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LOCALLY TRIVIAL FIBRATIONS WITH SINGULAR 1-DIMENSIONAL STEIN FIBER OVER q-COMPLETE SPACES

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Abstract. In connection with Serre's problem, we consider a locally trivial analytic fibration $\pi : E \longrightarrow B$ of complex spaces with typical fiber X. We show that if X is a Stein curve and B is q-complete, then E is q-complete.

§1. Introduction

Let $\pi : E \to B$ be a locally trivial analytic fibration of complex spaces with Stein fiber X of dimension n.

The following question was raised by Serre [17]:

Under the above assumptions, does it follow that E is Stein if B is Stein?

The answer is 'Yes' for $n = 0$ (Stein [23] and Le Barz [12]) and $n = 1$ (Mok [13]. In fact, some partial results were previously proved by various authors, Siu [19], Sibony [18], Hirschowitz [9], etc).

However, for $n \geq 2$ there are counterexamples to Serre's question (see Skoda [21], Demailly [7], and Coeuré–Loeb [3]).

Related to this circle of ideas we study the case when the base B is q-complete. The normalization is chosen such that Stein spaces correspond to 1-complete spaces.

For $n = 0$, *i.e.*, E is a topological covering of B, Ballico [2] proved the q-completeness of E. This is a particular case of a result due to Vâjâitu [24] which gives that if $\pi : Y \to Z$ is a locally trivial analytic fibration with hyperconvex fibre and Z is q-complete, then Y is q-complete. (A complex space S is said to be *hyperconvex* if S is Stein and has a negative exhaustion function which is continuous and plurisubharmonic.)

For $n = 1$, in order to generalize Mok's result, Vâjâitu [26] showed if X is non-singular, E is q-complete if B is q-complete. It remained the open problem when X is a singular Stein curve.

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Our main result gives a complete answer to this question. It can be stated as follows.

THEOREM 1. *Let $\pi : E \to B$ be a locally trivial analytic fibration with typical fiber X. If X is a Stein curve and B is q-complete, then E is q-complete, too.*

We remark that for $q = 1$ the above theorem can be deduced from the case when the fiber is non-singular. This is due to the fact that the class of Stein spaces is invariant under finite holomorphic surjections (see Narasimhan [14]).

When $q > 1$ the situation is drastically different because it is not known (see Colţoiu [6]) if the following holds:

Let $p : Y \to Z$ be a finite surjective holomorphic map of complex spaces. Assume Y is q-complete. Does it follow that Z is q-complete?

(When Z is q-complete, then Y is q-complete. See Vâjâitu [25].)

To avoid this difficulty we use essentially an approximated extension of q-convex functions defined on complex subspaces with control of the directions of positivity of the extended function. Also the quasi-plurisubharmonic functions (Peternell [15]; see also Demailly [8]) will play an important rôle in the proof.

§2. Preliminaries

Throughout this paper all complex spaces are assumed to be reduced and with countable topology.

Let Y be a complex space and T_yY denotes the (Zariski) tangent space of Y at y. Set $TY = \cup_{y\in Y} T_yY$.

A subset $\mathcal{M} \subset TY$ is said to be *a linear set over Y (of codimension $\leq q-1$)* if for every point $y \in Y$, $\mathcal{M}_y := \mathcal{M} \cap T_yY \subset T_yY$ is a complex vector subspace (of codimension $\leq q-1$). If $W \subset Y$ is an open subset, we have an obvious definition of the restriction $\mathcal{M}_{|W}$.

Let $\pi : Z \to Y$ be an analytic morphism of complex spaces and $\mathcal{M}$ a linear set over Y. For every $z \in Z$ we have an induced $\mathbf{C}$-linear map of complex vector spaces $\pi_{*,z} : T_zZ \to T_yY$, where $y = \pi(z)$. We set

$$\pi^*\mathcal{M} := \bigcup_{z\in Z} (\pi_{*,z})^{-1}(\mathcal{M}_y).$$

Clearly, $\pi^*\mathcal{M}$ is a linear set over Z. Moreover, if codim $(\mathcal{M}) \leq q-1$, then codim $(\pi^*\mathcal{M}) \leq q-1$.

A (local) chart of Y at a point $y \in Y$ is a holomorphic embedding $\iota : U \to \widehat{U}$, where $U \ni y$ is an open subset of Y and $\widehat{U}$ is an open subset of some euclidean space $\mathbf{C}^n$. Holomorphic embedding means that $\iota(U)$ is an analytic subset of $\widehat{U}$ and the induced map $\iota : U \to \iota(U)$ is biholomorphic.

Suppose $\iota : U \to \widehat{U}$ is a local chart at y; then the differential map $\iota_{*,y} : T_yY \to \mathbf{C}^n$ of ι at y is an injective homomorphism of complex vector spaces.

Let $D \subset \mathbf{C}^n$ be an open subset. A function $\varphi \in C^\infty(D, \mathbf{R})$ is said to be *q-convex* if the quadratic form

$$L(\varphi, z)(\xi) = \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(z)\xi_i\bar{\xi}_j,\ \xi \in \mathbf{C}^n,$$

has at least $n-q+1$ positive eigenvalues for every $z \in D$, or equivalently, there exists a family $\{M_z\}_{z\in D}$ of $(n-q+1)$-dimensional complex vector subspaces of $\mathbf{C}^n$ such that $L(\varphi, z)_{|M_z}$ is a positive definite form for all $z \in D$.

Let Y be a complex space. A function $\varphi \in C^\infty(Y, \mathbf{R})$ is said to be *q-convex* if every point of Y admits a local chart $\iota : U \to \widehat{U} \subset \mathbf{C}^n$ such that there is an extension $\widehat{\varphi} \in C^\infty(\widehat{U}, \mathbf{R})$ of $\varphi_{|U}$ which is q-convex on $\widehat{U}$. (This definition does not depend on the local embeddings.)

We say that Y is *q-complete* if there exists a q-convex function $\varphi \in C^\infty(Y, \mathbf{R})$ which is exhaustive, *i.e.*, the sublevel sets $\{\varphi < c\}, c \in \mathbf{R}$, are relatively compact in Y.

The following is due to Peternell [15].

Definition 1. Let Y be a complex space, $W \subset Y$ an open set, $\mathcal{M}$ a linear set over W, and $\varphi \in C^\infty(W, \mathbf{R})$.

(a) Let $y \in W$. Then we say that φ is *weakly 1-convex with respect to* $\mathcal{M}_y$ if there are: a local chart $\iota : U \to \widehat{U}$ of Y with $y \in U \subset W$, $\widehat{U} \subset \mathbf{C}^n$ open set, and an extension $\widehat{\varphi} \in C^\infty(\widehat{U}, \mathbf{R})$ of $\varphi_{|U}$ such that $L(\widehat{\varphi}, \iota(y))(\iota_{*,y}\xi) \geq 0$ for every $\xi \in \mathcal{M}_y$.

We say that φ is *weakly 1-convex with respect to* $\mathcal{M}$ if φ is weakly 1-convex with respect to $\mathcal{M}_y$ for every $y \in W$.

(b) The function φ is said to be *1-convex with respect to* $\mathcal{M}$ if every point of W admits an open neighborhood $U \subset W$ such that there exists a 1-convex function θ on U with $\varphi - \theta$ weakly 1-convex with respect to $\mathcal{M}_{|U}$.

It is not difficult to see that the extension $\widehat{\varphi}$ of φ is irrelevant for the above definition. In particular, if the functions φ and ψ are (weakly) 1-convex with respect to $\mathcal{M}$, so is their sum $\varphi + \psi$.

DEFINITION 2. Let Y be a complex space and $\mathcal{M}$ a linear set over Y. We denote by $\mathcal{B}(Y, \mathcal{M})$ the set of all $\varphi \in C^o(Y, \mathbf{R})$ such that every point of Y admits an open neighborhood D on which there are functions $f_1, \ldots, f_k \in C^\infty(D, \mathbf{R})$ which are 1-convex with respect to $\mathcal{M}_{|_D}$ and

$$\varphi_{|_D} = \max\{f_1, \ldots, f_k\}.$$

From [24] and [4] we quote

PROPOSITION 1. *Let $\mathcal{M}$ be a linear set over a complex space Y and $\varphi \in \mathcal{B}(Y, \mathcal{M})$. Then for every $\eta \in C^o(Y, \mathbf{R})$, $\eta > 0$, there exists $\widetilde{\varphi} \in C^\infty(Y, \mathbf{R})$ which is 1-convex with respect to $\mathcal{M}$ and*

$$\varphi \leq \widetilde{\varphi} < \varphi + \eta.$$

In particular, if $\mathcal{M}$ has codimension $\leq q - 1$, then $\widetilde{\varphi}$ is q-convex.

From [15] we have:

PROPOSITION 2. *Let Y be a complex space and $\varphi \in C^\infty(Y, \mathbf{R})$ a q-convex function. Then there is a linear set $\mathcal{M}$ over Y of codimension $\leq q-1$ such that φ is 1-convex with respect to $\mathcal{M}$.*

Motivated by Propositions 1 and 2, we say that a complex space Y is 1-*complete with respect to a linear set $\mathcal{M}$ over Y* if there exists an exhaustion function $\varphi \in \mathcal{B}(Y, \mathcal{M})$. Consequently a complex space Y is q-complete if, and only if, Y is 1-complete with respect to some linear set $\mathcal{M}$ of codimension $\leq q - 1$.

Let us recall that a Stein space S is said to be *hyperconvex* if there is a smooth plurisubharmonic exhaustion function $\varphi : S \to (-\infty, 0)$.

In [24] the following result has been proved:

PROPOSITION 3. *Let $\pi : E \to B$ be a locally trivial analytic fibration with hyperconvex fibre. If B is 1-complete with respect to a linear set $\mathcal{M}$ over B, then E is 1-complete with respect to $\pi^*\mathcal{M}$. In particular if B is q-complete, then E is q-complete.*

From this it follows

COROLLARY 1. *Let $\pi : E \to B$ be a covering space with q-complete base B. Let $\mathcal{M}$ be a linear set over B of codimension $\leq q-1$ such that B is 1-complete with respect to $\mathcal{M}$. Then there exists $\mu : E \to \mathbf{R}$ a smooth exhaustion function which is 1-convex with respect to $\pi^*\mathcal{M}$. In particular E is q-complete.*

We shall also need the following result of M. Peternell ([15], Satz 3.1).

PROPOSITION 4. *Let Y be a complex space and $A \subset Y$ a closed analytic subset. Then there is a function $h \in C^\infty(Y, \mathbf{R})$ such that:*

a) $h \geq 0$, $\{h = 0\} = A$.

b) *For every $y \in Y$ there exists an open neighborhood U of y and $\theta \in C^\infty(U, \mathbf{R})$ such that*

$$\log(h_{|_{U \setminus A}}) + \theta_{|_{U \setminus A}}$$

is 1-convex.

Remark 1. The function $\log h$ is locally equal to the sum of a plurisubharmonic function and a smooth function. Such a function is called in Demailly [8] a *quasi-plurisubharmonic function*.

§3. Construction of an auxiliary fibration

We recall that a complex space X is called *hyperbolic* (in the sense of Kobayashi) if the Kobayashi semidistance d_X is a distance. See the book of S. Lang [11].

Examples and properties.

1) $\mathbf{C} \setminus \{p, q\}$ with $p, q \in \mathbf{C}$, $p \neq q$, is hyperbolic.

2) Any open subset of a hyperbolic space is hyperbolic.

3) Let $\pi : X' \to X$ be a covering of complex spaces. Then X' is hyperbolic if and only if X is hyperbolic.

4) Any relatively compact open subset of $\mathbf{C}^n$ is hyperbolic.

A proof of these facts may be found in Kobayashi [10] and Lang [11].

Let us recall also the following result. (See Siu [20], p. 176 and Royden [16], p. 311.)

Lemma 1. *Let F be a hyperbolic manifold and W a connected complex space. Let $f : W \times F \to F$ be a holomorphic map such that for some $w_o \in W$ the restriction of f to $\{w_o\} \times F$ is biholomorphic onto F. Then f is independent of the variable in W, i.e., $f(w,x) = f(w_o,x)$ for all $w \in W$ and $x \in F$.*

Let now X be a Stein space of pure dimension 1 (Stein curve) and W a connected complex space.

We assume that a biholomorphic map $\Phi : W \times X \to W \times X$ is given such that the diagram

$$\begin{array}{ccc} W \times X & \xrightarrow{\;\;\Phi\;\;} & W \times X \\ & {\scriptstyle pr}\searrow \quad \swarrow{\scriptstyle pr} & \\ & W & \end{array}$$

is commutative. So, for every $w \in W$ we have an automorphism of X, $\Phi_w : X \to X$, given by $\Phi_w(x) = \Phi(w,x)$.

Let $\nu : \widetilde{X} \to X$ be the normalization map. Every $\Phi_w, w \in W$, lifts to a unique automorphism $\widetilde{\Phi}_w$ of $\widetilde{X}$ such that the diagram

$$\begin{CD} \widetilde{X} @>{\widetilde{\Phi}_w}>> \widetilde{X} \\ @V{\nu}VV @VV{\nu}V \\ X @>>{\Phi_w}> X \end{CD}$$

is commutative.

The maps $\{\widetilde{\Phi}_w\}_{w \in W}$ define a unique map $\widetilde{\Phi} : W \times \widetilde{X} \to W \times \widetilde{X}$ and we have a commutative diagram

$$\begin{CD} W \times \widetilde{X} @>{\widetilde{\Phi}}>> W \times \widetilde{X} \\ @V{\mathrm{id} \times \nu}VV @VV{\mathrm{id} \times \nu}V \\ W \times X @>>{\Phi}> W \times X. \end{CD} \tag{1}$$

We show

Lemma 2. *$\widetilde{\Phi}$ is biholomorphic.*

This will be proved in two steps.

Step 1. $\widetilde{\Phi}$ *is biholomorphic if* X *is irreducible.*

We may assume that $S := \mathrm{Sing}(X) \neq \emptyset$. Put $\widetilde{S} = \nu^{-1}(S)$. Consider the commutative diagram of isomorphisms

$$\begin{array}{ccc} W \times (\widetilde{X} \setminus \widetilde{S}) & \xrightarrow{\widetilde{\Phi}_1} & W \times (\widetilde{X} \setminus \widetilde{S}) \\ {\scriptstyle \mathrm{id} \times \nu} \downarrow & & \downarrow {\scriptstyle \mathrm{id} \times \nu} \\ W \times (X \setminus S) & \xrightarrow{\Phi_1} & W \times (X \setminus S) \end{array} \tag{2}$$

where $\Phi_1 := \Phi_{|_{W \times (X \setminus S)}}$ and $\widetilde{\Phi}_1 := \widetilde{\Phi}_{|_{W \times (\widetilde{X} \setminus \widetilde{S})}}$.

If $\mathrm{card}(\widetilde{S}) \geq 2$, then $\widetilde{X} \setminus \widetilde{S}$ is hyperbolic (by Examples 1), 2), and 3) in the beginning of this section). It follows from Lemma 1 that the maps Φ_1 and $\widetilde{\Phi}_1$ in diagram (2) are independent of $w \in W$, therefore also the maps Φ and $\widetilde{\Phi}$ in diagram (1) are independent of w. In particular $\widetilde{\Phi}$ is biholomorphic.

Similarly, if $\mathrm{card}(\widetilde{S}) = 1$ and $\widetilde{X} \neq \mathbf{C}$, it follows that $\widetilde{X} \setminus \widetilde{S}$ is hyperbolic and $\widetilde{\Phi}$ is biholomorphic (being independent of $w \in W$).

It remains to study the case when X has only one singular point, say $S = \{x_o\}$, at which X is locally irreducible (therefore $\mathrm{card}(\widetilde{S}) = 1$) and $\widetilde{X} = \mathbf{C}$ is its normalization. We may assume that $\nu^{-1}(x_o) = 0 \in \mathbf{C}$. It follows then easily that $\widetilde{\Phi}(w, \widetilde{x}) = f(w) \cdot \widetilde{x}$ with $f \in \mathcal{O}^*(W)$, so obviously $\widetilde{\Phi}$ is biholomorphic.

Step 2. $\widetilde{\Phi}$ *is biholomorphic for arbitrary* 1*-dimensional Stein space* X*.*

Clearly we may assume that X has no isolated points. Let $X = \cup X_i$ be the decomposition of X into irreducible components. We claim first that for every index i there is a unique index j so that $\Phi(W \times X_i) = W \times X_j$. To show this, we let $\mathrm{Reg}(X) = \cup D_i$ be the decomposition into connected components with $X_i = \overline{D_i}$. Obviously, for each i there is a (unique) j such that $\Phi(W \times D_i) = W \times D_j$. Using the continuity of Φ the claim follows.

Now let $\widetilde{X} = \cup \widetilde{X_i}$ be the decomposition of $\widetilde{X}$ into connected components. Therefore $\nu(\widetilde{X_i}) = X_i$ and $\nu_{|_{\widetilde{X_i}}} : \widetilde{X_i} \to X_i$ is the normalization of X_i.

From the above claim each connected component $W \times \widetilde{X}_i$ of $W \times \widetilde{X}$ corresponds by $\widetilde{\Phi}$ to a (unique) connected component $W \times \widetilde{X}_j$. Thus we have a commutative diagram

$$
\begin{array}{ccc}
W \times \widetilde{X}_i & \xrightarrow{\widetilde{\Phi}_2} & W \times \widetilde{X}_j \\
{\scriptstyle \mathrm{id} \times \nu}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \times \nu} \\
W \times X_i & \xrightarrow[\Phi_2]{} & W \times X_j
\end{array} \tag{3}
$$

where $\Phi_2 := \Phi_{|_{W \times X_i}}$ and $\widetilde{\Phi}_2 := \widetilde{\Phi}_{|_{W \times \widetilde{X}_i}}$.

We fix some biholomorphic map $h : X_j \to X_i$ (*e.g.*, $h = \Phi_{w_o}^{-1}$ for some $w_o \in W$) and consider the commutative diagram

$$
\begin{array}{ccccc}
W \times \widetilde{X}_i & \xrightarrow{\widetilde{\Phi}_2} & W \times \widetilde{X}_j & \xrightarrow{\mathrm{id} \times \widetilde{h}} & W \times \widetilde{X}_i \\
{\scriptstyle \mathrm{id} \times \nu}\downarrow & & {\scriptstyle \mathrm{id} \times \nu}\downarrow & & {\scriptstyle \mathrm{id} \times \nu}\downarrow \\
W \times X_i & \xrightarrow[\Phi_2]{} & W \times X_j & \xrightarrow[\mathrm{id} \times h]{} & W \times X_i .
\end{array} \tag{4}
$$

By step 1), $(\mathrm{id} \times \widetilde{h}) \circ \widetilde{\Phi}_2$ is biholomorphic, so $\widetilde{\Phi}_2$ is biholomorphic. It follows that $\widetilde{\Phi}$ is biholomorphic and the Lemma 2 is completely proved.

LEMMA 3. *Let $\pi : E \to B$ be a locally trivial holomorphic fibration with fibre X a pure 1-dimensional Stein space.*

Then there exist $\pi' : E' \to B$ a locally trivial holomorphic fibration with fibre $\widetilde{X}$ = the normalization of X and a holomorphic map $\tau : E' \to E$ with the following properties:

1) *The diagram*

$$
\begin{array}{ccc}
E' & \xrightarrow{\tau} & E \\
& {\scriptstyle \pi'}\searrow \quad \swarrow{\scriptstyle \pi} & \\
& B &
\end{array}
$$

is commutative.

2) *For every $b \in B$ the induced map*

$$\tau_b : E'_b \longrightarrow E_b$$

is the normalization map $\nu : \widetilde{X} \to X$.

3) *Let*

$$A := \bigcup_{b \in B} \operatorname{Sing}(E_b)$$

and $A' := \tau^{-1}(A)$. Then A and A' are closed analytic subsets of E and E' respectively and $\tau_{|_{E' \setminus A'}} : E' \setminus A' \to E \setminus A$ is biholomorphic.

Proof. Let $(W_i)_{i \in I}$ be a locally finite open covering of B such that $E_{|_{W_i}}$ is trivial and $W_i \cap W_j$ is connected for every $i, j \in I$. We have the transition functions $\Phi_{ij} : (W_i \cap W_j) \times X \to (W_i \cap W_j) \times X$ which are biholomorphic and such that the diagram

$$\begin{array}{ccc} (W_i \cap W_j) \times X & \xrightarrow{\Phi_{ij}} & (W_i \cap W_j) \times X \\ & {\scriptstyle pr} \searrow \quad \swarrow {\scriptstyle pr} & \\ & W_i \cap W_j & \end{array}$$

is commutative.

Therefore we have induced maps $\widetilde{\Phi}_{ij} : (W_i \cap W_j) \times \widetilde{X} \to (W_i \cap W_j) \times \widetilde{X}$ which are biholomorphisms by the previous lemma.

Then clearly $\{\widetilde{\Phi}_{ij}\}$ define the required holomorphic fibration $\pi' : E' \to B$. All other required properties in the lemma are easily verified.

Remark 2. The above two lemmas are trivial if we assume B to be normal. In this case (B normal) it is clear that $\widetilde{\Phi}_{ij}$ are biholomorphic and it is not necessary to make the assumption that the fiber X is a 1-dimensional Stein space (X may be an arbitrary complex space).

§4. The proof of the main result

In this section we shall prove the subsequent Theorem $1'$ which clearly implies Theorem 1 already mentioned in the introduction.

THEOREM $1'$. *Let $\pi : E \to B$ be a locally trivial analytic fibration with Stein fibre X of dimension 1 and assume that B is 1-complete with respect to a linear set $\mathcal{M}$ (over B). Then E is 1-complete with respect to $\pi^*\mathcal{M}$.*

In particular, E is q-complete if B is q-complete.

Proof. When the fiber is non-singular Theorem $1'$ is proved in [26].

Subsequently we deal with the singular fibre X.

Let $\pi' : E' \to B$ be a fibration with the properties stated in Lemma 3.

Denote $p := \pi_{|_A} : A \to B$ which is a covering map. In fact A can be described locally over B as follows: Let $E_{|_U} \simeq U \times X$ be a local trivialization. Then $A \cap \pi^{-1}(U) \simeq U \times \mathrm{Sing}(X)$ and $\mathrm{Sing}(X)$ is a discrete subset of X, say $\mathrm{Sing}(X) = \{a_j\}_{j \in J}$, since X is one dimensional. By Corollary 1 there exists a smooth exhaustion function $\mu : A \to \mathbf{R}$ such that μ is 1-convex with respect to $p^*\mathcal{M}$.

We shall prove the following statement:

(♣) Let $\eta : A \to (0, \infty)$ be any continuous function. Then there exists an open neighborhood V of A in E and a smooth function $\widetilde{\mu} : V \to \mathbf{R}$ which is 1-convex with respect to $(\pi^*\mathcal{M})_{|_V}$ and

$$\mu \leq \widetilde{\mu} < \mu + \eta$$

on A.

To prove (♣) we follow an idea from (Colţoiu [5], Lemma 3); but we refine it in order to get extensions with controlled positivity directions which are necessary for our patching process.

We fix a non-negative smooth strictly subharmonic function $f : X \to \mathbf{R}$ such that $\mathrm{Sing}(X) = \{f = 0\}$.

Let also $\{U_i\}_{i \in I}$ and $\{W_i\}_{i \in I}$ be locally finite open coverings of B such that $U_i \subset\subset W_i \subset\subset B$ and E is trivial near $\overline{W}_i$. Now select $\theta_i \in C^\infty(B, \mathbf{R})$ with $\theta_i > 0$ on $\overline{U}_i$ and $\theta_i < 0$ on ∂W_i.

We have $E_{|_{W_i}} \simeq W_i \times X$ and $E_{|_{W_i}}$ contains the sequence of mutually disjoint closed analytic subsets $W_i \times \{a_j\}$, $j \in J$. On $W_i \times \{a_j\}$ we consider the restriction of μ and perturb it with $\epsilon_{ij}\theta_i \circ \pi$. More precisely, we define near $\overline{W}_i \times \{a_j\}$

$$\mu_{ij} = \mu + \epsilon_{ij}\theta_i \circ \pi$$

where $\epsilon_{ij} > 0$ are small enough constants to be chosen later. For every $x \in A$ we set $\mu_1(x) = \max\{\mu_{ij}(x)\, ;\, (i,j) \in H(x)\}$ where $H(x) = \{(i,j) \in I \times J\, ;\, x \in W_i \times \{a_j\}\}$. If ϵ_{ij} are small enough, then μ_1 is continuous on A, $\mu_1 \in \mathcal{B}(A, p^*\mathcal{M})$, and $\mu \leq \mu_1 < \mu + \eta$ on A.

Moreover, on $\partial(W_i \times \{a_j\})$ one has

$$(*) \qquad \mu_1 > \mu_{ij}$$

for every indices $(i,j) \in I \times J$.

We shall prove that μ_1 has an extension $\widetilde{\mu_1}$ to a neighborhood V of A such that $\widetilde{\mu_1} \in \mathcal{B}(V, \pi^*\mathcal{M})$ and this clearly will conclude the proof of (♣) in view of the approximation Proposition 1.

For this we choose open neighborhoods $D_j \subset\subset X$ of the points a_j such that $\overline{D_j} \cap \overline{D_{j'}} = \emptyset$ if $j \neq j'$. The functions μ_{ij} defined on $W_i \times \{a_j\}$ can be extended to smooth functions $\widetilde{\mu_{ij}}$ on $W_i \times D_j$ which are 1-convex with respect to $\pi^*\mathcal{M}$. Indeed, if p'_{ij} and p''_{ij} denote the projections of $W_i \times D_j$ on $W_i \times \{a_j\}$ and on D_j respectively, then one may set

$$\widetilde{\mu_{ij}} := \mu_{ij} \circ p'_{ij} + f_{|_{D_j}} \circ p''_{ij}.$$

Put

$$\Omega := \bigcup_{(i,j) \in I \times J} W_i \times D_j$$

and for $x \in \Omega$, $\widetilde{\mu_1}(x) = \sup\{\widetilde{\mu_{ij}}(x)\,;\,(i,j) \in \Gamma(x)\}$ where $\Gamma(x) = \{(i,j) \in I \times J\,;\, x \in W_i \times D_j\}$.

If $V \subset \Omega$ is a small enough open neighborhood of A, it follows then from $(*)$ that $\widetilde{\mu_1}$ is continuous on V and in fact $\widetilde{\mu_1} \in \mathcal{B}(V, \pi^*\mathcal{M})$, whence the proof of statement (♣).

We now go back to the proof of Theorem $1'$. Since Theorem $1'$ holds for E', there exists a smooth exhaustion function $\psi' : E' \to \mathbf{R}$ which is 1-convex with respect to $(\pi')^*\mathcal{M}$.

We fix some smooth function $\widetilde{\mu} > 0$ defined near $\overline{V}$, where V is a sufficiently small open neighborhood of A such that $\widetilde{\mu}_{|_{\overline{V}}}$ is proper and $\widetilde{\mu}$ is 1-convex with respect to $\pi^*\mathcal{M}$ near $\overline{V}$.

By Proposition 4 there is a quasi-plurisubharmonic function $\beta : E \to [-\infty, \infty)$ with $A = \{\beta = -\infty\}$. Also we may assume $\beta = 0$ on $E \setminus V$. Then $\beta' = \beta \circ \tau$ is quasi-plurisubharmonic on E' and $A' := \tau^{-1}(A) = \{\beta' = -\infty\}$. Since ψ' is a smooth exhaustion function on E' which is 1-convex with respect to $(\pi')^*\mathcal{M}$, there is a strictly increasing smooth convex function $\delta : (0,\infty) \to (0,\infty)$ such that $\delta \circ \psi' + \beta'$ is 1-convex with respect to $(\pi')^*\mathcal{M}$ on $E' \setminus A'$, and

$$\delta \circ \psi' + \beta' > \widetilde{\mu} \circ \tau$$

on $\tau^{-1}(\partial V)$.

Now E is covered by the open subsets $V_1 := V$ and $V_2 := E \setminus A$. On V_1 we consider the function $\varphi_1 = \widetilde{\mu}$ and on V_2 the function $\varphi_2 = \delta \circ \psi' \circ \tau^{-1} + \beta$ and we define the function $\psi_1 : E \to \mathbf{R}$ given by $\psi_1(x) := \max\{\varphi_k(x)\,;\, k \in K(x)\}$, where $K = \{1, 2\}$ and $K(x) = \{k \in K\,;\, x \in V_k\}$.

Then ψ_1 is a continuous exhaustion function on E and $\psi_1 \in \mathcal{B}(E, \pi^*\mathcal{M})$. Thus the proof of Theorem $1'$ is complete.

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