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THE BLOWUP OF SOLUTIONS FOR 3-D AXISYMMETRIC COMPRESSIBLE EULER EQUATIONS

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Abstract. In this paper, for three dimensional compressible Euler equations with small perturbed initial data which are axisymmetric, we prove that the classical solutions have to blow up in finite time and give a complete asymptotic expansion of lifespan.

§1. Introduction

There are many results on the lifespan of classical solutions for nonlinear wave equations with small initial data ([1], [2] etc.). However, few papers have treated the problem of the lifespan of solutions for higher dimensional compressible Euler equations. In fact, it is very difficult to determine whether the smooth solutions of compressible Euler equations blow up or not. In [3], T. Sideris gave an upper bound for the lifespan of three dimensional Euler equations under appropriate conditions. In the paper, we will discuss 3-D problem with spherically symmetric initial data. Generally speaking, the spherically symmetric initial data don't satisfy the conditions in [3], so we can't give the upper bound of lifespan in light of the result in [3]. For the 2-D isentropic Euler equations with rotationally invariant data which are a perturbation of size ε of a rest state, S. Alinhac [4] has established the lifespan of solution. For the general irrotational initial data (not spherically symmetric) which is a small perturbation, we can also determine the lifespan of classical solutions. This will be given in another paper.

Consider the following initial data problem for 3-D compressible Euler

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equations:

$$
\begin{cases} \partial_t\rho + \operatorname{div}(\rho v) = 0 \\ \partial_t v + (v\nabla)v = -\dfrac{\nabla p(\rho)}{\rho} = -\dfrac{c^2(\rho)}{\rho}\nabla\rho \\ \rho|_{t=0} = \bar\rho + \varepsilon\rho_0(x), \quad v|_{t=0} = \varepsilon v_0(x) \end{cases} \tag{1}
$$

where $\bar\rho > 0$ is a constant, $c^2(\rho) = dp/d\rho > 0$, $p(\rho) \in C^\infty$ for $\rho > 0$, $\varepsilon > 0$ sufficiently small, $v = (v_1, v_2, v_3)$, $x = (x_1, x_2, x_3)$, $\rho_0(x), v_0(x) \in C^\infty(R^3)$ and have compact supports in $|x| \le R_0$. Moreover we assume that $v_0(x) = v_1^0(x)x$, where v_1^0 is a smooth function in R^3, and $v_1^0(x)$, $\rho_0(x)$ depend only on r, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Main Theorem. *Under the above assumptions,* (1) *has a* C^∞ *solution for* $0 \le t < T_\varepsilon$*, where*

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} \varepsilon \ln T_\varepsilon = \tau_0 \\
&\quad = -\frac{2\bar c}{(\bar\rho c'(\bar\rho) + \bar c)\displaystyle\min_{|q|\le R_0}\Big[q^2\partial_q v_1^0(q) + 3qv_1^0(q) + \frac{\bar c}{\bar\rho}(q\partial_q\rho_0(q) + \rho_0(q))\Big]}
\end{aligned}
$$

and $\bar c = c(\bar\rho)$, T_ε *denotes the lifespan of smooth solution.*

Remark. It is easy to know $\min_{|q|\le R_0}[q^2\partial_q v_1^0(q) + 3qv_1^0(q) + \bar c(q\partial_q\rho_0(q) + \rho_0(q))/\bar\rho] < 0$ unless $v_1^0 \equiv 0$, $\rho_0 \equiv 0$. Moreover $\bar\rho c'(\bar\rho) + \bar c > 0$ is known (see [5]).

For proving Theorem, we note that the rotations of v are zero, then (1) will be reduced into a nonlinear wave equation. As in [2], by constructing an approximate solution and considering the difference of exact solution and approximate solution, we easily get the lower bound of lifespan. On the other hand, the spherical symmetricity of solution makes us to change (1) into a 2×2 system equation in two variables (r, t). Hence by using the properties of above approximate solution and imitating the proof in [4], [6], [7], we may obtain the estimate of upper bound for the lifespan. Theorem asserts that the solution of (1) blows up in finite time unless $v_0 \equiv 0$, $\rho_0 \equiv 0$ in spite of any small ε.

§2. The lower bound of lifespan T_ε

Under the assumptions of Theorem, we know the solution of (1) has such a form in $t < T_\varepsilon$: $\rho(x,t) = \rho(r,t)$, $v(x,t) = \tilde v(r,t)x$, where $\tilde v(r,t)$

is a smooth function of (r,t). Because of $\operatorname{rot} v(x,t) = 0$, there exists a function $\omega(x,t) \in C^\infty$, such that $v(x,t) = \nabla\omega(x,t)$, moreover $\omega(x,t)$ has compact support in x (by the finite propagation speed we know v has compact support), and depends only on r. If we denote $\omega(r,t) = \omega(x,t)$, then $\partial_r\omega(r,t) = r\tilde{v}(r,t)$. Substituing $v(x,t) = \nabla\omega$ into the second equation in (1), then we have:

$$\partial_t\nabla\omega + \nabla\big(\frac{1}{2}|\nabla\omega|^2\big) = -\nabla h(\rho)$$

where $h'(\rho) = c^2(\rho)/\rho$, and $h(\bar{\rho}) = 0$. Hence $\partial_t\omega + |\nabla\omega|^2/2 = -h(\rho)$.

Noting $h'(\rho) > 0$, and by implicit function theorem, we know that

$$\rho = h^{-1}\big(-\big(\partial_t\omega + \frac{1}{2}|\nabla\omega|^2\big)\big), \quad \bar{\rho} = h^{-1}(0)$$

Therefore, from the first eqution in (1), it turns out

$$\begin{aligned}\partial_t^2\omega + 2\sum_{k=1}^{3}\partial_k\omega\partial_t\partial_k\omega - \frac{h^{-1}(-(\partial_t\omega + \frac{1}{2}|\nabla\omega|^2))}{(h^{-1})'(-(\partial_t\omega + \frac{1}{2}|\nabla\omega|^2))}\Delta\omega \\ + \sum_{i,k=1}^{3}\partial_i\omega\partial_k\omega\partial_i\partial_k\omega = 0.\end{aligned}$$

Now we determine the initial data $\omega|_{t=0}$ and $\partial_t\omega|_{t=0}$.

Obviously, $\omega|_{t=0} = \varepsilon\int_{R_0}^{r} sv_1^0(s)\,ds$.

Since $\partial_t\nabla\omega|_{t=0} + \nabla(|\nabla\omega|^2)|_{t=0}/2 = -c^2(\rho)\nabla\rho|_{t=0}/\rho$, then we have

$$\begin{aligned}&\partial_t\partial_r\omega|_{t=0} + \frac{1}{2}\varepsilon^2\partial_r[r^2(v_1^0(r))^2] \\ &\quad = -\varepsilon\frac{\bar{c}^2}{\bar{\rho}}\partial_r\rho_0 - \varepsilon^2\int_0^1\big(\frac{c^2(\rho)}{\rho}\big)'|_{\rho=\bar{\rho}+\theta\varepsilon\rho_0}\,d\theta\rho_0\partial_r\rho_0.\end{aligned}$$

Set

$$g(x,\varepsilon) = -\int_{R_0}^{r}\Big[\int_0^1\big(\frac{c^2(\rho)}{\rho}\big)'\Big|_{\rho=\bar{\rho}+\theta\varepsilon\rho_0(s)}d\theta\Big]\rho_0(s)\partial_s\rho_0(s)\,ds - \frac{1}{2}r^2(v_1^0(r))^2,$$

then $g(x,\varepsilon)$ is smooth in x, ε and has compact support in $r \le R_0$. So $\partial_t\omega|_{t=0} = -\varepsilon\bar{c}^2\rho_0/\bar{\rho} + \varepsilon^2 g(x,\varepsilon)$.

Then for considering the lower bound of lifespan for (1), we only discuss the lower bound of lifespan for the following problem:

$$
(2)\qquad \begin{cases}
\partial_t^2\omega + 2\sum_{k=1}^{3}\partial_k\omega\partial_t\partial_k\omega - \dfrac{h^{-1}(-(\partial_t\omega+\frac{1}{2}|\nabla\omega|^2))}{(h^{-1})'(-(\partial_t\omega+\frac{1}{2}|\nabla\omega|^2))}\Delta\omega \\
\qquad\qquad + \sum_{i,k=1}^{3}\partial_i\omega\partial_k\omega\partial_i\partial_k\omega = 0 \\
\omega|_{t=0} = \varepsilon\displaystyle\int_{R_0}^{r} sv_1^0(s)\,ds \\
\partial_t\omega|_{t=0} = -\varepsilon\dfrac{\bar{c}^2}{\bar{\rho}}\rho_0 + \varepsilon^2 g(x,\varepsilon).
\end{cases}
$$

It is easy to know $\frac{h^{-1}(-(\partial_t\omega+|\nabla\omega|^2/2))}{(h^{-1})(-(\partial_t\omega+|\nabla\omega|^2/2))} = \bar{c}^2 - 2\bar{\rho}c'(\bar{\rho})\partial_t\omega/\bar{c} + O(|\nabla_{x,t}\omega|^2)$.
we set

$$
(3)\qquad \begin{cases}
\partial_t^2\omega_0 - \bar{c}^2\Delta\omega_0 = 0 \\
\omega_0|_{t=0} = \displaystyle\int_{R_0}^{r} sv_1^0(s)\,ds \\
\partial_t\omega_0|_{t=0} = -\dfrac{\bar{c}^2}{\bar{\rho}}\rho_0
\end{cases}
$$

and

$$
\begin{cases}
\dfrac{\partial U(s,q)}{\partial s} = -\dfrac{\bar{\rho}c'(\bar{\rho})+\bar{c}}{2\bar{c}^2}\left(\dfrac{\partial U(s,q)}{\partial q}\right)^2 \\
U(0,q) = F_0(q).
\end{cases}
$$

Where $F_0(q)$ is the Friedlander radiation field of (3). By [2, (6.2.12)], we have

$$
\begin{aligned}
\frac{\partial F_0(q)}{\partial q} &= \frac{1}{2}\Big[\bar{c}\frac{d}{dq}\Big(q\int_{R_0}^{q} sv_1^0(s)\,ds\Big) + q\frac{\bar{c}^2}{\bar{\rho}}\rho_0(q)\Big] \\
&= \frac{\bar{c}}{2}\Big(\int_{R_0}^{q} sv_1^0(s)\,ds + q^2v_1^0(q) + \frac{\bar{c}}{\bar{\rho}}q\rho_0(q)\Big).
\end{aligned}
$$

Set $\omega_a = \varepsilon[\chi(\varepsilon t)\omega_0 + (1-\chi(\varepsilon t))r^{-1}U(\varepsilon\ln(\varepsilon t), r-\bar{c}t)]$, where $\chi\in C^\infty(R)$ decreases, equals to 1 in $(-\infty,1)$, and equals to 0 in $(2,\infty)$. Then we have the following conclusion:

Lemma 1. *For sufficiently small $\varepsilon>0$, and $\varepsilon\ln T < b < \tau_0$, then*

(i) (2) *has a* C^∞ *solution for* $0 < t \le T$.

(ii) $|\partial^\beta_{x,t}(\omega - \omega_a)| \le \dfrac{C_{\beta,b}\varepsilon^2 \ln(1/\varepsilon)}{1+t}$, $|\beta| \le 2$.

(iii) $|\partial^\beta_{x,t}\nabla_{x,t}\omega_a| \le \dfrac{C_{\beta,b}\varepsilon}{1+t}$, $|\beta| \ge 0$.

Proof. (i) Noting $\partial^2 F_0(q)/\partial q^2 = \bar{c}[q^2\partial_q v^0_1(q) + 3qv^0_1(q) + \bar{c}q\partial_q\rho_0(q)/\bar{\rho} + \bar{c}\rho_0(q)/\bar{\rho}]/2$, as in [2, Theorem 6.5.7], we know (i) holds.

(ii) By [2, Lemma 6.5.6] and S.Klainerman's inequality in [8], we easily know (ii) holds.

(iii) Its proof is similar to that in [2, Lemma 6.5.5].

From Lemma 1 (i), it is easy to know that $\lim\limits_{\varepsilon\to 0} \varepsilon \ln T_\varepsilon \ge \tau_0$.

Remark. We state that $\min_{|q|\le M}[q^2\partial_q v^0_1(q) + 3qv^0_1(q) + \bar{c}q\partial_q\rho_0(q)/\bar{\rho}] < 0$ unless $v_0 \equiv 0$ and $\rho_0 \equiv 0$. In fact, if $\min_{|q|\le M}[q^2\partial_q v^0_1(q) + 3v^0_1(q) + \bar{c}q\partial_q\rho_0(q)/\bar{\rho}] \ge 0$, then $\partial^2 F_0(q)/\partial q^2 \ge 0$. Because $F_0(q)$ has compact support in $|q| \le M$, then $\partial F_0(q)/\partial q \equiv 0$, hence $F_0(q) \equiv 0$. By [2, Theorem 6.2.2], we have $v^0_1 \equiv 0$, $\rho_0 \equiv 0$.

§3. The upper bound of lifespan T_ε

Assume that ω_a, b are defined as above, when $\varepsilon \ln T \le b$, we set $\rho_a = h^{-1}(-(\partial_t\omega_a + |\nabla\omega_a|^2/2))$. Since

$$\rho - \rho_a = -\int_0^1 (h^{-1})'\Big(-\theta\big(\partial_t u + \frac{1}{2}|\nabla u|^2\big) - (1-\theta)\big(\partial_t u_a + \frac{1}{2}|\nabla u_a|^2\big)\Big)d\theta \times\Big[\partial_t(u-u_a) + \frac{1}{2}(\nabla u + \nabla u_a)\nabla(u-u_a)\Big]$$

then by lemma 1, then we obtain

$$|\partial^\beta_{x,t}(\rho - \rho_a)| \le \frac{C_{\beta,b}\varepsilon^2\ln(1/\varepsilon)}{1+t}, \quad |\beta| \le 2 \tag{4}$$

and

$$|\partial^\beta_{x,t}(\rho_a - \bar{\rho})| \le \frac{C_{\beta,b}\varepsilon}{1+t}, \quad |\beta| \ge 0 \tag{5}$$

Set $\alpha(r,t) = \partial_r\omega(r,t)$, $\alpha_a(r,t) = \partial_r\omega_a(r,t)$. For the smooth solution of (1), we have

Lemma 2. (a) α, ρ *satisfy the following system*:

$$\left\{\begin{array}{l}\partial_t\rho + \alpha\partial_r\rho + \rho\big(\partial_r\alpha + \dfrac{2}{r}\alpha\big) = 0\\ \partial_t\alpha + \alpha\partial_r\alpha + \dfrac{c^2(\rho)}{\rho}\partial_r\rho = 0\end{array}\right. \tag{6}$$

(b) *Set* $\rho = \bar\rho + A(r,t)/r$, $\alpha = B(r,t)/r$, $Z_1 = (\partial_r A/\rho + \partial_r B/c)/2$, $Z_2 = (-\partial_r A/\rho + \partial_r B/c)/2$, *then* (Z_1, Z_2) *satisfy the following system*:

$$\left\{\begin{array}{l}\partial_t Z_1 + (\alpha + c)\partial_r Z_1 = Q_1\\ \partial_t Z_2 + (\alpha - c)\partial_r Z_2 = Q_2\end{array}\right. \tag{7}$$

where

$$\begin{aligned}Q_1 = &-\frac{Z_1^2}{r}(c + c'\rho) + \frac{Z_1Z_2}{r}\big(3c'\rho + \frac{B}{r}\big) + \frac{Z_2^2}{r}\big(c - \frac{B}{r}\big)\\ &+\frac{Z_1}{r^2}\Big[3B + \frac{cA}{\rho} + \frac{c'A}{2} + \frac{c'B\rho}{2c} + \big(\frac{c^2(\rho)}{\rho}\big)'\frac{\rho A}{c} - \frac{B\bar\rho}{2\rho} - \frac{AB}{2\rho r} + \frac{cr}{2\rho}(\rho - \bar\rho)\Big]\\ &+\frac{Z_2}{r^2}\Big[B - \frac{cA}{\rho} - \big(\frac{c^2(\rho)}{\rho}\big)'\frac{\rho A}{c} + \frac{c'B\rho}{2c} + \frac{cA}{\rho} + \frac{c'A}{2} + \frac{B\bar\rho}{2\rho} - \frac{cB}{2\rho r} - \frac{cr}{2} - \frac{\bar\rho cr}{2\rho}\Big]\\ &+\frac{1}{r^2}\Big(-\frac{cA}{2\rho} + B - \frac{\bar\rho B}{2\rho}\Big) + \frac{1}{r^3}\Big[-\frac{B^2}{c} - \big(\frac{c^2(\rho)}{\rho}\big)'\frac{A^2}{2c} - \frac{3AB}{2\rho} + \frac{\rho cB}{2\rho^2}\Big]\\ &+\frac{ABc}{2\rho^2r^4},\end{aligned}$$

$$\begin{aligned}Q_2 = &\frac{Z_1Z_2}{r}\big(3c'\rho - 2A - \frac{B}{r}\big) + \frac{Z_2^2}{r}\big(-2A + \frac{B}{r} - c'\rho\big)\\ &+\frac{Z_1}{r^2}\Big[\big(\frac{c^2(\rho)}{\rho}\big)'\frac{\rho A}{c} + \frac{cr}{2} + \frac{c'B\rho}{2c} - \frac{c'A}{2} + \frac{B\bar\rho}{2\rho} + \frac{\bar\rho}{2\rho}r + \frac{Bc}{2\rho r}\Big]\\ &+\frac{Z_2}{r^2}\Big[2B - \frac{cr}{2} - \big(\frac{c^2(\rho)}{\rho}\big)'\frac{\rho A}{c} - \frac{cA}{\rho} + \frac{c'B\rho}{2c} - \frac{c'A}{2} - \frac{B\bar\rho}{2\rho} + \frac{\bar\rho c}{2\rho}r + \frac{Bc}{2\rho r}\Big]\\ &+\frac{1}{r^2}\Big(-\frac{cA}{2\rho} - B + \frac{\bar\rho B}{2\rho}\Big) + \frac{1}{r^3}\Big[-\frac{B^2}{c} - \big(\frac{c^2(\rho)}{\rho}\big)'\frac{A^2}{2c} + \frac{3AB}{2\rho} - \frac{\bar\rho Bc}{2\rho^2}\Big]\\ &-\frac{ABc}{2\rho^2r^4}.\end{aligned}$$

Remark. It is very important to appear the factor $cr(\rho - \bar\rho)/2\rho$ in the coefficients of Z_1/r^2 in Q_1, or we can't give the upper bound of lifespan T_ε of solution to (1).

Proof. (a) It can be verified directly, we omit it.

(b) Since $Z_1+Z_2=\partial_r B/c=(\alpha+r\partial_r\alpha)/c$, $Z_1-Z_2=(r\partial_r\rho+\rho-\bar\rho)/\rho$, thus

$$\begin{aligned}
\partial_r\alpha &= \frac{c(Z_1+Z_2)}{r}-\frac{B}{r^2},\quad \partial_r\rho=\frac{\rho(Z_1-Z_2)}{r}-\frac{A}{r^2},\\
\partial_r^2\alpha &= \frac{c\partial_r(Z_1+Z_2)}{r}+\frac{c'\rho(Z_1^2-Z_2^2)}{r^2}-\frac{Ac'(Z_1+Z_2)}{r^3}-\frac{2c(Z_1+Z_2)}{r^2}+\frac{2B}{r^3},\\
\partial_r^2\rho &= \frac{\rho\partial_r(Z_1-Z_2)}{r}+\frac{\rho(Z_1-Z_2)^2}{r^2}-\frac{A(Z_1-Z_2)}{r^3}-\frac{2\rho(Z_1-Z_2)}{r^2}+\frac{2A}{r^3}.
\end{aligned}$$

then

$$\begin{aligned}
\partial_t(Z_1+Z_2) &= -\frac{B}{r}\partial_r(Z_1+Z_2)-c\partial_r(Z_1-Z_2)-\frac{c(Z_1-Z_2)}{r}\\
&\quad -\frac{c}{r}(Z_1+Z_2)^2+\frac{B}{r^2}(Z_1+Z_2)-\frac{2B^2}{cr^3}+\frac{2B(Z_1+Z_2)}{r^2}\\
&\quad -\Big(\frac{c^2(\rho)}{\rho}\Big)'\Big[\frac{\rho^2(Z_1-Z_2)^2}{cr}-\frac{2\rho A(Z_1-Z_2)}{cr^2}+\frac{A^2}{cr^3}\Big]\\
&\quad -\frac{c(Z_1-Z_2)^2}{r}+\frac{cA(Z_1-Z_2)}{\rho r^2}+\frac{2c(Z_1-Z_2)}{r}-\frac{cA}{\rho r^2}\\
&\quad +\frac{\rho c'(Z_1+Z_2)^2}{r}+\frac{c'\rho B(Z_1+Z_2)}{cr^2}
\end{aligned}\tag{8}$$

and

$$\begin{aligned}
\partial_t(Z_1-Z_2) &= -\frac{B}{r}(Z_1-Z_2)-c\partial_r(Z_1-Z_2)-\frac{2c}{r}(Z_1^2-Z_2^2)\\
&\quad +\frac{2cA(Z_1+Z_2)}{\rho r^2}-\frac{3AB}{\rho r^3}-\frac{B(Z_1-Z_2)^2}{r^2}+\frac{2B(Z_1-Z_2)}{r^2}\\
&\quad -\frac{c'\rho}{r}(Z_1^2-Z_2^2)+\frac{c'A(Z_1+Z_2)}{r^2}+\frac{2B}{r^2}-\frac{B\bar\rho(Z_1-Z_2)}{\rho r^2}\\
&\quad +\frac{\bar\rho Bc}{\rho^2r^3}-\frac{\bar\rho c(Z_1+Z_2)}{\rho r}-\frac{\bar\rho B}{\rho r^2}+\frac{B(z_1^2-Z_2^2)}{r^2}-\frac{Bc(Z_1+Z_2)}{\rho r^3}\\
&\quad +\frac{c(Z_1+Z_2)^2}{r}+\frac{B(Z_1+Z_2)}{r^2}+\frac{ABc}{\rho^2r^4}-\frac{Ac(Z_1+Z_2)}{\rho r^2}.
\end{aligned}\tag{9}$$

From (8) and (9), it is easy to obtain (7). Hence Lemma 2 is proved.

Denote $\Gamma_\lambda^\pm$ as the integral curves of $\partial_t\pm(\alpha+c)\partial_r$ passed through $(\lambda,0)$ in the plan (r,t), D as the strip domain bounded by $\Gamma_{R_0}^+$ and $\Gamma_{\sigma_0-1}^+$ where

σ_0 is taken such that $(q^2\partial_q v_1^0(q) + 3qv_1^0(q) + \bar{c}q\partial_q\rho_0(q)/\bar{\rho} + \bar{c}\rho_0(q)|_{q=\sigma_0}/\bar{\rho})$ is minimum. $D_\varepsilon = D \cap \{t \geq 1/2\varepsilon\}$. As in [4], [6], [7], for any T satisfying $1/2\varepsilon \leq T \leq e^{b/\varepsilon}$, $b < \tau_0$, we define

$$J(t) = \sup_{\substack{1/2\varepsilon \leq s \leq t \\ (r,s)\in D}} \int |Z_1(r,s)|\, dr, \quad M(t) = \sup_{\substack{1/2\varepsilon \leq s \leq t \\ (r,s)\in D}} (|A(r,s)| + |B(r,s)|),$$

and $V(t) = \sup_{\substack{1/2\varepsilon \leq s \leq t \\ (r,s)\in D}} s|Z_2(r,s)|$.

Lemma 3. *There exist some constants $J_1, M_1, V_1 > 0$, for sufficiently small $\varepsilon > 0$ and any t satisfying $1/2\varepsilon \leq t \leq T \leq e^{b/\varepsilon}$, such that $J(t) \leq J_1\varepsilon, M(t) \leq M_1\varepsilon, V(t) \leq V_1\varepsilon^{1/2}$. Moreover $r \geq \bar{c}t/2$ in D_ε.*

Proof. Firstly we verify the lemma for $1/2\varepsilon \leq t \leq 1/\varepsilon$.

Set $\rho = \rho_a + \dot{\rho}$. It turns out

$$2Z_1 = r\big(\frac{\partial_r\rho_a + \partial_r\dot{\rho}}{\rho} + \frac{\partial_r\alpha}{c}\big) + \big(\frac{\rho_a - \bar{\rho} + \dot{\rho}}{\rho} + \frac{\alpha}{c}\big)$$

and

$$2Z_2 = r\big(-\frac{\partial_r\rho_a + \partial_r\dot{\rho}}{\rho} + \frac{\partial_r\alpha}{c}\big) + \big(-\frac{\rho_a - \bar{\rho} + \dot{\rho}}{\rho} + \frac{\alpha}{c}\big).$$

Since

$$\begin{aligned} t\Big|2Z_2 - r\big(\frac{\partial_r\alpha_a}{c} - \frac{\partial_r\rho_a}{\rho}\big)\Big| &\\ \leq t\Big|r\big(-\frac{\partial_r\dot{\rho}}{\rho} + \frac{\partial_r\alpha - \partial_r\alpha_a}{c}\big) + \big(-\frac{\rho_a - \bar{\rho} + \dot{\rho}}{\rho} + \frac{\alpha}{c}\big)\Big| &\\ \leq cte\big(\varepsilon + \varepsilon\ln\frac{1}{\varepsilon}\big). & \end{aligned}$$

Noting that

$$\frac{\partial_r\alpha_a}{c} - \frac{\partial_r\rho_a}{\rho} = \frac{(\rho - \bar{\rho})\partial_r\alpha_a}{\rho c} - \frac{(c - \bar{c})\partial_r\rho_a}{\rho c} + \frac{\bar{\rho}\partial_r\alpha_a - \bar{c}\partial_r\rho_a}{\rho c}$$

and

$$\begin{aligned} \bar{\rho}\partial_r\alpha_a - \bar{c}\partial_r\rho_a = \frac{\bar{\rho}}{\bar{c}}\varepsilon(\bar{c}\partial_r + \partial_t)\partial_r\big(\omega_a - \frac{F_0(r - \bar{c}t)}{r}\big) &\\ +\varepsilon\bar{\rho}\Big[\frac{2F_0(r - \bar{c}t)}{r^3} - \frac{\partial_q F_0(r - \bar{c}t)}{r^2}\Big] &\\ +O(|\nabla_{x,t}\omega_a|^2) + O(|\nabla\omega_a\nabla\partial_r\omega_a|). & \end{aligned}$$

Thus, by

$$\Big|\partial_{x,t}^{\beta}\big(\omega_a - \frac{F_0(r-\bar{c}t)}{r}\big)\Big| \le \frac{C_\beta}{(1+t)^2}, \quad |\beta| \ge 0, \big(r \ge \frac{\bar{c}}{2}t\big)$$

we know

$$t|Z_2| \le tr\Big|\frac{\partial_r \alpha_a}{c} - \frac{\partial_r \rho_a}{\rho}\Big| + cte\big(\varepsilon + \varepsilon \ln\frac{1}{\varepsilon}\big) \le cte\varepsilon^{1/2}$$

i.e.,

$$V(t) \le cte\varepsilon^{1/2}.$$

On the other hand, owing to the width of D is finite and $|Z_1| \le cte\varepsilon$, then

$$J(t) \le cte\varepsilon, \quad M(t) = \sup_{\substack{1/2\varepsilon \le s \le t \\ (r,s)\in D}} (|r\alpha| + |r(\rho - \bar{\rho})|) \le cte\varepsilon.$$

We choose $M_1 = (2\bar{c}^2 + 2\bar{c}\bar{\rho})J_1$, $V_1 = J_1$, *s.t.* $J(1/2\varepsilon) \le J_1\varepsilon/4$, $M(1/2\varepsilon) \le M_1\varepsilon/4$ and $V(1/2\varepsilon) \le J_1\varepsilon^{1/2}/4$.

Now we verify the lemma as $1/2\varepsilon \le t \le T' \le T$. For this aim, as in [4], we first claim that:

(i) On $\Gamma_\lambda^+ \subset D$, then $|r - \bar{c}t - \lambda| \le cte$. In particular, for $t \ge 1/2\varepsilon$ and ε small enough, then $r \ge 3\bar{c}t/4$.

In fact, along $\Gamma_\lambda^+ \subset D$, $dr - \bar{c}dt = (\alpha + c - \bar{c})\,dt$, then

$$\begin{aligned} |r - \bar{c}t - \lambda| &\le cte \int_0^t |\alpha + c - \bar{c}|\,dt \\ &\le cte \int_0^t \frac{\varepsilon}{1+t}\,dt \le cte\varepsilon \ln(1+t) \le cte. \end{aligned} \tag{10}$$

(ii) If $(r,t) \in \Gamma_\mu^- \cap D_\varepsilon$, $(r',t') \in \Gamma_\mu^- \cap D_\varepsilon$, $t' \le t$, then $t - t' \le cte$. In particular, $t' \ge t/2$ for sufficiently small ε.

In fact,

$$\Big|\frac{d}{dt}(r + \bar{c}t - \mu)\Big| \le \frac{cte\varepsilon}{1+t}$$

and

$$|r(t) + \bar{c}t - \mu - (r(t') + \bar{c}t' - \mu)| \le cte\varepsilon \ln\frac{1+t}{1+t'} \le cte.$$

Then imitating the proof in [4], we know the statement (ii) holds.

Choosing $\mu = -\alpha - c$, along the integtal curve in D, we write

$$d(Z_1(dr + \mu dt)) = [(\alpha + c + \mu)\partial_r Z_1 + Z_1\partial_r\mu - Q_1]dr \wedge dt = \widetilde{Q} dr \wedge dt$$

where

$$\begin{aligned}\widetilde{Q} = &-\frac{Z_1Z_2}{r}(c + 2c'\rho + \frac{B}{r}) - \frac{Z_2^2}{r}(c - \frac{B}{r})\\
&-\frac{Z_1}{r^2}\Big[2B + \frac{cA}{\rho} - \frac{cA}{2} + \frac{c'\rho B}{2c} + (\frac{c^2(\rho)}{\rho})'\frac{\rho A}{c} - \frac{B\bar{\rho}}{2\rho} - \frac{AB}{2\rho r} + \frac{cr}{2\rho}(\rho - \bar{\rho})\Big]\\
&-\frac{Z_2}{r^2}\Big[B - \frac{cA}{\rho} - (\frac{c^2(\rho)}{\rho})'\frac{\rho A}{c} + \frac{c'B\rho}{2c} + \frac{cA}{\rho} + \frac{cA}{2} + \frac{B\bar{\rho}}{2\rho} - \frac{Bc}{2\rho r} - \frac{cr}{2} - \frac{\bar{\rho}cr}{2\rho}\Big]\\
&-\frac{1}{r^2}(-\frac{cA}{2\rho} + B - \frac{\bar{\rho}B}{2\rho}) - \frac{1}{r^3}\Big[-\frac{B^2}{c} - (\frac{c^2(\rho)}{\rho})'\frac{A^2}{2c} - \frac{3AB}{2\rho} + \frac{\bar{\rho}Bc}{2\rho^2}\Big]\\
&-\frac{cAB}{2\rho^2 r^4}.\end{aligned}$$

As in [4], we have

(i) $\displaystyle\int_{(r,t)\in D} |Z_1(r,t)|\, dr \le J(\frac{1}{2\varepsilon}) + \int_{\substack{1/2\varepsilon\le s\le t\\(r,s)\in D}} |\widetilde{Q}|\, drds$

(ii) $\displaystyle cte\int_{\Gamma^-(x,t)\cap D_\varepsilon} |Z_1(r,t)|\, dr \le J(\frac{1}{2\varepsilon}) + \int_{\substack{1/2\varepsilon\le s\le t\\(r,s)\in D}} |\widetilde{Q}|\, drds$

where $\Gamma^-(x,t)$ is the integral curve of $\partial_t - (\alpha + c)\partial_r$ passed through $(x,t) \in D_\varepsilon$.

In D_ε, for sufficiently small ε, we have

$$\begin{aligned}|\widetilde{Q}| \le cte\Big[&|Z_1|(\frac{V_1\varepsilon^{1/2}}{t^2} + \frac{M_1\varepsilon}{t^2} + \frac{M_1^2\varepsilon^2}{t^3} + \frac{M_1V_1\varepsilon^{3/2}}{t^3} + \frac{\varepsilon}{t^2})\\
&+\frac{V_1^2\varepsilon}{t^3}(1 + \frac{M_1\varepsilon}{t}) + \frac{V_1\varepsilon^{1/2}}{t^3}(M_1\varepsilon + t) + \frac{M_1\varepsilon}{t^2} + \frac{M_1^2\varepsilon^2}{t^3} + \frac{M_1^2\varepsilon^2}{t^4}\Big].\end{aligned}$$

Hence

$$\int_{\substack{1/2\varepsilon\le s\le t\\(r,s)\in D}} |\widetilde{Q}|\, drds \le cteJ_1\varepsilon^{3/2},\ \ J(t) \le J(\frac{1}{2\varepsilon}) + \int_{\substack{1/2\varepsilon\le s\le t\\(r,s)\in D}} |\widetilde{Q}|\, drds \le \frac{1}{2}J_1\varepsilon.$$

In order to estimate $M(t)$, we note that

$$\begin{cases}\partial_t A + (\alpha - c)\partial_r A = -2\rho c Z_1 + \dfrac{AB}{r^2} - \dfrac{\rho B}{r}\\[2mm]
\partial_t B + (\alpha - c)\partial_r B = -2c^2 Z_1 + \dfrac{B^2}{r^2} + \dfrac{c^2(\rho)}{\rho}\dfrac{A}{r}.\end{cases}$$

Then, for ε small enough, we have

$$(|A|+|B|)(r,t) \le \frac{3}{4}M_1\varepsilon.$$

Similarly, $V(t) \le V_1\varepsilon^{1/2}/2$.

Therefore, by continuous induction, we know that Lemma 3 holds.

Along $\Gamma^+_{\sigma_0} \subset D$, we define $w(t) = -Z_1(r(t),t)$. It is easy to know that w satisfies the equation:

$$w'(t) = a_0(t)w^2 + a_1(t)w + a_2(t) \tag{11}$$

where

$$\begin{aligned}
a_0(t) &= \Big(\frac{\rho c' + c}{r}\Big)(r(t),t),\\
a_1(t) &= \frac{Z_2}{r}\Big(3c'\rho + \frac{B}{r}\Big) + \frac{1}{r^2}\Big[3B + \frac{cA}{\rho} + \frac{c'A}{2} + \frac{c'B\rho}{2c} + \Big(\frac{c^2(\rho)}{\rho}\Big)'\frac{\rho A}{c}\\
&\qquad - \frac{B\bar\rho}{2\rho} - \frac{AB}{2\rho r} + \frac{cr}{2\rho}(\rho - \bar\rho)\Big],\\
a_2(t) &= -\frac{Z_2}{r}\Big(c - \frac{B}{r}\Big) - \frac{Z_2^2}{r}\Big[B - \frac{cA}{\rho}\Big(\frac{c^2(\rho)}{\rho}\Big)'\frac{\rho A}{c} + \frac{\bar cB\rho}{2c} + \frac{cA}{\rho} + \frac{c'A}{2}\\
&\qquad + \frac{B\rho}{2\rho} - \frac{Bc}{2\rho r} - \frac{cr}{2} - \frac{\bar\rho cr}{2\rho}\Big] - \frac{1}{r^2}\Big(-\frac{cA}{2\rho} + B - \frac{\bar\rho B}{2\rho}\Big)\\
&\qquad - \frac{1}{r^3}\Big[-\frac{B^2}{c} - \Big(\frac{c^2(\rho)}{\rho}\Big)'\frac{A^2}{2c} - \frac{3AB}{2\rho} + \frac{\bar\rho Bc}{2\rho^2}\Big] - \frac{cAB}{2\rho^2 r^4}.
\end{aligned}$$

Lemma 3 and the above calculation give out the following estimates

$$a_0(t) = \frac{\bar\rho c'(\bar\rho) + \bar c}{\bar c t} + O(t^{-2}), \quad |a_1(t)| \le cte\frac{\varepsilon^{1/2}}{t^2}, \quad |a_2(t)| \le cte\frac{\varepsilon^{1/2}}{t^2}$$

and

$$\begin{aligned}
w\Big(\frac{1}{2\varepsilon}\Big) &= -\frac{1}{2}\Big[\frac{(\rho_a - \bar\rho)(r(\frac{1}{2\varepsilon}), \frac{1}{2\varepsilon}) + r(\frac{1}{2\varepsilon})\partial_r\rho_a(r(\frac{1}{2\varepsilon}), \frac{1}{2\varepsilon})}{\rho_a(r(\frac{1}{2\varepsilon}), \frac{1}{2\varepsilon})}\\
&\qquad + \frac{\alpha_a(r(\frac{1}{2\varepsilon}), \frac{1}{2\varepsilon}) + r(\frac{1}{2\varepsilon})\partial_r\alpha_a(r(\frac{1}{2\varepsilon}), \frac{1}{2\varepsilon})}{c(\rho_a(r(\frac{1}{2\varepsilon}), \frac{1}{2\varepsilon}))}\Big] + o(\varepsilon).
\end{aligned} \tag{12}$$

As ε is small enough, from the proof in Lemma 3, we know $r(1/2\varepsilon) \geq (\bar{c}/2)(1/2\varepsilon)$. So it turns out:

$$\Big|\partial^{\beta}_{x,t}\big(\omega_0 - \frac{F_0(r-\bar{c}t)}{r}\big)\Big|_{r=r(1/2\varepsilon),t=1/2\varepsilon} \leq \frac{cte}{(1+\frac{1}{2\varepsilon})^2} \leq cte\varepsilon^2, \;\; |\beta| \geq 0$$

and

$$\alpha_a\Big(r\big(\frac{1}{2\varepsilon}\big), \frac{1}{2\varepsilon}\Big) = \varepsilon\Big[\frac{\partial_q F_0(r(\frac{1}{2\varepsilon}) - \frac{\bar{c}}{2\varepsilon})}{r(\frac{1}{2\varepsilon})} - \frac{F_0(r(\frac{1}{2\varepsilon}) - \frac{\bar{c}}{2\varepsilon})}{r^2(\frac{1}{2\varepsilon})}\Big] + o(\varepsilon),$$

$$\begin{aligned}\rho_a - \bar{\rho} = {}& \frac{\varepsilon\bar{\rho}}{\bar{c}}\frac{\partial_q F_0(r(\frac{1}{2\varepsilon}) - \frac{\bar{c}}{2\varepsilon})}{r(\frac{1}{2\varepsilon})} - \frac{\varepsilon\bar{\rho}}{\bar{c}} r\big(\frac{1}{2\varepsilon}\big)\Big[-\frac{\partial_q^2 F_0(r(\frac{1}{2\varepsilon}) - \frac{\bar{c}}{2\varepsilon})}{r(\frac{1}{2\varepsilon})} \\ & + \frac{\partial_q F_0(r(\frac{1}{2\varepsilon}) - \frac{\bar{c}}{2\varepsilon})}{r^2(\frac{1}{2\varepsilon})}\Big] + o(\varepsilon).\end{aligned}$$

Substituting them into (12), we obtain

$$w\big(\frac{1}{2\varepsilon}\big) = -\frac{\varepsilon\partial_q^2 F_0(r(\frac{1}{2\varepsilon}) - \frac{\bar{c}}{2\varepsilon})}{\bar{c}} + o(\varepsilon).$$

By (10), $|r(1/2\varepsilon) - \bar{c}/2\varepsilon - \sigma_0| \leq cte\varepsilon\ln(1+1/2\varepsilon)$, then

$$w\big(\frac{1}{2\varepsilon}\big) = -\frac{\varepsilon}{2}\Big[\sigma_0^2\partial_q v_1^0(\sigma_0) + 3\sigma_0 v_1^0(\sigma_0) + \frac{\bar{c}}{\bar{\rho}}(\sigma_0\partial_q\rho_0(\sigma_0) + \rho_0(\sigma_0))\Big] + o(\varepsilon)$$

Denote $K = (\int_{1/2\varepsilon}^T |a_2(t)|\,dt)\exp(\int_{1/2\varepsilon}^T |a_1(t)|\,dt)$, and note the equation (11) satisfies the conditions of [2, Lemma 1.3.2.] in $1/2\varepsilon \leq t \leq T$, so we obtain:

$$\Big(\int_{1/2\varepsilon}^T a_0(t)\,dt\Big)\exp\Big(-\int_{1/2\varepsilon}^T |a_1(t)|\,dt\Big) < \big(w\big(\frac{1}{2\varepsilon}\big) - K\big)^{-1}$$

that is

$$\begin{aligned}&\frac{\bar{\rho}c'(\bar{\rho}) + \bar{c}}{\bar{c}}(\ln T + \ln 2\varepsilon + O(\varepsilon))e^{-cte\varepsilon^{3/2}} \\ &\quad < \Big[-\frac{\varepsilon}{2}\big(\sigma_0^2\partial_q v_1^0(\sigma_0) + 3\sigma_0 v_1^0(\sigma_0) + \frac{\bar{c}}{\bar{\rho}}\sigma_0\partial_q\rho_0(\sigma_0) + \frac{\bar{c}}{\bar{\rho}}\rho_0(\sigma_0)\big) \\ &\qquad + o(\varepsilon) + cte\varepsilon^{3/2}\Big]^{-1}.\end{aligned}$$

Let $\varepsilon \to 0$, we get

$$\lim_{\varepsilon\to 0}\varepsilon\ln T_\varepsilon \leq \tau_0. \tag{13}$$

Combine (13) with the conclusion in §2, we know Theorem holds.

References

[1] S. Klainerman, *Uniform decay estimate and the Lorentz invariance of the classical wave equation*, Comm. Pure Appl. math., **38** (1985), 321–332.

[2] L. Hömander, *Nonlinear hyperbolic differential equations*, Lectures, 1986–1987.

[3] T. Sideris, *Formation of solution singularities in three dimensional compressible fluids*, Comm. Math. Phys., **101** (1985), 475–487.

[4] S. Alinhac, *Temps de vie des solutions regulieres de equations d'Euler compressibles axisymetriques en dimendion deux*, Invent. Math., **111** (1993), 627–667.

[5] R. Courant, K. O. Friedrichs, Supersonic flow and shock waves, Wiley Interscience, New York, 1949.

[6] L. Hömander, *The lifespan of classical solutions of nonlinear hyperbolic equations*, Mittag-Leffler report No. 5 (1985).

[7] F. John, *Blow-up of radial solutions of* $u_{tt} = c^2(u_t)\Delta u$ *in three space dimensions*, Mat. Apl. Comput., **4** (1985), No. 1, 3–18.

[8] S. Klainerman, *Remarks on the global Sobelev inequalities in the Minkowski space* R^{n+1}, Comm. Pure Appl. Math., **40** (1987), 111–117.

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