t_+, t_- \rangle \rtimes \langle i \rangle \simeq \mathbb{Z}^2 \rtimes \mathfrak{S}_2$, where $\mathfrak{S}_2$ is the symmetric group of degree 2. Let $\widetilde{G}$ be the subgroup of the birational automorphisms of $X(c)$ generated by T_+, T_-, I. So we have a natural morphism $\varphi : \widetilde{G} \to G$ of groups by the commutative diagrams above. We can check

$$T_+ \circ T_- = T_- \circ T_+ = S^2 = \mathrm{Id}, \quad \text{and} \quad S \circ T_+ \circ S = T_-$$

so that φ is an isomorphism. Namely the (extended) affine Weyl group G of type $\tilde{A}_1$ birationally operates on the scheme $X(c) = \mathbb{A}_L^2$ in such a way that the diagram

$$\begin{array}{ccc} X(c) & \xrightarrow{\phi_g} & X(c) \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\ \operatorname{Spec} L & \xrightarrow{\psi_g} & \operatorname{Spec} L \end{array}$$

is commutative for every $g \in G$, where ϕ_g is the birational automorphism of $X(c)$ induced the element $g \in G$ and ψ_g is the $K(t)$-automorphism of $\operatorname{Spec} L$ defined by the operation of the element $g \in G$. We projectify the affine plane $X(c) = \mathbb{A}_L^2$ with derivation. We prepare four copies W_i $(1 \le i \le 4)$ of the affine plane $\mathbb{A}_L^2$ and glue them by the following rule to get the projective model $Z(c)$ that is denoted by $\Sigma_{(\varepsilon)}^{(2)}$ in [O1] with $\varepsilon = c$. Let (y_i, z_i) be the coordinate system of W_i $(1 \le i \le 4)$ so that $W_i = \operatorname{Spec} L[y_i, z_i]$.

(i) A point $(y_1, z_1) \in W_1$ and a point $(y_2, z_2) \in W_2$ are identified if

$$y_1 = y_2 \quad \text{and} \quad z_1 z_2 = 1.$$

(ii) A point $(y_1, z_1) \in W_1$ and a point $(y_3, z_3) \in W_3$ are identified if

$$y_1 y_3 = 1 \quad \text{and} \quad z_1 = c y_3 - y_3^2 z_3.$$

We notice here that the latter condition is equivalent to $z_3 = c y_1 - y_1^2 z_1$.

(iii) A point $(y_3, z_3) \in W_3$ and a point $(y_4, z_4) \in W_4$ are identified if

$$y_3 = y_4 \quad \text{and} \quad z_3 z_4 = 1.$$

The projections

$$W_i \longrightarrow \mathbb{A}^1_L, \quad (y_i, z_i) \longmapsto y_i$$

for $1 \leq i \leq 4$ glue together and give a morphism

$$Z(c) \longrightarrow \mathbb{P}^1_L.$$

Namely $Z(c)$ is a $\mathbb{P}^1_L$-bundle over $\mathbb{P}^1_L$ or $Z(c)$ is a rational ruled surface known to be isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The curves $z_2 = 0$ in W_2 and $z_4 = 0$ in W_4 also glue together and give a section D_1 of the ruled surface $Z(c) \to \mathbb{P}^1_L$ such that $D_1^2 = 2$. We embed the affine plane $X(c)$ over L in $Z(c)$ by identifying it with W_1 by an L-isomorphism

$$L[y_1, z_1] \longrightarrow L[q_c, p_c], \quad y_1 \longmapsto q_c, \;\; z_1 \longmapsto p_c.$$

The derivation $\delta(c)$ of $L[q_c, p_c]$ defines a rational derivation on the scheme $Z(c)$. We notice here that since $\delta(t) = 1$, the rational derivation on $Z(c)$ is not a vector field on an variety over L even on an open set of $Z(c)$, for $Z(c)$ is defined over the field $L = K(t, c)$. Okamoto constructed the space of initial conditions of the second Painlevé equation $P_{II}(\alpha)$ by blowing up the projective surface $Z(c)$ over L at 8 points. They are infinitely near points of $(y_4, z_4) = (0, 0)$ on $W_4 \subset Z(c)$. Namely they are the point $(y_4, z_4) = (0, 0)$ in W_4 and 7 other points lying on the exceptional divisor that is contracted to the point $(y_4, z_4) = (0, 0) \in W_4$. Let us denote the thus obtained surface by $\mathcal{X}(c)$. On $\mathcal{X}(c)$ there are 8 curves D_i isomorphic to $\mathbb{P}^1_L$ with $D_i^2 = -2$ $(1 \leq i \leq 8)$. The space of initial conditions of the second Painlevé equation is defined as $\mathcal{X}(c) - \bigcup_{i=1}^{8} D_i$ (cf. [O1, Chap. III, §1]).

THEOREM 2.11. *The rational operation of the group G on $X(c)$ gives a regular operation of G on the projective model $\mathcal{X}(c)$ of $X(c)$.*

We prove the Theorem in §3. To explain the construction of $\mathcal{X}(c)$, we need the notation. Let W be the affine plane A^2_M with a coordinate system (y, z), i.e., $W = \operatorname{Spec} M[y, z]$, where M is a field. The blow-up $p : \widetilde{W} \to W$ of W at $(y, z) = (0, 0)$ is by definition

$$\widetilde{W} = \{(y, z; (x_0, x_1)) \in W \times \mathbb{P}^1_M \mid yx_0 = zx_1\}$$

and the morphism $p : \widetilde{W} \to W$ is induced by the projection $W \times \mathbb{P}^1_M \to W$. Let us denote by $W(y)$ the open subset

$$\{(y, z; (x_0, x_1)) \in \widetilde{W} \mid x_0 \neq 0\}$$

of $\widetilde{W}$. Then writing $x_1/x_0 = Y$, we have an isomorphism

$$W(y) \longrightarrow \mathbb{A}^2_M, \quad (y, z; (x_0, x_1)) \longmapsto \left(\frac{x_1}{x_0}, z\right) = (Y, z).$$

In fact the inverse map $\mathbb{A}^2_M \to W(y)$ is given by $(Y, z) \mapsto (Yz, z; (1, Y))$. Namely the open subset $W(y) \in \widetilde{W}$ is isomorphic to the affine plane $\mathbb{A}^2_M$ with the coordinate system (Y, z). Similarly we write $x_0/x_1 = Z$. Then the open subset

$$W(z) := \{(y, z; (x_0, x_1)) \in \widetilde{W} \mid x_1 \neq 0\}$$

of $\widetilde{W}$ is isomorphic to $\mathbb{A}^2_M$ by sending $(y, z; (x_0, x_1))$ to $(y, x_0/x_1) = (y, Z)$. In other words the open subset $W(z) \subset \widetilde{W}$ is isomorphic to the affine plane $\mathbb{A}^2_M$ with the coordinate system (y, Z). So $\widetilde{W}$ is covered by the two open subsets $W(y)$, $W(z)$ isomorphic to the affine plane $\mathbb{A}^2_M$. On the open subset $W(y) \simeq \mathbb{A}^2_L$ of $\widetilde{W}$ the projection

$$p : \widetilde{W} \longrightarrow W, \quad (y, z; (x_0, x_1)) \longmapsto (y, z)$$

is written in terms of the coordinate system (Y, z) as

$$(Y, z) \longmapsto (Yz, z)$$

and similarly on the other open subset $W(z)$ of $\widetilde{W}$ the projection $p : \widetilde{W} \to W$ is written in terms of the coordinate system (y, Z) on $\mathbb{A}^2_M$ as

$$(y, Z) \longmapsto (y, yZ).$$

Here is the construction of $\mathcal{X}(c)$. The center a_1 of the first blow-up of $Z(c)$ is the point $(y_4, z_4) = (0, 0)$ on $W_4 = \mathbb{A}^2$. Let $Z_1(c)$ be the blow-up of $Z(c)$ at $(y_4, z_4) = (0, 0)$. Ignoring the index 4, we use the above convention for W_4 and $M = L$. We have the blow-up

$$p_4 : \widetilde{W}_4 \longrightarrow W_4$$

and $\widetilde{W}_4$ is covered by two open subsets $W_4(y)$, $W_4(z)$ both isomorphic to $\mathbb{A}^2_L$ with the coordinate systems (Y, z) and (y, Z). So $Z_1(c)$ is covered by 5 open subsets isomorphic to $\mathbb{A}^2$: W_1, W_2, W_3, $W_4(y)$, $W_4(z)$. Then the center a_2 of the second blow-up of $Z_1(c)$ is the point $(y, Z) = (0, 0)$ on $W_4(z) = \mathbb{A}^2$. We denote by $Z_2(c)$ the thus obtained surface. To simplify the notation, we set $W_5 = W_4(z)$ that is the affine plane, and we denote the coordinate system (y, Z) of the affine plane W_5 by (y, z) which we should not confuse with the coordinate system on W_4. So we get the blow-up

$$p_5 : \widetilde{W}_5 \longrightarrow W_5$$

of W_5 at $(y, z) = (0, 0)$. $\widetilde{W}_5$ is covered by two open subsets $W_5(y)$ and $W_5(z)$ with the coordinate systems (Y, z) and (y, Z) respectively both isomorphic to the affine plane $\mathbb{A}^2$. Therefore $Z_2(c)$ is covered by 6 affine planes:

$$W_1,\ W_2,\ W_3,\ W_4(y),\ W_5(y),\ W_5(z).$$

This procedure is repeated to get the third and fourth centers a_3, a_4. Namely $W_5(z)$ is the affine plane with the coordinate system (y, Z) and we set $W_6 = W_5(z)$ and denote Z by z so that the coordinate system on the affine plane W_6 is (y, z). The center a_3 of the third blow-up is the point $(y, z) = (0, 0)$ of the affine plane $W_6 = W_5(z) \subset Z_2(c)$. So we blow up $Z_2(c)$ at the point $(y, z) = (0, 0)$ in W_6 to obtain $Z_3(c)$. Locally on W_6, we get the blow-up

$$p_6 : \widetilde{W}_6 \longrightarrow W_6,$$

which is covered by two open subsets $W_6(y)$ and $W_6(z)$ that are affines planes with coordinate systems (Y, z) and (y, Z). So $Z_3(c)$ is covered by 7 affine planes

$$W_1,\ W_2,\ W_3,\ W_4(y),\ W_5(y),\ W_6(y),\ W_6(z).$$

The center a_4 of the fourth blow-up to get $Z_4(c)$ is the point $(y, Z) = (0, 0)$ on $W_6(z) \subset Z_3(c)$. So it is convenient to denote Z by z and we set $W_7 =$

$W_6(z)$ so that W_7 is the affine plane with the coordinate system (y, z). Locally we have the blow-up

$$p_7 : \widetilde{W}_7 \longrightarrow W_7.$$

Then $\widetilde{W}_7$ is covered by two open subsets $W_7(y)$ and $W_7(z)$ isomorphic to the affine plane with the coordinate systems (Y, z) and (y, Z). So $Z_4(c)$ is covered by 8 affine planes:

$$W_1,\ W_2,\ W_3,\ W_4(y),\ W_5(y),\ W_6(y),\ W_7(y),\ W_7(z).$$

The center of the fifth blow-up is the point $(y, Z) = (0, 1/2)$ of $W_7(z)$. So we denote $W_7(z)$ by W_8. Now we introduce a new coordinate system (y, z) on the affine plane $W_7(z)$ by setting $y = y$, $Z = (z + 1)/2$. We use this new coordinate system on W_8 so that the center a_5 of the fifth blow-up is the point $(y, z) = (0, 0)$ on W_8 in terms of the new coordinate system. Therefore we get the blow-up

$$Z_5(c) \longrightarrow Z_4(c).$$

Locally on W_8 we have the blow-up

$$p_8 : \widetilde{W}_8 \longrightarrow W_8$$

and $\widetilde{W}_8$ is covered by two open subsets $W_8(y)$ and $W_8(z)$ that have the coordinate systems (Y, z) and (y, Z) respectively. So $Z_5(c)$ is covered by 9 affine planes:

$$W_1,\ W_2,\ W_3,\ W_4(y),\ W_5(y),\ W_6(y),\ W_7(y),\ W_8(y),\ W_8(z).$$

The center a_6 of the sixth blow-up to get $Z_6(c)$ is the point $(y, Z) = (0, 0)$ of $W_8(z)$. Therefore to simplify the notation, we set $W_9 = W_8(z)$ and we denote Z by z so that W_9 is the affine plane with the coordinate system (y, z). We get the blow-up

$$Z_6(c) \longrightarrow Z_5(c)$$

and locally we have a blow-up morphism

$$p_9 : \widetilde{W}_9 \longrightarrow W_9.$$

So the surface $\widetilde{W_9}$ is covered by two open subsets $W_9(y)$ and $W_9(z)$ isomorphic to the affine plane with coordinate systems (Y, z) and (z, Y) respectively. The surface $Z(6)$ is covered by 10 affine planes:

$$W_1,\ W_2,\ W_3,\ W_4(y),\ W_5(y),\ W_6(y),\ W_7(y),\ W_8(y),\ W_9(y),\ W_9(z).$$

The center of the seventh blowing up is $(y, Z) = (0, -t/2)$ on $W_9(z) \subset Z_6(c)$. So we set $W_{10} = W_9(z)$ and introduce z by $Z = (z - t)/2$ and we use a new coordinate system (y, z) on the affine plane W_{10}. So the center a_7 of the seventh blow-up is $(y, z) = (0, 0)$ on W_{10}. In this way we get the blow-up

$$Z_7(c) \longrightarrow Z_6(c).$$

Locally we have the blow-up

$$p_{10} : \widetilde{W_{10}} \longrightarrow W_{10}$$

and the surface $\widetilde{W_{10}}$ is covered by open subsets $W_{10}(y)$ and $W_{10}(z)$ both isomorphic to the affine plane $\mathbb{A}^2$ with the coordinate systems (Y, z) and (y, Z). So $Z_7(c)$ is covered by 11 affine planes:

$$W_1, W_2, W_3, W_4(y), W_5(y), W_6(y), W_7(y), W_8(y), W_9(y), W_{10}(y), W_{10}(z).$$

The center of the eighth and hence the last blow-up is the point $(y, Z) = (0, -2c - 1)$ on $W_{10}(z)$. So we get

$$Z_8(c) \longrightarrow Z_7(c).$$

This the definition of $Z_8(c) = \mathcal{X}(c)$.

§3. Proof of the Theorem

The group G is a Coxeter group generated by two reflections i and j:

$$i : K(c) \longrightarrow K(c),\ \ i(c) = -c, \quad \text{and} \quad j : K(c) \longrightarrow K(c),\ \ j(c) = -1 - c.$$

So we have to show that the birational automorphisms of $\mathcal{X}(c)$ corresponding to i, j are in fact biregular automorphisms of $\mathcal{X}(c)$. Let us first study the reflection j. The operation of the reflection j comes from the transformation

$$J(c, -1 - c) = I(c + 1, -1 - c) \circ T_+(c, c + 1) : Sol_2(c) \longrightarrow Sol_2(-1 - c),$$
$$(q, p) \longmapsto (-q, -2q^2 - p - t).$$

Keeping the notation of §2, we consider a differential L-morphism

$$L\{Q,P\} \longrightarrow L(q_c,p_c) \tag{3.1}$$

sending

$$Q \longmapsto -q_c, \quad P \longmapsto -2q_c^2 - p_c - t.$$

Since $J(c,-1-c)(q_c,p_c) = (-q_c, -2q_c^2 - p_c - t) \in Sol_2(-1-c)$, the morphism (3.1) factors through the residue class morphism

$$L\{Q,P\} \longrightarrow L\{Q,P\}/I(-1-c) = R(-1-c)$$

so that we have a differential L-morphism

$$R(-1-c) \longrightarrow K(c). \tag{3.2}$$

Since $J(-1-c,c)\circ J(c,-1-c)(q,p) = (q,p)$ for a generic solution (q,p) of $S_2(c)$ over L, the L-morphism (3.2) is birational. Geometrically we have a differential L-birational map

$$J_X(c,-1-c) : X(c) = \operatorname{Spec} R(c) \dashrightarrow X(-1-c) = \operatorname{Spec} R(-1-c)$$

and therefore differential L-birational maps

$$\begin{aligned} &J_Z(c,-1-c) : Z(c) \dashrightarrow Z(-1-c), \quad \text{and} \\ &J_{\mathcal{X}}(c,-1-c) : \mathcal{X}(c) \dashrightarrow \mathcal{X}(-1-c), \end{aligned} \tag{3.3}$$

as $Z(c)$ and $\mathcal{X}(c)$ are models of $X(c) = \operatorname{Spec} R(c)$. Since we have a natural differential $K(t)$-isomorphism

$$R(c) = L[q_c,p_c] \longrightarrow R(-c-1) = L[q_{-1-c},p_{-1-c}],$$
$$q_c \longmapsto q_{-1-c}, \;\; p_c \longmapsto p_{-1-c}, \;\; c \longmapsto -1-c,$$

and thus a $K(t)$-isomorphism $X(-1-c) = \operatorname{Spec} R(-1-c) \to X(c) = \operatorname{Spec} R(c)$. This isomorphism gives further differential $K(t)$-isomorphisms

$$J'_Z : Z(-1-c) \longrightarrow Z(c) \quad \text{and} \quad J'_{\mathcal{X}} : \mathcal{X}(-c-1) \longrightarrow \mathcal{X}(c) \tag{3.4}$$

Composing the morphisms (3.3), (3.4), we get birational maps

$$J_Z : Z(c) \dashrightarrow Z(c) \quad \text{and} \quad J_{\mathcal{X}} : \mathcal{X}(c) \dashrightarrow \mathcal{X}(c)$$

that is by definition the birational operation of j on $\mathcal{X}(c)$. We must show that $J = J_{\mathcal{X}}$ is an isomorphism. Since the morphisms (3.4) are isomorphisms, we have to show that the birational map $J_{\mathcal{X}}(c, -1-c) : \mathcal{X}(c) \cdots\to \mathcal{X}(-1-c)$ is biregular. To this end we look for the minimum resolution of the rational map

$$J_Z(c, -1-c) : Z(c) \cdots\to Z(-1-c).$$

Indeed we see below that the blow-up $\mathcal{X}(c) \to Z(c)$ is the minimum resolution of the birational map $J_Z(c, -1-c) : Z(c) \cdots\to Z(-1-c)$.

Lemma 3.5. *The blow-up $\mathcal{X}(c) \to Z(c)$ is the minimum resolution of the birational map $J_Z(c, -1-c) : Z(c) \cdots\to Z(-1-c)$.*

Let us admit for a moment Lemma 3.5. We have a regular map

$$J_{\mathcal{X}Z}(c, -1-c) : \mathcal{X}(c) \longrightarrow Z(-1-c)$$

such that

$$J_Z(c, -1-c) \circ p = J_{\mathcal{X}Z}(c, -1-c),$$

$p : \mathcal{X}(c) \to Z(c)$ being the blow-up morphism. Since

$$J(-1-c, c) \circ J(c, -1-c) = \mathrm{Id}_{\mathcal{X}(c)},$$

the blow-up $\mathcal{X}(-1-c) \to Z(-1-c)$ is the minimum resolution of the birational map

$$J_Z(c, -1-c)^{-1} = J_Z(-1-c, c) : Z(-1-c) \cdots\to Z(c).$$

Therefore the birational map $J_{\mathcal{X}}(c, -1-c) : \mathcal{X}(c) \to \mathcal{X}(-1-c)$ is biregular. So it remains to prove Lemma 3.5.

Proof of Lemma 3.5. To study the rational map $J_Z(c, -1-c) : Z(c) \cdots\to Z(-1-c)$, we simplify the notation. The ruled surfaces $Z(c)$ and $Z(-1-c)$ are defined by coverings W_i $(1 \le i \le 4)$. To distinguish the covering of $Z(c)$ from that of $Z(-1-c)$, we denote the covering of $Z(c)$ by W_i, $1 \le i \le 4$, and the covering of $Z(-1-c)$ by $\overline{W}_i$, $1 \le i \le 4$. So it follows from the definition that the L-morphism $J_Z(c, -1-c) : Z(c) \cdots\to Z(-1-c)$ is defined by

$$J_{11} : W_1 \longrightarrow \overline{W}_1, \quad (y_1, z_1) \longmapsto (\overline{y}_1, \overline{z}_1) = (-y_1, -2y_1^2 - z_1 - t).$$

So $J_Z(c,-1-c)$ is regular on the open subset W_1 of $Z(c)$. As $y_2 = y_1$, $z_2 = z_1^{-1}$, on W_2 we have

$$J_{21} : W_2 \cdots\to \overline{W}_1, \quad (y_2, z_2) \longmapsto (\overline{y}_1, \overline{z}_1) = \left(-y_2, -2y_2^2 - \frac{1}{z_2} - t\right).$$

So J_{21} is not defined on $W_2 \cap \{z_2 = 0\}$. On the other hand since $\overline{y}_2 = \overline{y}_1$, $\overline{z}_2 = \overline{z}_1^{-1}$, we have

$$J_{22} : W_2 \longrightarrow \overline{W}_2, \quad (y_2, z_2) \longmapsto \left(-y_2, \frac{z_2}{-2y_2z_2 - 1 - tz_2}\right).$$

So on $W_2 \cap \{-2y_2^2 z_2 - 1 - tz_2 = 0\}$ the map $J_Z(c,-1-c)$ is not defined and the set of base points of $J_Z(c,-1-c)$ on W_2 is

$$\{2y_2^2 + 1 + tz_2 = 0\} \cap \{z_2 = 0\} = \emptyset.$$

Namely $J_Z(c,-1-c)$ is regular on W_2. Similarly it follows from the definition of the ruled surface $Z(c)$ that we have on W_3

$$J_{31} : W_3 \cdots\to \overline{W}_1,\ (y_3, z_3) \longmapsto \left(-\frac{1}{y_3}, \frac{-2-(cy_3 - y_3^2 z_3)y_3^3 - ty_3^2}{y_3^2}\right),$$

$$J_{32} : W_3 \cdots\to \overline{W}_2,\ (y_3, z_3) \longmapsto \left(-\frac{1}{y_3}, \frac{y_3^2}{-2-(c - y_3z_3)y_3^3 - ty_3^2}\right),$$

$$J_{33} : W_3 \cdots\to \overline{W}_3,\ (y_3, z_3) \longmapsto \left(-y_3, \frac{2 - z_3y_3^4 + ty_3^2 + (2c+1)y_3^2}{y_3^4}\right)$$

and

$$J_{34} : W_3 \cdots\to \overline{W}_4,\ (y_3, z_3) \longmapsto \left(-y_3, \frac{y_3^4}{2 - z_3y_3^4 + ty_3^2 + (2c+1)y_3^2}\right).$$

So the rational map $J_Z(c,-1-c) : Z(c) \cdots\to Z(-1-c)$ is regular on the open subset W_3 of $Z(c)$. Similarly an easy calculation shows

$$J_{41} : W_4 \cdots\to \overline{W}_1,\ (y_4, z_4) \longmapsto \left(-\frac{1}{y_4}, \frac{-2z_4 - (cz_4 - y_4)y_4^3 - ty_4^2 z_4}{y_4^2 z_4}\right)$$

$$J_{42} : W_4 \cdots\to \overline{W}_2,\ (y_4, z_4) \longmapsto \left(-\frac{1}{y_4}, \frac{y_4^2 z_4}{-2z_4 - (cz_4 - y_4)y_4^3 - ty_4^2 z_4}\right)$$

$$J_{43} : W_4 \cdots\to \overline{W}_3,\ (y_4, z_4) \longmapsto \left(-y_4, \frac{2z_4 - y_4^4 + ty_4^2 z_4 + (2c+1)y_4^3 z_4}{y_4^4 z_4}\right)$$

and

$$J_{44} : W_4 \cdots\to \overline{W}_4,\ (y_4, z_4) \longmapsto \left(-y_4, \frac{y_4^4 z_4}{2z_4 - y_4^4 + ty_4^2 z_4 + (2c+1)y_4^3 z_4}\right).$$

We conclude now that on W_4 the rational map

$$J_Z(c, -1-c) : Z(c) \cdots\to Z(-1-c)$$

has a base point at $(y_4, z_4) \in W_4$ for which

$$y_4^4 z_4 = 0 \quad \text{and} \quad 2z_4 - y_4^4 + ty_4^2 z_4 + (2c+1)y_4^3 z_4 = 0.$$

Namely $(y_4, z_4) = (0, 0) \in W_4$ is the unique base point of the rational map $J_Z(c, -1-c) : Z(c) \cdots\to Z(-1-c)$. This point is the center a_1 of the first blow-up in the construction of $\mathcal{X}(c)$ as well as in the resolution of the rational map $J_Z(c, -1-c)$. We have proved

SUBLEMMA 3.6. *The resolution of the rational map*

$$J_Z(c, -1-c) : Z(c) \cdots\to Z(-1-c)$$

is equivalent to the resolution of the rational function

$$F : W_4 \cdots\to \mathbb{P}^1, \quad (y_4, z_4) \longmapsto (y_4^4 z_4, 2z_4 - y_4^4 + ty_4^2 z_4 + (2c+1)y_4^3 z_4)$$

on W_4.

We use the notation of the construction of the model $\mathcal{X}(c)$. We blow up W_4 at $a_1 = (y_4, z_4) = (0, 0)$ to get

$$p_4 : \widetilde{W_4} \longrightarrow W_4.$$

$\widetilde{W_4}$ is covered by the two affine planes $W_4(y)$ and $W_4(z)$ with the coordinate systems respectively (Y, z) and (z, Y). In terms of these coordinate systems the morphism p_4 is written as

$$W_4(y) = \mathbb{A}^2 \longrightarrow W_4 = \mathbb{A}^2, \quad (Y, z) \longmapsto (y_4, z_4) = (Yz, z) \tag{3.7}$$

on $W_4(y)$ and

$$W_4(z) = \mathbb{A}^2 \longrightarrow W_4 = \mathbb{A}^2, \quad (y, Z) \longmapsto (y_4, z_4) = (y, yZ) \tag{3.8}$$

on $W_4(z)$. So we substituting (3.7) and (3.8), the rational function F on $\widetilde{W}_4$ is given on $W_4(y)$ by

$$\begin{aligned} W_4(y) = \mathbb{A}^2 \cdots\to \mathbb{P}^1, \\ (Y, z) \longmapsto (Y^4z^4, 2 - Y^4z^3 + tY^2z^2 + (2c+1)Y^3z^3) \end{aligned} \tag{3.9}$$

and on $W_4(z)$ by

$$\begin{aligned} W_4(z) = \mathbb{A}^2 \cdots\to \mathbb{P}^1, \\ (y, Z) \longmapsto (y^4Z, 2Z - y^3 + ty^2Z + (2c+1)y^3Z). \end{aligned} \tag{3.10}$$

It follows from (3.10) that the unique base point of the rational function F on $W_4(z)$ is $(y, Z) = (0, 0)$ and by (3.9) the point $(Y, z) = (0, 0) \in W_4(y)$ is not a base point. Thus $(y, Z) = (0, 0)$ on $W_4(z)$ is the unique base point of the rational function F on $\widetilde{W}_4$. This point coincides with the center a_2 of the second blow-up in the construction of $\mathcal{X}(c)$. Now we set $W_5 = W_4(z)$ and denote Y again by y. We blow up $W_5 = \mathbb{A}^2$ with coordinate system (y, z) at $a_2 = (y, z) = (0, 0) \in W_5$ to solve the singularity of the rational map (3.10), which is given by

$$\begin{aligned} W_4(z) = \mathbb{A}^2 \cdots\to \mathbb{P}^1, \\ (y, z) \longmapsto (y^4z, 2z - y^3 + ty^2z + (2c+1)y^3z) \end{aligned} \tag{3.11}$$

because we denote Y by y. Let $p_5 : \widetilde{W}_5 \to W_5$ be the blow-up so that $\widetilde{W}_5$ is covered by the two affine planes $W_5(y)$ and $W_5(z)$ with the coordinate systems (Y, z) and (y, Z). The morphism

$$p_5 : \widetilde{W}_5 \longrightarrow W_5$$

is given on the open subsets of $\widetilde{W}_5$ by

$$W_5(y) = \mathbb{A}^2 \longrightarrow W_5 = \mathbb{A}^2, \quad (Y, z) \longmapsto (Yz, z)$$

on $W_5(y)$ and

$$W_5(z) = \mathbb{A}^2 \longrightarrow W_5 = \mathbb{A}^2, \quad (y, Z) \longmapsto (y, yZ)$$

on $W_5(z)$. So if we substitute $(y, z) = (Yz, z)$ and $(y, z) = (y, yZ)$ into (3.11), we get the expressions of the rational function F:

$$W_5(y) = \mathbb{A}^2 \cdots\to \mathbb{P}^1, \quad (Y, z) \longmapsto (Y^4z^4, 2 - Y^2z - tY^2z^2 + (2c+1)Y^3z^3)$$

on $W_5(y)$ and

$$W_5(z) = \mathbb{A}^2 \cdots\to \mathbb{P}^1, \quad (y, Z) \longmapsto (y^4Z, 2Z - y^2 + ty^2Z + (2c+1)y^3Z)$$

on $W_5(z)$. So $(y, Z) = (0, 0) \in W_5(z)$ is the unique base point of the rational function F on $\widetilde{W_5}$. This point coincides with the center a_3 of the third blow-up in the construction of $\mathcal{X}(c)$. Now we use the notation of the construction of $\mathcal{X}(c)$. We denote Z by z so that $W_6 = W_5(z)$ is the affine plane with coordinate system (y, z). We blow up W_6 at $a_3 = (y, z) = (0, 0)$. Then the rational function F on W_6 is written on the open subsets $W_6(y)$ and $W_6(z)$ as

$$W_6(y) = \mathbb{A}^2 \cdots\to \mathbb{P}^1, \quad (Y, z) \longmapsto (Y^4z^4, 2 + O(1)),$$

where $O(1)$ is an element of the ideal (y, z) of $L[Y, z]$, and

$$W_6(z) = \mathbb{A}^2 \cdots\to \mathbb{P}^1, \quad \left(y^4Z, 2Z - y + ty^2Z + (2c+1)y^3Z\right).$$

So $(y, Z) = (0, 0)$ is the unique base point of the rational function F on $\widetilde{W_6}$. This point is the center a_4 of the fourth blow-up in the construction of $\mathcal{X}(c)$. Hence we denote Z by z so that $W_7 = W_6(z)$ is the affine plane with coordinate system (y, z). We blow up W_7 at $a_4 = (y, z) = (0, 0)$ to get $\widetilde{W_7} \to W_7$. Then on $\widetilde{W_7}$ the representations of the rational function F are as follows.

$$W_7(y) \cdots\to \mathbb{P}^1, \quad (Y, z) \longmapsto (Y^4z^4, 2 + O(1))$$

and

$$W_7(z) \cdots\to \mathbb{P}^1, \quad (y, Z) \longmapsto (y^4Z, 2Z - y + ty^2Z + (2c+1)y^3Z),$$

where $O(1)$ is an element of the ideal (Y, z) of $L[Y, z]$. Hence the unique base point of the rational function F on $\widetilde{W_7}$ is $(y, Z) = (0, 0)$ on $W_7(z)$ that is the center a_5 of the fifth blow-up in the construction of $\mathcal{X}(c)$. We denote Z by z so that $W_7(z) = W_8$ is the affine plane with coordinate system (y, z). We blow up W_8 at $(y, z) = (0, 0)$. We then have local expressions of the rational function F on the open subsets $W_7(y)$ and $W_7(z)$ of $\widetilde{W_7}$. Namely

$$W_7(y) \cdots\to \mathbb{P}^1, \quad (Y, z) \longmapsto (Y^4z^4, 2 + O(1)),$$

where $O(1)$ is an element of the ideal (Y, z) of $L[Y, z]$, and

$$W_7(z) \cdots\to \mathbb{P}^1, \quad (y, Z) \longmapsto \left(y^4Z, 2Z - 1 + ty^2Z + (2c+1)y^3Z\right).$$

Therefore the unique base point of the rational function F on $\widetilde{W}_8$ is the point $(y, Z) = (0, 1/2)$ on $W_7(z)$. So we blow up $W_7(z)$ at $a_5 = (y, Z) = (0, 1/2)$. We introduced the coordinate system (y, z) on $W_7(z)$ such that $Z = (z + 1)/2$ and denote the affine plane $W_7(z)$ with this coordinate system (y, z) by W_8. So on $W_8 = \mathbb{A}^2$ the rational function F is expressed as

$$W_8 \cdots\rightarrow \mathbb{P}^1,$$
$$(y, z) \longmapsto \left(y^4 z + y^4, 2z + ty^2 + (2c+1)y^3 + ty^2 z + (2c+1)y^3 z\right).$$

On the open subsets $W_9(y)$ and $W_9(z)$ of the blow-up $\widetilde{W}_8 \to W_8$, the rational function F is written as

$$W_8(y) \cdots\rightarrow \mathbb{P}^1, \quad (Y^4 z^4 + Y^4 z^3, 2 + O(1)),$$

where $O(1)$ is an element of the ideal (Y, z) of $L[Y, z]$, and

$$W_8(z) \cdots\rightarrow \mathbb{P}^1,$$
$$(y, Z) \longmapsto \left(y^4 Z + y^3, 2Z + ty + (2c+1)y^2 + ty^2 z + (2c+1)y^3 z\right).$$

So the unique base point of the rational function F on $\widetilde{W}_8$ is the point $(y, Z) = (0, 0)$ in $W_8(z)$. This is the center a_6 of the sixth blow-up in the construction of $\mathcal{X}(c)$. So we denote Z by z and blow up the affine plane $W_8(z) = W_9$ with coordinate system (y, z) at $a_6 = (y, z) = (0, 0)$. Then the local expression of the rational function F on the open subsets $W_9(y)$ and $W_9(z)$ are as follows.

$$W_9(y) \cdots\rightarrow \mathbb{P}^1, \quad (Y, z) \longmapsto (Y^4 z^4 + Y^4 z^3, 2 + O(1)),$$

where $O(1)$ is an element of the ideal (Y, z) of $L[Y, z]$, and

$$W_9(z) \cdots\rightarrow \mathbb{P}^1,$$
$$(y, Z) \longrightarrow \left(y^4 Z + y^2, 2Z + t + (2c+1)y + ty^2 z + (2c+1)y^3 z\right).$$

So the unique base point of the rational function F on $\widetilde{W}_9$ is the point $(y, Z) = (0, -t/2)$ on $W_9(z)$. This point is the center of the seventh blow-up in the construction of $\mathcal{X}(c)$. We introduced z by $Z = (z - t)/2$ and the affine plane $W_9(z)$ with the coordinate system (y, z) was denoted by W_{10}. So the rational function F on W_{10} is given by

$$W_{10} \cdots\rightarrow \mathbb{P}^1,$$
$$(y, z) \longmapsto \left(y^4(z - t) + 2y^2, 2z + 2(2c+1)y + (t + (2c+1)y)y^2(z - t)\right).$$

We blow up W_{10} at $a_7 = (y, z) = (0, 0)$ to get $\widetilde{W}_{10} \to W_{10}$. The local representation of the rational function F on $\widetilde{W}_{10}$ are

$$W_{10} \cdots\to \mathbb{P}^1, \quad (Y, z) \longmapsto (O(1), 2 + O(1)'),$$

where $O(1)$, $O(1)'$ are elements of the ideal (Y, z) of $L[Y, z]$, and

$$W_{10}(z) \cdots\to \mathbb{P}^1,$$
$$(y, Z) \longmapsto \big(y^3(yZ - t) + 2y, 2Z + 2(2c + 1) + (t + (2c + 1)y)y(yZ - t)\big).$$

So the unique base point of the rational function F on $\widetilde{W}_{10}$ is the point $(y, Z) = (0, -2c - 1)$ of $\widetilde{W}_{10}(z)$. This is the center of the eighth blow-up in the construction of $\mathcal{X}(c)$. We introduced z by $Z = z - 2c - 1$ and denoted the affine plane $W_{10}(z)$ with this new coordinate system (y, z) by W_{11}. Then we blew up W_{11} at $(y, z) = (0, 0)$ to get $\widetilde{W}_{11} \to W_{11}$. The rational function F on W_{11} is written as

$$W_{11} \cdots\to \mathbb{P}^1,$$
$$(y, z) \longmapsto \big(y^3(y(z - 2c - 1) - t) + 2y,$$
$$2z + (t + (2c + 1)y)y(y(z - 2c - 1) - t)\big).$$

Now the local expressions of the rational function F on $\widetilde{W}_{11}$ are

$$W_{11}(y) \cdots\to \mathbb{P}^1, \quad (y, z) \longmapsto (O(1), 2 + O(1)'),$$

and

$$W_{11}(z) \cdots\to \mathbb{P}^1,$$
$$(y, Z) \longmapsto \big(y^2(y(yZ - 2c - 1) - t) + 2,$$
$$2Z + (t + (2c + 1)y)(y(yZ - 2c - 1) - t)\big), \tag{3.12}$$

where $O(1)$, $O(1)'$ are elements of the ideal (Y, z) of $L[Y, z]$. So the point $(Y, z) = (0, 0)$ of $W_{11}(y)$ is not a base point of the rational function F. We show that there is no base point of F on $W_{11}(z)$ either, i.e., F has no base point on $\mathcal{X}(c)$. In fact let $(y, Z) \in W_{11}(z)$ be a base point of the rational function F. Then $y \neq 0$. Equating the coordinates of $\mathbb{P}^1$ in (3.12) equals to 0, we have

$$y^2(y(yZ - 2c - 1) - t) + 2 = 0 \tag{3.13}$$

and

$$2Z + (t + (2c+1)y)(y(yZ - 2c - 1) - t) = 0. \tag{3.14}$$

It follows from (3.13)

$$y(yZ - 2c - 1) - t = -\frac{2}{y^2}. \tag{3.15}$$

We substitute (3.15) into (3.14) to get

$$2Z + (t + (2c+1)y)\left(-\frac{2}{y^2}\right) = 0$$

and hence

$$-Zy^2 + (2c+1)y + t = 0, \tag{3.16}$$

which contradicts (3.15). So Lemma 3.5 is proved. □

Now we have to study the operation of the reflection i. The operation of the reflection i comes from the transformation $I(c, -c)$ in §2. In fact keeping the notation in §2, we consider a differential L-morphism

$$L\{Q, P\} \longrightarrow L(q_c, p_c) \tag{3.17}$$

sending

$$Q \longmapsto q_c - \frac{c}{p_c}, \quad P \longmapsto p_c.$$

Since

$$\left(q_c - \frac{c}{p_c}, p_c\right)$$

is a solution of the system $S_2(c)$, the morphism (3.1) factors through the residue class morphism

$$L(c,t)\{Q, P\} \longrightarrow L\{Q, P\}/I(-c) = R(-c)$$

so that we have a differential L-morphism

$$R(-c) \cdots\rightarrow K(c). \tag{3.18}$$

Since $I \circ I(q, p) = (q, p)$ for a generic solution (q, p) of $S_2(c)$ over L, the L-morphism (3.18) is birational. Namely we have a differential L-birational morphism

$$X(c) = \operatorname{Spec} R(c) \cdots\rightarrow \operatorname{Spec} R(-c) = X(-c) \tag{3.19}$$

as we have a natural L-isomorphism

$$R(c) = L(q_c, p_c) \longrightarrow R(-c) = L(q_{-c}, p_{-c})$$

sending

$$q_c \longmapsto q_{-c}, \quad p_c \longmapsto p_{-c}, \quad c \longmapsto -c$$

and thus a natural L-morphism

$$X(-c) = \operatorname{Spec} R(-c) \longrightarrow X(c) = \operatorname{Spec} R(c). \tag{3.20}$$

Composing the morphisms (3.19) and (3.20), we get the birational map $I : X(c) \cdots\rightarrow X(c)$. It follows from the construction that the isomorphism (3.20) induces an isomorphism $Z(-c) \to Z(c)$. So we have to show that the birational morphism $I_{\mathcal{X}}(c, -c) : \mathcal{X}(c) \to \mathcal{X}(-c)$ arising from (3.19) is biregular. The L-birational map (3.19) defines an L-birational map

$$I_Z(c, -c) : Z(c) \cdots\rightarrow Z(-c) \tag{3.21}$$

of the projective surfaces. We denote the charts $W_i \simeq \mathbb{A}^2_L$ of $Z(c)$ by $W_i(c)$ and the coordinate system of the affine plane $W_i(c) = \mathbb{A}^2$ by (y_i, z_i) and the coordinate system of the affine plane $W_i(-c) = \mathbb{A}^2$ by $(\overline{y}_i, \overline{z}_i)$ for $1 \leq i \leq 4$. So the rational map $I_Z(c, -c)$ is given by a rational map

$$W_1(c) = \mathbb{A}^2 \cdots\rightarrow W_1(-c) = \mathbb{A}^2, \quad (y_1, z_1) \longmapsto (\overline{y}_1, \overline{z}_1) = \left(y_1 - \frac{c}{z_1}, z_1\right)$$

in terms of the charts $W_1(c)$ and $W_1(-c)$. Since

$$\overline{y}_3 = \frac{1}{\overline{y}_1}, \quad \overline{z}_3 = -c\overline{y}_1 - \overline{y}_1^2\overline{z}_1, \tag{3.22}$$

it follows from (3.22) that the rational map (3.21) gives

$$W_1(c) = \mathbb{A}^2 \cdots\rightarrow W_3(-c) = \mathbb{A}^2,$$
$$(y_1, z_1) \longmapsto (\overline{y}_3, \overline{z}_3) = \left(\frac{z_1}{y_1 z_1 - c}, -y_1(y_1 z_1 - c)\right).$$

So the rational map (3.21) has no base point on $W_1(c)$. Substituting $y_1 = y_2$, $z_1 = 1/z_2$, we conclude that the rational map (3.21) gives rational maps

$$W_2(c) \cdots\rightarrow W_1(-c), \quad (y_2, z_2) \longmapsto \left(y_2 - cz_2, \frac{1}{z_2}\right).$$

and

$$W_2(c) \cdots\rightarrow W_2(-c), \quad (y_2, z_2) \longmapsto (y_2 - cz_2, z_2).$$

So there is no base point of rational map (3.21) on $W_2(c)$ either. A similar calculation shows that the rational map (3.21) is written as a rational map

$$W_3(c) \cdots\rightarrow W_1(-c), \quad (y_3, z_3) \longmapsto \left(\frac{z_3}{y_3 z_3 - 1}, (c - y_3 z_3) y_3\right)$$

and

$$W_3(c) \cdots\rightarrow W_3(-c), \quad (y_3, z_3) \longmapsto \left(-\frac{c - y_3 z_3}{z_3}, z_3\right).$$

Therefore there is no base point of the rational map (3.21) on $W_3(c)$. Similarly in terms of $W_4(c)$ and $W_4(-c)$, the rational map (3.21) gives an isomorphism

$$W_4(c) \longrightarrow W_4(-c), \quad (y_4, z_4) \longmapsto (y_4 - cz_4, z_4) \tag{3.23}$$

so that there is no base point of the rational map (3.21) on $W_4(c)$ either. Namely the rational map (3.21) is indeed regular. Now since $I_Z(-c, c) \circ I_Z(c, -c) = \mathrm{Id}$, the rational map (3.21) is biregular. We have to show that the rational map (3.21) induces the biregular morphism $\mathcal{X}(c) \to \mathcal{X}(-c)$. Let $C_0(c)$ be the Zariski closure of the curve

$$\left\{(y_4, z_4) \in W_4 \mid 2z_4 - y_4^4 + ty_4^4 z_4 + (2c+1) y_4^3 z_4 = 0\right\}$$

in $Z_0(c)$, $E_i(c) \subset Z_i(c)$ the exceptional curve of the i-th blow-up $Z_i(c) \to Z_{i-1}(c)$, and $C_i(c)$ the proper transform of $C_0(c)$ by the blow-up

$$Z_i(c) \longrightarrow Z_{i-1}(c) \longrightarrow \cdots \longrightarrow Z_0(c) = Z(c)$$

for $1 \le i \le 8$. In the course of the proof of Lemma 3.5, we have proved

LEMMA 3.24. *The center a_1 of the first blow-up is the point $(y_4, z_4) = (0, 0) \in W_4$ and the center a_i of the i-th blow-up $Z_i(c) \to Z_{i-1}(c)$ is the intersection of the curve $C_{i-1}(c)$ and the exceptional divisor E_{i-1} for $2 \le i \le 8$. Namely we have $a_i = E_{i-1} \cap C_{i-1}(c)$ for $2 \le i \le 8$.*

We denote the center a_i on $Z_{i-1}(c)$ by $a_i(c)$ and the center a_i on $Z_{i-1}(-c)$ by $a_i(-c)$ to distinguish them. Therefore we must show that at each step of blow-up the center $a_{i+1}(c)$ on $Z_i(c)$ is mapped to the center

$a_{i+1}(-c)$ on $Z_i(-c)$. In fact (3.23) shows that the first center a_1 on $Z(c)$ is mapped to the first center on $Z(-c)$. To see the image of the successive centers, in view of Lemma 3.24 let us consider the image of the curve

$$C_0(c) \cap W_4 = \{2z_4 - y_4^4 + ty_4^2 z_4 + (2c+1)y_4^3 z_4 = 0\} \subset W_4 \subset Z(c).$$

Since $I_Z(c,-c)(y_4, z_4) = (y_4 - cz_4, z_4) = (\overline{y}_4, \overline{z}_4)$, we substitute

$$y_4 = \overline{y}_4 + cq\overline{z}_4, \quad z_4 = \overline{z}_4$$

to get

$$\begin{aligned}(3.25) \qquad 2\overline{z}_4 - \overline{y}_4^4 - t\overline{y}_4^2\overline{z}_4 + (-2c+1)\overline{y}_4^3\overline{z}_4 + (2t + 3c\overline{y}_4)\overline{y}_4\overline{z}_4^2 \\ +3(ct + (2c+1)c\overline{y}_4)\overline{z}_4^3 + (-2c+1)c^3\overline{z}_4^4 = 0.\end{aligned}$$

The Zariski closure of this curve is the image $\overline{C}$ of the curve $C_0(c)$. We show that the birational map $I_i : Z_i(c) \cdots\rightarrow Z_i(-c)$ induced by (3.21) is a biregular isomorphism for $0 \le i \le 8$ by induction. More precisely we prove by induction on j that $I_j : Z_j(c) \rightarrow Z_j(-c)$ is a biregular isomorphism and $I_j(a_{j+1}(c)) = a_j(-c)$ for $0 \le j \le 7$, which implies $I_8 = I_{\mathcal{X}}(c,-c) : \mathcal{X}(c) = Z_8(c) \rightarrow \mathcal{X}(-c) = Z_8(-c)$ is a biregular isomorphism. In fact we have seen that the assertion holds for $i = 0$. Hence we assume that for a number i with $0 \le i \le 7$, the birational map $I_j : Z_j(c) \cdots\rightarrow Z_j(-c)$ is a biregular isomorphism and $I_j(a_{j+1}(c)) = a_{j+1}(-c)$ for $0 \le j \le i$. Then the birational map $I_{i+1} : Z_{i+1}(c) \cdots\rightarrow Z_{i+1}(-c)$ is a biregular isomorphism. We have to show that $I_{i+1}(a_{i+2}(c)) = a_{i+2}(-c)$. Since $a_{i+2}(c) = Z_{i+1}(c) \cap C_{i+1}(c)$ and since $I_j(E_j(c)) = E_j(-c)$ for $j \le i+1$, we have to show that the proper transform $\overline{C}_l(c)$ of $\overline{C}_0(c)$ and the curve $C_l(-c)$, which is the proper transform of $C_0(-c)$, under the blow-up

$$Z_l(-c) \longrightarrow Z_{i-1}(-c) \longrightarrow \cdots \longrightarrow Z_0(-c) = Z(-c)$$

both intersect with $E_l(-c)$ at the same point for $1 \le l \le 7$. This follows from the defining equation (3.25) of the curve $\overline{C}_0(c)$ and from the defining equation

$$2\overline{z}_4 - \overline{y}_4^2 - t\overline{y}_4^2\overline{z}_4 + (-2c+1)\overline{y}_4^3\overline{z}_4 = 0$$

of the curve $C(-c)$ on $W_4(-c)$.

§4. Equivariant fibration over the affine line

We proved that the affine Weyl group of type $\tilde{A}_1$ operates regularly on the projective surface $\mathcal{X}(c)$ in such a way that the fibration $\mathcal{X}(c) \to \operatorname{Spec} L = \operatorname{Spec} K(t)(c)$ is equivariant. It is, however, more natural to look for an equivariant model

$$\mathcal{Y}[c] \longrightarrow \operatorname{Spec} K(t)[c]$$

of

$$\mathcal{X}(c) \longrightarrow \operatorname{Spec} L = \operatorname{Spec} K(t)(c)$$

such that the affine Weyl group operates biregularly on $\mathcal{Y}[c]$. We constructed the projective surface $\mathcal{X}(c)$ over the field $K(t)(c)$. The construction works over the ring $K(t)[c]$ so that we get a model

$$\mathcal{X}[c] \longrightarrow \operatorname{Spec} K(t)[c]$$

that is a scheme with derivation. So the affine Weyl group operates on the scheme $\mathcal{X}[c]$ with derivation birationally such that the fibration

$$\mathcal{X}[c] \longrightarrow \operatorname{Spec} K(t)[c]$$

is equivariant. Namely first we construct $Z[c]$ over $\operatorname{Spec} K(t)[c] = \mathbb{A}^1_{K(t)}$ by gluing 4 copies of $\mathbb{A}^2_{K(t)} \times_{K(t)} \operatorname{Spec} K(t)[c] = \mathbb{A}^3_{K(t)}$, which we denote by $W_i \times \mathbb{A}^1_{K(t)}$ or by $W_i[c]$, $1 \le i \le 4$, by the same rule as in the construction of the ruled surface $Z(c)$. Then we blow up $Z[c]$ 8 times along the sections of $Z[c] \to \mathbb{A}^1_{K(t)}$. So $W_i[c]$ is the affine space $\mathbb{A}^3_{K(t)}$ with the coordinate system (y_i, z_i, c), $1 \le i \le 4$. Similarly we have rational maps over $\operatorname{Spec} K(t)[c]$

$$J_{\mathcal{X}}[c, -1-c] : \mathcal{X}[c] \cdots\to \mathcal{X}[-1-c]$$

and

$$I_{\mathcal{X}}[c, -c] : \mathcal{X}[c] \cdots\to \mathcal{X}[-c].$$

In other words, locally on $W_1[c]$, $W_1[-c]$ and $W_1[-1-c]$, $J_{\mathcal{X}}[c, -1-c]$, $I_{\mathcal{X}}[c, -c]$ are given respectively by

$$J_{\mathcal{X}}[c, -1-c] : W_1[c] \cdots\to W_1[-1-c],$$
$$(y_1, z_1, c) \longmapsto (-y_1, -2y_1^2 - z_1 - t, -1-c)$$

and

$$I_{\mathcal{X}}[c,-c] : W_1[c] \cdots\to W_1[-c], \quad (y_1, z_1, c) \longmapsto \left(y_1 - \frac{c}{z_1}, z_1, -c\right).$$

The argument of §3 allows us to prove the following

LEMMA 4.1. *$J_{\mathcal{X}}[c,-1-c] : \mathcal{X}[c] \cdots\to \mathcal{X}[-1-c]$ is a biregular isomorphism.*

The curves

$$\{(y_1, z_1, c) \in W_1[c] \mid z_1 = c = 0\}$$

on $W_1[c]$ and

$$\{(y_3, z_3, c) \in W_3[c] \mid z_3 = c = 0\}$$

glue together to define a curve F on $\mathcal{X}[c]$ isomorphic to $\mathbb{P}^1_{K(t)}$, for $z_3 = cy_1 - y_1^2 z_1$ on $W_1[c] \cap W_3[c]$. Unfortunately the rational map $I_{\mathcal{X}}[c,-c] : \mathcal{X}[c] \cdots\to \mathcal{X}[-c]$ is not biregular.

LEMMA 4.2. *The base locus of the rational map $I_{\mathcal{X}}[c,-c] : \mathcal{X}[c] \cdots\to \mathcal{X}[-c]$ is the curve F.*

Proof. Locally on $W_1[c]$ and $W_1[-c]$, the rational map $I_{\mathcal{X}}[c,-c]$ is given by

$$W_1[c] \cdots\to W_1[-c], \quad (y_1, z_1, c) \longmapsto \left(y_1 - \frac{c}{z_1}, z_1, -c\right) \tag{4.3}$$

and on $W_3[c]$ and $W_3[-c]$

$$W_3[c] \cdots\to W_3[-c], \quad (y_3, z_3, c) \longmapsto \left(y_3 - \frac{c}{z_3}, z_3, -c\right).$$

The argument of §3 shows that there is no base point outside $W_1[c] \cup W_3[c]$. So the rational map $I_{\mathcal{X}}[c,-c]$ is not regular only on the curve F. □

Since $I_{\mathcal{X}}[c,-c]$ is not regular, we have to modify the model $\mathcal{X}[c] \to \mathbb{A}^1_{K(t)}$. To this end we blow up $\mathcal{X}[c]$ along the curve F to get $\mathcal{X}^1[c] \to \mathcal{X}[c]$.

LEMMA 4.4. *The rational map*

$$I_{\mathcal{X}}[c,-c] : \mathcal{X}[c] \cdots\to \mathcal{X}[-c]$$

induces a biregular isomorphism

$$I^1_{\mathcal{X}}[c,-c] : \mathcal{X}^1[c] \longrightarrow \mathcal{X}^1[-c].$$

Proof. Locally on $W_1[c]$, we blew up $W_1[c]$ along $z_1 = c = 0$. The blow-up of $W_1[c]$ along the curve $z_1 = c = 0$ is by definition

$$\widetilde{W_1}[c] = \{(y_1, z_1, c; (x_0, x_1)) \in W_1[c] \times \mathbb{P}^1_{K(t)} \mid z_1 x_0 = c x_1\}$$

and the projection $p : \widetilde{W_1}[c] \to W_1[c]$ is the restriction to $\widetilde{W_1}[c]$ of the projection $p_1 : W_1[c] \times \mathbb{P}^1_{K(t)} \to W_1[c]$ on to the first factor. Let us denote by $\widetilde{W_1}[c](z_1)$ the open subset

$$\{(y_1, z_1, c; (x_0, x_1)) \in \widetilde{W_1}[c] \mid x_0 \neq 0\}$$

of $\widetilde{W_1}[c]$. Then setting $Z_1 = x_1/x_0$, we have an isomorphism

$$W_1[c][z_1] \longrightarrow \mathbb{A}^3_{K(t)}, \quad (y_1, z_1, c; (x_0, x_1)) \longmapsto (y, x_1/x_0, c) = (y_1, Z_1, c).$$

Namely $W_1[c](z_1)$ is the affine space with the coordinate system (y_1, Z_1, c). Similarly if we denote x_0/x_1 by C, then the open subset

$$W[c](c) := \{(y_1, z_1, c; (x_0, x_1)) \in \widetilde{W_1}[c] \mid x_1 \neq 0\}$$

is isomorphic to $\mathbb{A}^3_{K(t)}$ by sending $(y_1, z_1, c; (x_0, x_1))$ to $(y_1, z_1, x_0/x_1)$. So $W_1[c](c)$ is isomorphic to the affine space $\mathbb{A}^3_{K(t)}$ with the coordinate system (y_1, z_1, C). So the blow-up $\widetilde{W_1}[c]$ is covered by two open subsets $W_1[c](z_1)$ and $W_1[c](c)$ isomorphic to $\mathbb{A}^3_{K(t)}$. On each open subsets the projection p is given by

$$W_1[c](z) \longrightarrow W_1[c], \quad (y, Z, c) \longmapsto (y, cZ, c)$$

on $W_1[c](z)$ and

$$W_1[c](c) \longrightarrow W_1[c], \quad (y, z, C) \longmapsto (y, z, zC)$$

on $W_1[c](c)$. Let us check that the rational map $\mathcal{X}^1[c] \cdots\to \mathcal{X}^1[-c]$ is regular on $W_1[c](z)$. Locally we have an expression

$$\begin{gathered} W_1[c](z) \longrightarrow W_1[c] \cdots\to W_1[-c], \\ (y, Z, c) \longmapsto (y, Zc, c) \longmapsto \left(y - \frac{1}{Z}, Zc, -c\right) \end{gathered}$$

so that the rational map $W_1[c](z) \cdots\to W_1[-c](z)$ is given by

$$(y, Z, c) \longmapsto \left(y - \frac{1}{Z}, -Z, -c\right)$$

and $W_1[c](z) \cdots\to W_3[-c]$ by

$$(y, Z, c) \longmapsto \left(\frac{1}{yZ-1}, -c\Big(y - \frac{1}{Z}\Big) - \Big(y - \frac{1}{Z}\Big)^2 Zc, -c\right)$$
$$= \left(\frac{1}{yZ-1}, cy(1-yZ), -c\right).$$

This shows that the rational map

$$I^1_{\mathcal{X}}[c,-c] : \mathcal{X}^1[c] \cdots\to \mathcal{X}^1[-c]$$

is regular on $W_1[c](z)$. On the other hand the composite rational map

$$W_1[c](c) \longrightarrow W_1[c] \cdots\to W_1[-c]$$

is given by

$$(y, z, C) \longmapsto (y, z, zC) \longmapsto (y - C, z, -zC)$$

and hence $W_1[c](c) \cdots\to W_1[-c](-c)$ is given by

$$(y, z, C) \longrightarrow (y - C, z, -zC).$$

So the rational map

$$I^1_{\mathcal{X}}[c,-c] : \mathcal{X}^1[c] \cdots\to \mathcal{X}^1[-c]$$

is regular on $W_1[c](C)$ and consequently on $\widetilde{W_1}[c]$. We have a local expression of

$$I^1_{\mathcal{X}} : \mathcal{X}^1[c] \cdots\to \mathcal{X}^1[c]$$

in terms of $W_3[c]$ and $W_3[-c]$:

$$W_3[c] \cdots\to W_3[-c], \quad (y_3, z_3, c) \longmapsto \left(-\frac{c - y_3 z_3}{z_3}, z_3, c\right) = \left(y_3 - \frac{c}{z_3}, z_3, c\right)$$

as we have seen in §3. So the above argument shows that the rational map $I^1_{\mathcal{X}}[c,-c] : \mathcal{X}^1[c] \cdots\to \mathcal{X}^1[-c]$ is regular locally on the blow-up of $W_3[c]$ too. If we notice here that in the construction of $\mathcal{X}^1[c]$ the centers $a_i[c]$ are on $W_4[c]$ and hence outside of the $W_1[c] \cup W_3[c]$ and that $I_{\mathcal{X}}[c,-c]$ induces an isomorphism $W_4[c] \to W_4[-c]$ mapping the centers $a_i[c]$ to the centers $a_i[-c]$, the rational map

$$I^1_{\mathcal{X}}[c,-c] : \mathcal{X}^1[c] \cdots\to \mathcal{X}^1[-c]$$

is regular. Since

$$I^1_{\mathcal{X}}[-c,c] \circ I^1_{\mathcal{X}}[c,-c] = \mathrm{Id},$$

$I^1_{\mathcal{X}}[c,-c]$ is biregular. □

Now

$$I^1_{\mathcal{X}}[c, -c] : \mathcal{X}^1[c] \longrightarrow \mathcal{X}^1[-c]$$

is biregular but the birational map

$$J^1_{\mathcal{X}}[c, -1-c] : \mathcal{X}^1[c] \cdots\!\!\rightarrow \mathcal{X}^1[-1-c]$$

corresponding to $J_{\mathcal{X}}$ is not regular. To remedy this, we have to blow up $\mathcal{X}^1[c]$ at infinitely many curves that are mutually disjoint. So the resulting object is not a scheme any more but a pro-scheme, i.e., the projective limit of schemes. In our case what we are going to get is a complex manifold if $K = \mathbb{C}$. Similarly as we obtained $\mathcal{X}[c]$, we can construct the ruled surface $Z(c)$ and its blow-ups $Z_i(c)$, $1 \le i \le 8$ over $\mathbb{A}^1_{K(t)}$. We denote the corresponding varieties by $Z[c]$ and $Z_i[c]$, the exceptional divisors on $Z_i[c]$ by $E_i[c]$ for $1 \le i \le 8$ so that $\mathcal{X}[c] = Z_8[c]$. The proper transform of $E_i[c]$ on $\mathcal{X}[c] = Z_8[c]$ is denoted by $D_{i+1}[c]$ for $1 \le i \le 8$. The proper transform by the blow-up morphism $p : \mathcal{X}[c] \to Z[c]$ of the divisor

$$\overline{\{z_2 = 0 \text{ on } W_2[c]\}} = \overline{\{z_4 = 0 \text{ on } W_4[c]\}}$$

on $Z[c]$ is denoted by $D_1[c]$. For $\overline{c} \in \overline{K}$, we denote by $\mathcal{X}[\overline{c}]$ the reduction of $\mathcal{X}[c]/K(t)[c]$ at $c = \overline{c}$, i.e., $\mathcal{X}[\overline{c}]$ is the fiber $\mathcal{X}[c]_{\overline{c}}$ over the rational point $c = \overline{c}$ of $\mathbb{A}^1_{K(t)} = \operatorname{Spec} K(t)[c]$. $\mathcal{X}[\overline{c}]$ is a projective surface over $K(t)[\overline{c}] = K(t)$ with the derivation $D(\overline{c})$ that is written as

$$D(\overline{c}) = \frac{\partial}{\partial t} + \left(y_1^2 + z_1 + \frac{t}{2}\right)\frac{\partial}{\partial y_1} + (-2y_1 z_1 + c)\frac{\partial}{\partial z_1}$$

on the open subset $W_1(\overline{c})$. The following result is due to Okamoto ([O1, Chap. III, §1]).

Proposition 4.5. *For every $\overline{c} \in \overline{K}$ regarded as a $\overline{K}$-rational point of $\mathbb{A}^1_{K(t)}$ so that we can speak of the reduction $\mathcal{X}[\overline{c}]$, the Zariski open subset $\mathcal{X}[\overline{c}] - \bigcup_{i=1}^{8} D_i[c]$ is the set of points P of $\mathcal{X}[c]$ where the rational vector field $D[\overline{c}]$ is regular at P. Namely we have*

$$\mathcal{X}[\overline{c}] - \bigcup_{i=1}^{8} D_i[\overline{c}] = \{P \in \mathcal{X}[\overline{c}] \mid D(\overline{c})\mathcal{O}_P \subset \mathcal{O}_P\}.$$

Corollary 4.6. *Let* $\overline{c}_1, \overline{c}_2 \in \overline{K}$ *and* $f : \mathcal{X}[\overline{c}_1] \to \mathcal{X}[\overline{c}_2]$ *be an isomorphism of schemes with derivation. Then*

$$f\left(\bigcup_{i=1}^{8} D_i[\overline{c}_1]\right) = \bigcup_{i=1}^{8} D_i[\overline{c}_2]$$

and f *induces an isomorphism*

$$\left(\mathcal{X}[\overline{c}_1] - \bigcup_{i=1}^{8} D_i[\overline{c}_1]\right) \longrightarrow \left(\mathcal{X}[\overline{c}_2] - \bigcup_{i=1}^{8} D_i[\overline{c}_2]\right).$$

Proof. This is a direct consequence of the Proposition. ∎

Lemma 4.7. *For* $\overline{c} \in \overline{K}$, *the reduction*

$$J_{\mathcal{X}}(\overline{c}, -1-\overline{c}) : \mathcal{X}[\overline{c}] \longrightarrow \mathcal{X}[-1-\overline{c}]$$

of the birational map $J_{\mathcal{X}}[c, -1-c]$ *is an isomorphism of schemes with derivation.*

Proof. This is a consequence of Lemma 4.1. ∎

Lemma 4.8. *For* $\overline{c} \in \overline{K}$, *the reduction*

$$I_{\mathcal{X}}(\overline{c}, -\overline{c}) : \mathcal{X}[\overline{c}] \longrightarrow [-\overline{c}]$$

of the birational map $I_{\mathcal{X}}[c, -c]$ *is an isomorphism of schemes with derivation.*

Proof. The birational map $I[c, -c]$ has the base locus F lying over the point $c = 0$ but the reduction $I_{\mathcal{X}}[0, 0]$ is equivalent to the identity. Now the lemma follows from Lemma 4.2. ∎

Lemma 4.9. *For every integer* n, *the open subset*

$$\mathcal{X}^0[n] := \mathcal{X}[n] - \bigcup_{i=1}^{8} D_i[n]$$

contains a curve $F[n]$ *that is isomorphic to* $\mathbb{P}^1_{K(t)}$, *tangent to the vector field* $D(n)$. *Moreover the curve* $F[n]$ *is the unique complete curve on* $\mathcal{X}^0[n]$ *tangent to the vector field* $D[n]$.

Proof. We know that the assertion of the corollary holds for $n = 0$ (see [UW]). Now the assertion follows from Corollary 4.6, Lemmas 4.7 and 4.8. ∎

Now we blow up $\mathcal{X}[c]$ at the infinitely many curves $F[n]$, $n \in \mathbb{Z}$ to get $\mathcal{Y}[c] \to \mathcal{X}[c]$.

THEOREM 4.10. *The affine Weyl group of type $\tilde{A}_1$ regularly operates on $\mathcal{Y}[c]$ such that the fibration $\mathcal{Y}[c] \to \operatorname{Spec} K(t)[c]$ is equivariant.*

Proof. It follows from Lemmas 4.1 and 4.4 that $I_{\mathcal{X}}(c, -c)$ and $J_X(c, -1-c)$ induce biregular morphisms

$$I_{\mathcal{Y}}[c, -c] : \mathcal{Y}[c] \longrightarrow \mathcal{Y}[-c] \quad \text{and} \quad J_{\mathcal{Y}}[c, -1-c] : \mathcal{Y}[c] \longrightarrow \mathcal{Y}[-1-c].$$

For two variables c, c' over $K(t)$, we have a natural isomorphism $\mathcal{Y}[c] \to \mathcal{Y}[c']$ covering the isomorphism

$$f^* : \operatorname{Spec} K(t)[c] \longrightarrow \operatorname{Spec} K(t)[c'],$$

where $f : K(t)[c'] \to K(t)[c]$ is the $K(t)$-isomorphism sending c' to c. So combining morphisms of this type with $I_{\mathcal{Y}}[c, -c]$ and $Y_{\mathcal{Y}}[c, -1-c]$, we get the morphisms

$$I_{\mathcal{Y}} : \mathcal{Y}[c] \longrightarrow \mathcal{Y}[-c] \quad \text{and} \quad J_{\mathcal{Y}} : \mathcal{Y}[c] \longrightarrow \mathcal{Y}[-1-c]$$

covering respectively i^* and j^*, where i, j are automorphisms of $K(t)[c]$ such that $i(c) = -c$, $j(c) = -1-c$. $I_{\mathcal{Y}}$ and $J_{\mathcal{Y}}$ are operations of i and j. So the theorem is proved. ∎

What is the fiber $\mathcal{Y}[0]$ over $c = 0$ of the fibration $\mathcal{Y}[c] \to \mathbb{A}^1_{K(t)}$? $\mathcal{Y}[0]$ has two irreducible components $\mathcal{X}[0]$ and the exceptional divisor contracted to the curve F. We describe the differential equation on the exceptional divisor. Let us work over $W_1[c](c)$ that is the affine space with the coordinate system (y_1, z_1, C), where $c = z_1 C$. So on $W_1[c](c)$ we have,

$$(4.11) \qquad \begin{cases} \dfrac{dy_1}{dt} = y_1^2 + z_1 + \dfrac{t}{2}, \\ \dfrac{dz_1}{dt} = -2y_1 z_1 + C z_1, \\ \dfrac{dC}{dt} = -2y_1 C + C. \end{cases}$$

The fibration $W_1[c](c) \to \mathbb{A}^1$ is given by $(y_1, z_1, C) \mapsto c = z_1 C$. So

$$\begin{aligned}&\mathcal{Y}[0] \cap W_1[c](c)\\&\quad = \{(y_1, z_1, C) \in W_1[c](c) \mid C = 0\} \cup \{(y_1, z_1, C) \in W_1[c](c) \mid z_1 = 0\}.\end{aligned}$$

Here

$$\mathcal{X}[0] \cap W_1[c] \simeq \{(y_1, z_1, C) \in W_1[c](c) \mid C = 0\}$$

and

$$(\text{the exceptional fibre}) \cap W_1[c](c) = \{(y_1, z_1, C) \in W_1[c](c) \mid z_1 = 0\}$$

Therefore on

$$(\text{the exceptional fibre}) \cap W_1[c](c)$$

that is the affine plane with the coordinate system (y_1, C) the differential equation (4.11) reduces to the system

$$\begin{cases} \dfrac{dy_1}{dt} = y_1^2 + \dfrac{t}{2}, \\ \dfrac{dC}{dt} = -2y_1 C + C. \end{cases}$$

So the differential equation on the exceptional divisor is of little interest.

References

[MMT] T. Matano, A. Matsumiya and K. Takano, *On some Hamiltonian structures on Painlevé systems II*, to appear, J. Math. Soc. Japan.

[Mu] Y. Murata, *Rational solutions of the second and the fourth Painlevé equations*, Funkcial. Ekvac., **28** (1985), 1–32.

[O1] K. Okamoto, *Sur les feuilletages associeés aux équation du second ordre à points critiques fixes de P. Painlevé*, Jap. J. Math., **5** (1979), 1–79.

[O2] ———, *Studies on the Painlevé equations*, Math. Ann., **275** (1986), 221–255.

[ST] T. Shioda and K. Takano, *On some Hamiltonian structures of Painlevé systems I*, Funkcial. Ekvac., **40** (1997), 271–291.

[US] H. Umemura and M. Saito, *Painlevé equations and deformations of rational surfaces with rational double points*, preprint.

[UW] H. Umemura, and H. Watanabe, *Solutions of the second and fourth Painlevé equations*, Nagoya Math. J., **148** (1997), 151–198.

Graduate School of Mathematics
Nagoya University
Nagoya, 464-8602
Japan
umemura@math.nagoya-u.ac.jp