SOME PROPERTIES OF ISOLATING BLOCKS
FOR PLANAR SYSTEMS

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Abstract. In this paper, some qualitative properties of trajectories inside an isolating block for planar differential equations are obtained.

§1. Introduction

Consider the differential system defined in the plane

$$
(1.1) \qquad
\begin{aligned}
\frac{dx}{dt} &= X(x,y), \\
\frac{dy}{dt} &= Y(x,y).
\end{aligned}
$$

Suppose $X, Y \in C^1$. Let the vector field $V \equiv (X,Y)$ define a flow $f(p,t)$. Let $B \subset R^2$ be the closure of a bounded and connected open set with the boundary ∂B. In general, B is assumed to be multiply connected. Let $L_1, \cdots, L_n$ denote its boundary components, where $L_i \cap L_j = \phi$ for $i \neq j$, and L_1 the external boundary. Each of them is a smooth simple closed curve. Let $\text{int}B$ denote the interior of B. We define three subsets b^+, b^- and τ as follows:

$$
b^+ = \{p \in \partial B | \exists \varepsilon > 0 \quad \text{with} \quad f(p, (-\varepsilon, 0)) \cap B = \phi\},
$$

$$
b^- = \{p \in \partial B | \exists \varepsilon > 0 \quad \text{with} \quad f(p, (0, \varepsilon)) \cap B = \phi\},
$$

$$
\tau = \{p \in \partial B | V \quad \text{is tangent to} \quad B \text{ at } p\}.
$$

DEFINITION 1.1. ([1]) If $b^+ \cap b^- = \tau$ and $b^+ \cup b^- = \partial B$, then B is called an isolating block for the flow defined by (1.1).

It follows from the above definition that if B is an isolating block, then all the tangencies to B must be external.

DEFINITION 1.2. Suppose $B \subset R^2$ is an isolating block for the flow defined by (1.1). If a trajectory Γ_1 of (1.1) enters B at M_1 (a strict entrance point) of the external boundary L_1 of B and then leaves B at M_2 (a strict exit point) of L_1 so that int$B$ is divided into two disconnected regions B_1 and B_2, then Γ_1 is called a cut trajectory of B.

Conley and Easton in [2] have studied generally properties of the isolating block. A special property of planar isolating blocks and an application to the existence of connecting orbits have discussed in [3] (See Lemma 1 of [3]). In the present paper, we shall discuss qualitative properties of planar flows inside the isolating block and give some new results.

§2. The main results

Suppose B is an isolating block for flow defined by (1.1) and a critical point $Q \in$ int$B$. Our main aim is to study the existence of elliptic regions of Q (See [4, p.295] for the definition). Therefore, we consider a bounded sectorial region D contained in B with boundary consisting of the critical point Q, two semi-trajectory arcs $f(M_1, R^+), f(M_2, R^-)$ and the closed subarc M_1mM_2 of ∂B from M_1 to M_2, and such that when $t \to +\infty$ (or $-\infty$), $f(M_1, t)$ (or $f(M_2, t)$) tends to Q, and $M_1 \in b^+, M_2 \in b^-$. Such a sectorial region is said to be adjacent to ∂B.

DEFINITION 2.1. Suppose D is a bounded sectorial region adjacent to ∂B, as stated above. D is said to be inadmissible if there are a trajectory $\Gamma \subset D$ which tends to Q as $t \to \pm\infty$ and a circle ρ of radius r small enough with the centre Q such that the interior of each of the curvilinear triangles $Qm_1\gamma_1$ and $Qm_2\gamma_2$ is a parabolic sector of Q ([5,p.164]) in ρ, where it is assumed that ρ intersects QM_1, Γ and QM_2 at m_1, γ_1, γ_2 and m_2, respectively (Fig.1).

THEOREM 2.1. *Let B be an isolating block for flow defined by (1.1) and a critical point $Q \in$ int$B$. Let D be an inadmissible sectorial region adjacent to ∂B and let D do not contain any internal boundary components of ∂B. Let $D_1 = D\backslash\overline{G}$, where G is the region enclosed by Γ and Q. Then there must be at least one critical point of (1.1) in D_1 (Fig.1).*
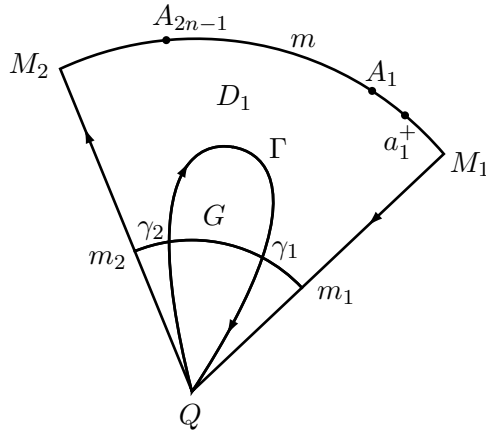
(Figure 1)

Proof. Since $M_1 \in b^+$ and $M_2 \in b^-$, there must be at least one tangency to B on the segmental arc $M_1 m M_2$. Let $A_1, A_2, \cdots, A_{2n-1}$ be the tangencies to be arranged in numerical order on the arc $M_1 m M_2$. Then the points on the segmental arcs $M_1 A_1, A_2 A_3, \cdots, A_{2n-2} A_{2n-1}$ (In what follows, we shall denote these arcs by $1^+, 2^+, \cdots, n^+$ respectively) of $M_1 m M_2$ are the strict entrance points of B, while the points on the segmental arcs $A_1 A_2, A_3 A_4, \cdots, A_{2n-1} M_2$ (We shall denote them by $1^-, 2^-, \cdots, n^-$ respectively) of $M_1 m M_2$ are the strict exit points of B ([5, p.37]). Take arbitrarily an integer k such that $1 < k < n$. Consider three consecutive tangencies $A_{2k}, A_{2k-1}, A_{2k-2}$, and four relevant segmental arcs $(k+1)^+, k^-, k^+, (k-1)^-$. It is easy to see that the positive semi-trajectory originating from every point in a small neighbourhood of A_{2k-1} on k^+ must leave D_1 from some point on the segmental arc k^-, while the positive semi-trajectory originating from every point in a small neighbourhood of A_{2k-2} on k^+ must leave D_1 from some point on the segmental arc $(k-1)^-$. Therefore, there must be a nonempty set $\beta_k^+ \subset k^+$ such that for each point $x \in \beta_k^+$, the positive semi-trajectory $f(x, R^+)$ leaves D_1 neither from a point on k^- nor from a point on $(k-1)^-$ for increasing time. Similarly, there must be a nonempty set $\beta_k^- \subset k^-$ such that for each point $x \in \beta_k^-$, the negative semi-trajectory $f(x, R^-)$ leaves D_1 neither from a point on k^+ nor from a point on $(k+1)^+$ for all $t < 0$. In other words, each of these semi-trajectories can not leave D_1 from a point on the adjacent segmental arcs. Consider case $k = 1$. It is easy to see that if the positive semi-trajectory $f(x, R^+)$ originating from a

point $x \in 1^+$ tends to Q in D as $t \to +\infty$, then it must enter the parabolic sector $Qm_1\gamma_1$(see Fig.1). So, from the continuity (the solutions depend continuously on initial conditions) it follows that the positive semi-trajectory originating from every point in a small neighbourhood of x on 1^+ also tends to Q in D as $t \to +\infty$. The same argument implies that the positive semi-trajectory through a point x of M_1A_1 sufficiently close to M_1 must tend to Q in D as $t \to +\infty$. Thus there is a maximal open segmental arc M_1h of M_1A_1 such that for every point $x \in M_1h$, the positive semi-trajectory $f(x, R^+)$ tends to Q in D as $t \to +\infty$, while the positive semi-trajectory $f(h, R^+)$ does not tend to Q in D as $t \to +\infty$. On the other hand, the same argument used in case $1 < k < n$ implies that the positive semi-trajectory originating from every point in a small neighbourhood of A_1 on 1^+ must leave D_1 from some point on the segmental arc 1^-. Moreover, the set of such points is an open set on the segmental arc M_1A_1. This implies that $h \neq A_1$, and the positive semi-trajectory $f(h, R^+)$ can neither tend to Q in D nor leave D_1 from a point on 1^- for increasing time. Hence, for $k = 1$, we have proved that there is a nonempty set $\beta_1^+ \subset 1^+$ such that for each point $x \in \beta_1^+$, the positive semi-trajectory $f(x, R^+)$ can neither tend to Q in D nor leave D_1 from a point on 1^- for increasing time. For $k = n$, a similar conclusion holds.

Choose arbitrarily n points $a_i^+ \in \beta_i^+ (i = 1, 2, \cdots, n)$. We now can prove that there is at least one among the positive semi-trajectories $\{f(a_i^+, R^+)|i = 1, 2, \cdots, n\}$ such that it stays in D_1 for all $t > 0$ and does not tend to Q as $t \to +\infty$. The following proof proceeds by reduction to absurdity. Suppose that each of the semi-trajectories $\{f(a_i^+, R^+)|i = 1, 2, \cdots, n\}$ either leaves D_1 from some point on the segmental arc M_1mM_2 for increasing time or tends to Q as $t \to +\infty$. Thus, since the positive semi-trajectory $f(a_1^+, R^+)$ can not tend to Q(note $a_1^+ \in \beta_1^+$), it must leave D_1 from a point on M_1mM_2 for increasing time. Let it leave D_1 from some point on k^-, where $1 < k \leq n$. But this means that each of $\{f(a_i^+, R^+)|i = 2, \cdots, k\}$ can not tend to Q as $t \to +\infty$ for, otherwise it must meet $f(a_1^+, R^+)$ at a point for increasing time and which contradicts uniqueness of solutions. Therefore, each of $\{f(a_i^+, R^+)|i = 2, \cdots, k\}$ must leave D_1 from a point on M_1mM_2 for increasing time. However, we note that the semi-trajectory $f(a_2^+, R^+)$ can not leave D_1 from a point on 1^- or 2^- because $a_2^+ \in \beta_2^+$, hence, it can only leave D_1 from a point on $i^-(i \geq 3)$. This implies that the positive semi-trajectory $f(a_3^+, R^+)$ can not leave D_1 from a point on 1^- for increasing time for, otherwise it must meet $f(a_2^+, R^+)$ and which

contradicts uniqueness of solutions, hence, it can only leave D_1 from a point on $i^-(i \geq 4)$. Further, this also implies that the semi-trajectory $f(a_4^+, R^+)$ can not leave D_1 from a point on 2^-, hence it can only leave D_1 from a point on $i^-(i \geq 5)$ for increasing time. Repeating an argument used above, one implies that the semi-trajectory $f(a_i^+, R^+)$ can only leave D_1 from a point on $j^-(k \geq j \geq i+1)$ (i.e., on the segmental arc with greater subscript). From this, it follows that the semi-trajectory $f(a_k^+, R^+)$ can not leave D_1 from a point on M_1mM_2 for increasing time. Moreover, as stated above, it can not also tend to Q as $t \to +\infty$, therefore, this contradicts the preceding hypothesis. Thus, there must be a positive semi-trajectory γ^+ such that it can neither tend to Q nor leave D_1 from a point on M_1mM_2 for increasing time. By the Poincaré-Bendixson theory of planar systems, the ω-limit set of γ^+ must contain critical points or closed orbits. Further, since a closed orbit contains at least one critical point of (1.1) in its interior, this implies that there must be at least one critical point of (1.1) in D_1. Hence Theorem 2.1 is proved.

COROLLARY 1. *If the sectorial region D in Theorem 2.1 contains the internal boundary components $L_{i_1}, \cdots, L_{i_k}$ of ∂B, then the conclusion of Theorem 2.1 still holds provided we set $D_1 = D \backslash (\overline{G} \cup \overline{G}_{i_1} \cup \cdots \cup \overline{G}_{i_k})$, where $G_{i_1}, \cdots, G_{i_k}$ are the regions enclosed by $L_{i_1}, \cdots, L_{i_k}$ respectively.*

Proof of Corollary 1. We know from the proof of Theorem 2.1 that there is at least one among the positive semi-trajectories $\{f(a_i^+, R^+) | i = 1, 2, \cdots, n\}$, say $f(a_j^+, R^+)$, such that it stays in $D \backslash \overline{G}$ for all $t > 0$ and does not tend to Q as $t \to +\infty$, where $a_j^+ \in \beta_j^+ \subset j^+$, A_{2j-1} and A_{2j-2} are two tangencies close to a_j^+.

Suppose $f(a_j^+, R^+)$ intersects L_{i_0} at b_j, where L_{i_0} is one of the internal boundary components $\{L_{i_1}, \cdots, L_{i_k}\}$ and b_j is a strict exit point of B. Consider the segmental arc $A_{2j-2}a_j^+A_{2j-1}$ and its segmental subarc $a_j^+A_{2j-1}$. Let $\tilde{a} = \{x \in A_{2j-2}a_j^+A_{2j-1} |$ the point where $f(x, R^+)$ intersects L_{i_0} is a strict exit point of $B\}$. From the theorem of continuity (the solutions depend continuously on initial conditions) it follows that $\tilde{a}$ is an open set on the segmental arc $A_{2j-2}a_j^+A_{2j-1}$. Since A_{2j-1} is a tangency to B, the positive semi-trajectory originating from every point in a small neighbourhood of A_{2j-1} on $a_j^+A_{2j-1}$ must leave B from some point on the segmental arc M_1mM_2. Hence there must be at least one boundary point of the set $\tilde{a}$ on $a_j^+A_{2j-1}$. Let a_0 be a boundary point close to a_j^+. Then either there is a

point $p \in f(a_0, R^+)$ such that p is a tangency to L_{i_0} or $f(a_0, R^+)$ tends to a critical point of (1.1) in D_1 as $t \to +\infty$. In the former case, it follows that there is an internal tangency to B. But this is impossible because B is an isolating block. In the latter case, it follows that Corollary 1 holds.

If $f(a_j^+, R^+)$ does not meet any one of the internal boundary components $\{L_{i_1}, \cdots, L_{i_k}\}$, then by the Poincaré-Bendixson theory of planar systems it follows that there must be at least one critical point of (1.1) in D_1, where $D_1 = D \backslash (\overline{G} \cup \overline{G}_{i_1} \cup \cdots \cup \overline{G}_{i_k})$. Corollary 1 is proved.

Remark 1. Suppose Q is a unique critical point of (1.1) in B. Then, Theorem 2.1 means that the fact that there are no internal tangencies to B can imply that there are no certain type of elliptic regions of the critical point Q.

Using exactly the same argument used in the proof of Theorem 2.1, we can prove the following theorem. We suppose that the symbols $a_i^{\pm}, i^{\pm}, \beta_i^{\pm}$ have the same meanings as in Theorem 2.1.

THEOREM 2.2. *Let B be an isolating block for flow defined by* (1.1). *Let B_1 be the region enclosed by the trajectory arc M_1M_2 of the cut trajectory Γ_1 of B and the segmental arc M_1mM_2 of the external boundary L_1 of B* (Fig.2). *Let $A_1, A_2, \cdots, A_{2n-1}$ be the tangencies to be arranged in numerical order on the arc M_1mM_2. If $n \geq 2$, then there must be a point $a_i^+ \in i^+$ and a point $a_i^- \in i^-$ such that the semi-trajectories $f(a_i^+, R^+)$ and $f(a_i^-, R^-)$ stay in B_1 for all $t > 0$ and $t < 0$ respectively $(i = 1, 2, \cdots, n)$.*

Proof. First we note, by Definition 1.2, it follows that the positive semi-trajectory originating from every point in a small neighbourhood of M_1 on 1^+ must leave B_1 from a point on n^- for increasing time. For $k = n$, a similar conclusion holds. Thus, one can consider 1^+ and n^- as two adjacent segmental arcs.

We proceed by induction. First suppose $n = 2$. That is, there are three tangencies A_1, A_2, A_3 on the arc M_1mM_2. It is easy to see that the positive (or negative) semi-trajectory originating from any point on β_i^+ (or β_i^-) $(i = 1, 2)$ stays in B_1 for all $t > 0$ (or $t < 0$). So, when $n = 2$, Theorem 2.2 holds.

Let $k > 2$ be an arbitrary positive integer. Let us now make the inductional hypothesis that Theorem 2.2 is true for $2 \leq n \leq k - 1$ (i.e., for all those odd numbers which are not greater than $2k - 3$). We need to show that it is also true for $n = k$ (i.e., for the odd number $2k - 1$).

In fact, from $n = k - 1$ to $n = k$, two tangencies A_{2k-2} and A_{2k-1} are added to the arc M_1mM_2. The following proof proceeds by reduction to absurdity. Suppose that there is some segmental arc j^+ such that for every point $x \in j^+$, the positive semi-trajectory $f(x, R^+)$ leaves B_1 from a point on M_1mM_2 for increasing time. Take arbitrarily a point $a_j^+ \in \beta_j^+$, then, the positive semi-trajectory $f(a_j^+, R^+)$ must leave B_1 from the point a_l' on l^- for increasing time. The trajectory arc $a_j^+a_l'$ divides the segmental arc M_1mM_2 into three segmental arcs: The segmental arcs $a_j^+a_l'$, $M_1a_j^+$ and M_2a_l' (Fig.2). From the fact that the semi-trajectory $f(a_j^+, R^+)$ leaves B_1 neither from a point on the adjacent segmental arcs nor from a point on any entrance segmental arc i^+ for increasing time, it follows that there are at least three tangencies on the segmental arc $a_j^+a_l'$ of M_1mM_2, while the amount of tangencies on the arcs $M_1a_j^+$ and M_2a_l' of M_1mM_2 is not less than 2. Thus the number of tangencies on the arc $a_j^+a_l'$ of M_1mM_2 is not greater than $2k-1-2 = 2k-3$. Furthermore, since a_j^+ is a strict entrance point of B_1 and a_l' is a strict exit point of B_1, the trajectory arc $a_j^+a_l'$ possesses the same property as the arc M_1M_2 of Γ_1. By the inductional hypothesis it follows that there must be a point q on the arc $a_j^+A_{2j-1}$ of M_1mM_2 such that the positive semi-trajectory $f(q, R^+)$ stays for all $t > 0$ in the region enclosed by the segmental arc $a_j^+a_l'$ of M_1mM_2 and the trajectory arc $a_j^+a_l'$, hence in B_1. But since $a_j^+A_{2j-1} \subset j^+$, this contradicts the preceding hypothesis. Hence we have proved that for each $i \in \{1, 2, \cdots, n\}$, there must be a point $a_i^+ \in i^+$ such that $f(a_i^+, R^+)$ stays in B_1 for all $t > 0$. Similarly, we can prove that for each $i \in \{1, 2, \cdots, n\}$, there must be a point $a_i^- \in i^-$ such that $f(a_i^-, R^-)$ stays in B_1 for all $t < 0$. Thus Theorem 2.2 is proved.
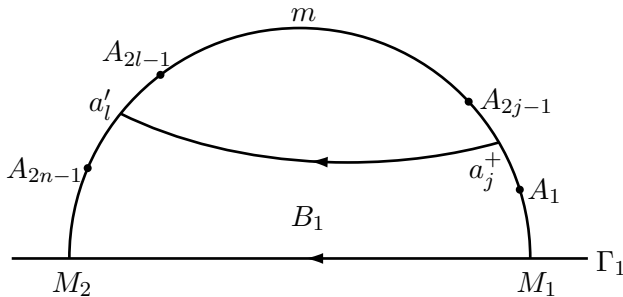
(Figure 2)

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