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INTEGRAL REPRESENTATIONS OF POSITIVE DEFINITE FUNCTIONS ON CONVEX SETS OF CERTAIN SEMIGROUPS OF RATIONAL NUMBERS

KOJI FURUTA

ABSTRACT. H. Glöckner proved that an operator-valued positive definite function on an open convex subset of $\boldsymbol{Q}^N$ is a restriction of the Laplace transform of an operator-valued measure on $\boldsymbol{R}^N$. We generalize this result to a function on an open convex subset of a certain subsemigroup of $\boldsymbol{Q}^2$.

1. Introduction

Let $\overrightarrow{m} = \{m_n\}_{n=1}^{\infty}$ be a sequence of integers greater than or equal to 2, and let $S(\overrightarrow{m})$ be the subsemigroup of the additive semigroup, $\boldsymbol{Q}$, of rational numbers, as defined by

$$S(\overrightarrow{m}) = \left\{ \frac{k}{m_1 \cdots m_n} : k \in \boldsymbol{Z},\, n \geq 1 \right\},$$

where $\boldsymbol{Z}$ denotes the set of all integers. For example, if $m_n = n+1$ for $n \geq 1$, we have $S(\overrightarrow{m}) = \boldsymbol{Q}$, and if $m_n = 2$ for $n \geq 1$, then $S(\overrightarrow{m})$ is the set of all dyadic rational numbers.

Let Ω denote an open convex subset of $\boldsymbol{R}^N$ $(N \geq 1)$ and let φ be a real-valued function on $\Omega \cap \Pi_{k=1}^N S_k(\overrightarrow{m})$, where $S_k(\overrightarrow{m}) = S(\overrightarrow{m})\,(1 \leq k \leq N)$. We say that φ is *positive definite* if

$$\sum_{i,j=1}^{n} c_i c_j \varphi(r_i + r_j) \geq 0$$

for all $n \geq 1$, $c_1, c_2, \ldots, c_n \in \boldsymbol{R}$ and $r_1, r_2, \ldots, r_n \in \Pi_{k=1}^N S_k(\overrightarrow{m})$, such that $2r_i \in \Omega \cap \Pi_{k=1}^N S_k(\overrightarrow{m})$ for $1 \leq i \leq n$. In [7], N. Sakakibara proved that $[0,\infty[\cap S(\overrightarrow{m})$ is a perfect semigroup, that is, every positive definite function on $[0,\infty[\cap S(\overrightarrow{m})$ has a unique representation as an integral of multiplicative functions. In [4], we obtained an integral representation of a positive definite function on $\Omega \cap \Pi_{k=1}^N S_k(\overrightarrow{m})$ in the case where $N = 1$. In this note, we show that every positive definite function has an integral representation in the case where $N = 2$. We also give a condition for

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a function of $\Omega \cap \Pi_{k=1}^{2} S_k(\overrightarrow{m})$ into $B(\mathcal{H})$, where $B(\mathcal{H})$ is the set of bounded linear operators on a Hilbert space $\mathcal{H}$, to have an integral representation (Theorem 2.2). As will be seen, our methods are applicable to any number of dimensions. The result we obtain represents a generalization of a result by H. Glöckner ([6], Theorem 18.5). For integral representations of continuous positive definite functions on open convex sets of $\boldsymbol{R}^2$, we refer to A. Devinatz [3].

2. Integral representations of positive definite functions

Define the function χ on $S(\overrightarrow{m})$ as follows (cf. [7]):

if the sequence $\overrightarrow{m} = \{m_n\}_{n=1}^{\infty}$ contains no even numbers, set

$$\chi\left(\frac{k}{m_1 \cdots m_n}\right) = (-1)^k, \qquad \frac{k}{m_1 \cdots m_n} \in S(\overrightarrow{m});$$

if $\overrightarrow{m}$ contains only finitely many even numbers, we may suppose that $m_1, \ldots, m_p$ are even and that m_q $(q > p)$ are odd. Then, we set

$$\chi\left(\frac{k}{m_1 \cdots m_p m_{p+1} \cdots m_n}\right) = (-1)^k, \qquad \frac{k}{m_1 \cdots m_p m_{p+1} \cdots m_n} \in S(\overrightarrow{m}).$$

It is clear that χ is well-defined and multiplicative, i.e., $\chi(r_1 + r_2) = \chi(r_1)\chi(r_2)$ for $r_1, r_2 \in S(\overrightarrow{m})$. In fact, the functions $r \in S(\overrightarrow{m}) \mapsto e^{rx}$ and $r \in S(\overrightarrow{m}) \mapsto \chi(r)e^{rx}$ (where $x \in \boldsymbol{R}$) are the semicharacters of $S(\overrightarrow{m})$ [7].

For a convex subset Ω of $\boldsymbol{R}^N$, let us denote by $E_+(\Omega, \boldsymbol{R}^N)$ the set of all positive Radon measures, μ, on $\boldsymbol{R}^N$ such that the function $x = (x_1, \ldots, x_N) \in \boldsymbol{R}^N \mapsto e^{r \cdot x} = e^{r_1 x_1 + \cdots + r_N x_N}$ is μ-integrable for all $r = (r_1, \ldots, r_N) \in \Omega$; by $E(\Omega, \boldsymbol{R}^N)$, let us denote the set of all signed Radon measures, μ, such that $|\mu| \in E_+(\Omega, \boldsymbol{R}^N)$. The σ-algebra of all Borel sets in $\boldsymbol{R}^N$ is denoted by $\mathcal{B}(\boldsymbol{R}^N)$.

In order to state our results, we need the following ([4], Theorem 2.1):

Theorem 2.1. *Let $a, b \in \boldsymbol{R} \cup \{-\infty, \infty\}$ such that $a < b$ and let $\overrightarrow{m} = \{m_n\}_{n=1}^{\infty}$ be a sequence of integers $m_n \geq 2$. Let φ be a positive definite function on $]a, b[\cap S(\overrightarrow{m})$.*

(1) *If the sequence $\overrightarrow{m}$ contains at most finitely many even numbers, then there exist positive Radon measures $\mu, \nu \in E_+(]a, b[, \boldsymbol{R})$ such that*

$$\varphi(r) = \int_{\boldsymbol{R}} e^{rx} d\mu(x) + \int_{\boldsymbol{R}} \chi(r) e^{rx} d\nu(x), \quad r \in]a, b[\cap S(\overrightarrow{m}).$$

Moreover, the pair (μ, ν) is uniquely determined by φ.

(2) *If the sequence $\overrightarrow{m}$ contains infinitely many even numbers, then there exists a uniquely determined measure $\mu \in E_+(]a, b[, \boldsymbol{R})$ such that*

$$\varphi(r) = \int_{\boldsymbol{R}} e^{rx} d\mu(x), \qquad r \in]a, b[\cap S(\overrightarrow{m}).$$

Proposition 2.1. *Let $a, b \in \boldsymbol{R} \cup \{-\infty, \infty\}$ such that $a < b$ and let $\overrightarrow{m} = \{m_n\}_{n=1}^{\infty}$ be a sequence of integers $m_n \geq 2$.*

(i) *If the sequence $\overrightarrow{m}$ contains at most finitely many even numbers, then the mapping $\mathcal{L}_1 : E(]a,\, b[, \boldsymbol{R}) \times E(]a,\, b[, \boldsymbol{R}) \to \boldsymbol{R}^{]a,b[\cap S(\overrightarrow{m})}$, defined by*

$$\mathcal{L}_1(\mu, \nu)(r) = \int_{\boldsymbol{R}} e^{rx} d\mu(x) + \int_{\boldsymbol{R}} \chi(r) e^{rx} d\nu(x), \quad r \in]a, b[\cap S(\overrightarrow{m}),$$

is injective.

(ii) *If the sequence $\overrightarrow{m}$ contains infinitely many even numbers, then the mapping $\mathcal{L}_2 : E(]a,\, b[, \boldsymbol{R}) \to \boldsymbol{R}^{]a,b[\cap S(\overrightarrow{m})}$, defined by*

$$\mathcal{L}_2(\mu)(r) = \int_{\boldsymbol{R}} e^{rx} d\mu(x), \quad r \in]a, b[\cap S(\overrightarrow{m}),$$

is injective.

Proof. (i) Suppose that $\mathcal{L}_1(\mu, \nu) = \mathcal{L}_1(\tilde{\mu}, \tilde{\nu})$ for $\mu, \nu, \tilde{\mu}, \tilde{\nu} \in E(]a, b[, \boldsymbol{R})$. Using the Jordan decomposition $\mu = \mu_1 - \mu_2$, $\nu = \nu_1 - \nu_2$, $\tilde{\mu} = \tilde{\mu_1} - \tilde{\mu_2}$, and $\tilde{\nu} = \tilde{\nu_1} - \tilde{\nu_2}$, we have

$$\int_{\boldsymbol{R}} e^{rx} d(\mu_1 + \tilde{\mu_2})(x) + \int_{\boldsymbol{R}} \chi(r) e^{rx} d(\nu_1 + \tilde{\nu_2})(x)$$
$$= \int_{\boldsymbol{R}} e^{rx} d(\tilde{\mu_1} + \mu_2)(x) + \int_{\boldsymbol{R}} \chi(r) e^{rx} d(\tilde{\nu_1} + \nu_2)(x),$$

for $r \in]a, b[\cap S(\overrightarrow{m})$. By Theorem 2.1, we have $\mu_1 + \tilde{\mu_2} = \tilde{\mu_1} + \mu_2$ and $\nu_1 + \tilde{\nu_2} = \tilde{\nu_1} + \nu_2$, such that $\mu = \tilde{\mu}$ and $\nu = \tilde{\nu}$. (ii) is proved analogously. □

Proposition 2.2. *Let $a_1,\ a_2,\ b_1, b_2 \in \boldsymbol{R} \cup \{\infty, -\infty\}$ such that $a_i < 0 < b_i$ $(i = 1, 2)$ and let $\overrightarrow{m} = \{m_n\}_{n=1}^{\infty}$ be a sequence of integers $m_n \geq 2$. Put $I_i =]a_i, b_i[$ $(i = 1, 2)$ and let φ be a positive definite function on $I_1 \times I_2 \cap \Pi_{k=1}^{2} S_k(\overrightarrow{m})$.*

(i) *If the sequence $\overrightarrow{m}$ contains at most finitely many even numbers, then there exist positive Radon measures $\kappa_i \in E_+(I_1 \times] -\varepsilon,\, \varepsilon[, \boldsymbol{R}^2)$ $(1 \leq i \leq 4)$, where $\varepsilon = min\{\frac{|a_2|}{2}, \frac{b_2}{2}\}$, such that*

$$\varphi(r) = \int_{\boldsymbol{R}^2} e^{r \cdot x} d\kappa_1(x) + \int_{\boldsymbol{R}^2} \chi(s) e^{r \cdot x} d\kappa_2(x)$$
$$+ \int_{\boldsymbol{R}^2} \chi(t) e^{r \cdot x} d\kappa_3(x) + \int_{\boldsymbol{R}^2} \chi(s)\chi(t) e^{r \cdot x} d\kappa_4(x) \quad (1)$$

for $r = (s, t) \in (I_1 \times] -\varepsilon,\, \varepsilon[) \cap \Pi_{k=1}^{2} S_k(\overrightarrow{m})$. The quadruple $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ is uniquely determined by φ.

(ii) *If the sequence* $\overrightarrow{m}$ *contains infinitely many even numbers, then there exists a uniquely determined measure* $\kappa \in E_+(I_1 \times I_2, \boldsymbol{R}^2)$*, such that*

$$\varphi(r) = \int_{\boldsymbol{R}^2} e^{r \cdot x} d\kappa(x), \quad \text{for } r \in I_1 \times I_2 \cap \Pi_{k=1}^2 S_k(\overrightarrow{m}). \tag{2}$$

Proof. (i) We use the technique that was used in the proof of Theorem 6.5.4 in [2]. For each $t \in]-\varepsilon, \varepsilon[\cap S(\overrightarrow{m})$, define the functions φ_1, φ_2 on $I_1 \cap S(\overrightarrow{m})$ by

$$\varphi_1(s) = \varphi(s, 2t), \quad \varphi_2(s) = \varphi(s, 2t) + \varphi(s, 0) - 2\varphi(s, t)$$

for $s \in I_1 \cap S(\overrightarrow{m})$, respectively. Then, φ_1 and φ_2 are both positive definite. Indeed, let $n \geq 1$, $c_1, \dots, c_n \in \boldsymbol{R}$ and $s_1, \dots, s_n \in S(\overrightarrow{m})$ such that $2s_i \in I_1$ $(1 \leq i \leq n)$. Then, $(2s_i,\ 2t) \in I_1 \times I_2$ and we have

$$\sum_{i,j=1}^{n} c_i c_j \varphi_1(s_i + s_j) = \sum_{i,j=1}^{n} c_i c_j \varphi(s_i + s_j, t + t) \geq 0.$$

As for φ_2, expressing the defining property of the positive definiteness of φ for $c_1, \dots, c_n, -c_1, \dots, -c_n \in \boldsymbol{R}$ and $(s_1, t), \dots, (s_n, t), (s_1, 0), \dots, (s_n, 0) \in \Pi_{k=1}^2 S_k(\overrightarrow{m})$, we have

$$\sum_{i,j=1}^{n} c_i c_j \varphi_2(s_i + s_j) = \sum_{i,j=1}^{n} c_i c_j \left(\varphi(s_i + s_j, 2t) + \varphi(s_i + s_j, 0) - 2\varphi(s_i + s_j, t)\right) \geq 0.$$

Therefore, by Theorem 2.1, there exist $\mu_t^i,\ \nu_t^i \in E_+(I_1, \boldsymbol{R})\ (i = 1, 2)$ such that

$$\varphi(s, 2t) = \int_{\boldsymbol{R}} e^{sx}\, d\mu_t^1(x) + \int_{\boldsymbol{R}} \chi(s) e^{sx}\, d\nu_t^1(x),$$

$$\varphi(s, 2t) + \varphi(s, 0) - 2\varphi(s, t) = \int_{\boldsymbol{R}} e^{sx}\, d\mu_t^2(x) + \int_{\boldsymbol{R}} \chi(s) e^{sx}\, d\nu_t^2(x).$$

For $t \in]-\varepsilon, \varepsilon[\cap S(\overrightarrow{m})$, we define

$$\mu_t = \frac{1}{2}(\mu_t^1 + \mu_0^1 - \mu_t^2), \quad \nu_t = \frac{1}{2}(\nu_t^1 + \nu_0^1 - \nu_t^2).$$

By Proposition 2.1, (μ_t, ν_t) is a unique pair of measures in $E(I_1, \boldsymbol{R})$ such that

$$\varphi(s, t) = \int_{\boldsymbol{R}} e^{su}\, d\mu_t(u) + \int_{\boldsymbol{R}} \chi(s) e^{su}\, d\nu_t(u) \quad \text{for } s \in I_1 \cap S(\overrightarrow{m}). \tag{3}$$

The mappings $t \mapsto \mu_t$ and $t \mapsto \nu_t$ are positive definite on $]-\varepsilon, \varepsilon[\cap S(\overrightarrow{m})$ in the sense that

$$\sum_{i,j=1}^{n} c_i c_j \mu_{t_i + t_j},\ \sum_{i,j=1}^{n} c_i c_j \nu_{t_i + t_j} \in E_+(I_1, \boldsymbol{R})$$

for $c_1, \ldots, c_n \in \boldsymbol{R}$ and $t_1, \ldots, t_n \in S(\overrightarrow{m})$ such that $2t_i \in]-\varepsilon, \varepsilon[$. To see this, we consider the function $\psi : I_1 \cap S(\overrightarrow{m}) \to \boldsymbol{R}$, defined by

$$\psi(s) = \int_{\boldsymbol{R}} e^{su} \left(\sum_{i,j=1}^{n} c_i c_j d\mu_{t_i+t_j} \right)(u) + \int_{\boldsymbol{R}} \chi(s) e^{su} \left(\sum_{i,j=1}^{n} c_i c_j d\nu_{t_i+t_j} \right)(u).$$

Then, ψ is positive definite because

$$\sum_{p,q=1}^{m} d_p d_q \psi(s_p + s_q) = \sum_{p,q=1}^{m} \sum_{i,j=1}^{n} (d_p c_i)(d_q c_j) \varphi(s_p + s_q, t_i + t_j) \geq 0$$

for $d_1, \ldots, d_m \in \boldsymbol{R}$ and $s_1, \ldots, s_m \in S(\overrightarrow{m})$ with $2s_p \in I_1\,(1 \leq p \leq m)$. By Theorem 2.1, there exists a unique pair (ρ, σ) of positive measures such that

$$\psi(s) = \int_{\boldsymbol{R}} e^{su}\, d\rho(u) + \int_{\boldsymbol{R}} \chi(s) e^{su}\, d\sigma(u).$$

From Proposition 2.1, it follows that $\sum_{i,j=1}^{n} c_i c_j \mu_{t_i+t_j} = \rho, \sum_{i,j=1}^{n} c_i c_j \nu_{t_i+t_j} = \sigma \in E_+(I_1, \boldsymbol{R})$. In particular, for any $A \in \mathcal{B}(\boldsymbol{R})$, the functions $t \mapsto \mu_t(A)$ and $t \mapsto \nu_t(A)$ are positive definite on $]-\varepsilon, \varepsilon[\cap S(\overrightarrow{m})$. For the present, let us consider the function $\mu_t(A)$. By Theorem 2.1, $\mu_t(A)$ can be uniquely represented as

$$\mu_t(A) = \int_{\boldsymbol{R}} e^{tv} d\tau_A^1(v) + \int_{\boldsymbol{R}} \chi(t) e^{tv} d\tau_A^2(v) \tag{4}$$

with $\tau_A^i \in E_+(]-\varepsilon, \varepsilon[, \boldsymbol{R})\,(i = 1, 2)$. The mappings $A \mapsto \tau_A^i\,(i = 1, 2)$ of $\mathcal{B}(\boldsymbol{R})$ into $E_+(]-\varepsilon, \varepsilon[, \boldsymbol{R})$ satisfy the following:

(a) $\tau_\emptyset^i = 0$;

(b) $\tau_{\cup_n A_n}^i = \sum_{n=1}^{\infty} \tau_{A_n}^i$, when $\{A_n\}_{n=1}^{\infty}$ is a sequence of disjoint sets in $\mathcal{B}(\boldsymbol{R})$;

(c) $\tau_A^i = \sup\{\tau_K^i : K \in \mathcal{K}(\boldsymbol{R}), K \subset A\}$, where $A \in \mathcal{B}(\boldsymbol{R})$ and $\mathcal{K}(\boldsymbol{R})$ denotes the set of all compact sets of $\boldsymbol{R}$.

Let us verify these properties.

(a) For every $t \in]-\varepsilon, \varepsilon[\cap S(\overrightarrow{m})$, we have

$$0 = \mu_t(\emptyset) = \int_{\boldsymbol{R}} e^{tv} d\tau_\emptyset^1(v) + \int_{\boldsymbol{R}} \chi(t) e^{tv} d\tau_\emptyset^2(v).$$

Substituting $t = 0$, we have $\tau_\emptyset^1(\boldsymbol{R}) + \tau_\emptyset^2(\boldsymbol{R}) = 0$, such that $\tau_\emptyset^i = 0\,(i = 1, 2)$.

(b) For $t \in]-\varepsilon, \varepsilon[\cap S(\overrightarrow{m})$, we have

$$\begin{aligned} \int_{\boldsymbol{R}} e^{tv} d\tau_{\cup_n A_n}^1(v) + \int_{\boldsymbol{R}} \chi(t) e^{tv} d\tau_{\cup_n A_n}^2(v) &= \mu_t\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu_t(A_n) \\ &= \sum_{n=1}^{\infty} \left(\int_{\boldsymbol{R}} e^{tv} d\tau_{A_n}^1(v) + \int_{\boldsymbol{R}} \chi(t) e^{tv} d\tau_{A_n}^2(v) \right). \end{aligned} \tag{5}$$

Setting $t = 0$, we obtain $\sum_{n=1}^{\infty}(\tau_{A_n}^1(\boldsymbol{R}) + \tau_{A_n}^2(\boldsymbol{R})) = \mu_0(\bigcup_{n=1}^{\infty} A_n) \leq \mu_0(\boldsymbol{R}) < +\infty$, which shows that $\sum_{n=1}^{\infty} \tau_{A_n}^i$ $(i = 1, 2)$ are Radon measures (cf. [2], Exercise 2.1.28). Furthermore, (5) implies that $\sum_{n=1}^{\infty} \tau_{A_n}^i \in E_+(]-\varepsilon, \varepsilon[, \boldsymbol{R})$ $(i = 1, 2)$ and (b) follows from Proposition 2.1.

(c) By (a) and (b), we see that, for each $A \in \mathcal{B}(\boldsymbol{R})$, the net $\{\tau_K^i : K \in \mathcal{K}(\boldsymbol{R}),\, K \subset A\}$ is increasing if the index set is ordered by inclusion $(i = 1, 2)$. By Exercise 2.1.29 in [2],

$$\tilde{\tau}_A^i = \sup\{\tau_K^i : K \in \mathcal{K}(\boldsymbol{R}),\, K \subset A\}$$

is a Radon measure and $\tilde{\tau}_A^i \leq \tau_A^i$, in particular, $\tilde{\tau}_A^i \in E_+(]-\varepsilon, \varepsilon[, \boldsymbol{R})$ $(i = 1, 2)$. For each t, we have

$$\begin{aligned}
\int_{\boldsymbol{R}} e^{tv} d\tilde{\tau}_A^1(v) + \int_{\boldsymbol{R}} \chi(t) e^{tv} d\tilde{\tau}_A^2(v) &= \lim_K \left(\int_{\boldsymbol{R}} e^{tv} d\tau_K^1(v) + \int_{\boldsymbol{R}} \chi(t) e^{tv} d\tau_K^2(v) \right) \\
&= \lim_K \mu_t(K) = \mu_t(A) \\
&= \int_{\boldsymbol{R}} e^{tv} d\tau_A^1(v) + \int_{\boldsymbol{R}} \chi(t) e^{tv} d\tau_A^2(v),
\end{aligned}$$

which shows that $\tilde{\tau}_A^i = \tau_A^i$ $(i = 1, 2)$.

By (a), (b), and (c), the functions $\Phi_i : \mathcal{B}(\boldsymbol{R}) \times \mathcal{B}(\boldsymbol{R}) \to \boldsymbol{R}$, defined by $\Phi_i(A, B) = \tau_A^i(B)$ $(i = 1, 2)$, are Radon bimeasures; thus, by Theorem 2.1.10 in [2], there exist Radon measures κ_i $(i = 1, 2)$ on $\boldsymbol{R}^2$ such that

$$\Phi_i(A, B) = \int_{\boldsymbol{R}^2} 1_A(u) 1_B(v) d\kappa_i(u, v) = \int_{\boldsymbol{R}} 1_B(v) d\tau_A^i(v) \quad \text{for } A, B \in \mathcal{B}(\boldsymbol{R}),$$

where 1_A denotes the indicator function on A. By standard arguments of integral theory, we have

$$\int_{\boldsymbol{R}^2} 1_A(u) h(v) d\kappa_i(u, v) = \int_{\boldsymbol{R}} h(v) d\tau_A^i(v)$$

for $A \in \mathcal{B}(\boldsymbol{R})$ and any τ_A^i-integrable function $h : \boldsymbol{R} \to \boldsymbol{R}$, in particular,

$$\int_{\boldsymbol{R}^2} 1_A(u) e^{tv} d\kappa_i(u, v) = \int_{\boldsymbol{R}} e^{tv} d\tau_A^i(v), \quad \text{for } A \in \mathcal{B}(\boldsymbol{R}),\ t \in]-\varepsilon, \varepsilon[\cap S(\overrightarrow{m}).$$

Combining this with (4), we have

$$\mu_t(A) = \int_{\boldsymbol{R}^2} 1_A(u) e^{tv} d\kappa_1(u, v) + \int_{\boldsymbol{R}^2} 1_A(u) \chi(t) e^{tv} d\kappa_2(u, v)$$

for every $A \in \mathcal{B}(\boldsymbol{R})$ and $t \in]-\varepsilon, \varepsilon[\cap S(\overrightarrow{m})$. Again, by standard similar arguments we have

$$\int_{\boldsymbol{R}} g(u) d\mu_t(u) = \int_{\boldsymbol{R}^2} g(u) e^{tv} d\kappa_1(u, v) + \int_{\boldsymbol{R}^2} g(u) \chi(t) e^{tv} d\kappa_2(u, v)$$

for any μ_t-integrable function $g : \boldsymbol{R} \to \boldsymbol{R}$. In particular, we have

$$\int_{\boldsymbol{R}} e^{su} d\mu_t(u) = \int_{\boldsymbol{R}^2} e^{su+tv} d\kappa_1(u,v) + \int_{\boldsymbol{R}^2} \chi(t) e^{su+tv} d\kappa_2(u,v)$$

for $s \in I_1 \cap S(\overrightarrow{m})$ and $t \in]-\varepsilon, \varepsilon[\cap S(\overrightarrow{m})$. Using a similar argument for the function $t \mapsto \nu_t(A)\,(A \in \mathcal{B}(\boldsymbol{R}))$, we obtain

$$\int_{\boldsymbol{R}} \chi(s) e^{su} d\nu_t(u) = \int_{\boldsymbol{R}^2} \chi(s) e^{su+tv} d\kappa_3(u,v) + \int_{\boldsymbol{R}^2} \chi(s)\chi(t) e^{su+tv} d\kappa_4(u,v)$$

with $\kappa_i \in E_+(I_1 \times]-\varepsilon, \varepsilon[, \boldsymbol{R}^2)\,(i = 3, 4)$. Thus, by (3) we obtain the desired representation of φ.

To prove the uniqueness of the representing measure, we suppose that signed measures $\kappa_i \in E(I_1 \times]-\varepsilon,\, \varepsilon[, \boldsymbol{R}^2)\,(1 \le i \le 4)$ satisfy

$$\int_{\boldsymbol{R}^2} e^{r\cdot x} d\kappa_1(x) + \int_{\boldsymbol{R}^2} \chi(s) e^{r\cdot x} d\kappa_2(x) + \int_{\boldsymbol{R}^2} \chi(t) e^{r\cdot x} d\kappa_3(x) + \int_{\boldsymbol{R}^2} \chi(s)\chi(t) e^{r\cdot x} d\kappa_4(x) \equiv 0 \tag{6}$$

for $r = (s,t) \in (I_1 \times]-\varepsilon,\, \varepsilon[) \cap \Pi_{k=1}^2 S_k(\overrightarrow{m})$. Letting $t \in 2S(\overrightarrow{m}) = \{2s :\, s \in S(\overrightarrow{m})\}$ in (6), we have

$$\int_{\boldsymbol{R}^2} e^{su+tv} d(\kappa_1 + \kappa_3)(u,v) + \int_{\boldsymbol{R}^2} e^{su+tv} \chi(s) d(\kappa_2 + \kappa_4)(u,v) = 0. \tag{7}$$

Let us define $\pi_i : \boldsymbol{R}^2 \to \boldsymbol{R}\,(i = 1, 2)$ by $\pi_1(u,v) = u$ and $\pi_2(u,v) = v$, respectively, and put $e_t = e_t(v) = e^{tv}$. Then, by (7), we see that the image measures $\omega_1 = (e_t(\kappa_1 + \kappa_3))^{\pi_1}$ and $\omega_2 = (e_t(\kappa_2 + \kappa_4))^{\pi_1}$ satisfy $\int_{\boldsymbol{R}} e^{su} d\omega_1(u) + \int_{\boldsymbol{R}} \chi(s) e^{su} d\omega_2(u) = 0$, which implies that $\omega_1 = \omega_2 = 0$. This means that, for any $A \in \mathcal{B}(\boldsymbol{R})$ and $t \in]-\varepsilon, \varepsilon[\cap 2S(\overrightarrow{m})$,

$$\int_{\boldsymbol{R}^2} 1_A(u) e^{tv} d(\kappa_1 + \kappa_3) = \int_{\boldsymbol{R}^2} 1_A(u) e^{tv} d(\kappa_2 + \kappa_4) = 0.$$

By Proposition 2.1, we have $(1_{A\times\boldsymbol{R}}(\kappa_1 + \kappa_3))^{\pi_2} = (1_{A\times\boldsymbol{R}}(\kappa_2 + \kappa_4))^{\pi_2} = 0$, which implies that $(\kappa_1 + \kappa_3)(A \times B) = (\kappa_2 + \kappa_4)(A \times B) = 0$ for $A, B \in \mathcal{B}(\boldsymbol{R})$. Therefore, $\kappa_1 + \kappa_3 = 0$, $\kappa_2 + \kappa_4 = 0$. Similarly, letting $t \in S(\overrightarrow{m}) \setminus 2S(\overrightarrow{m})$ in (6), we obtain $\kappa_1 - \kappa_3 = 0, \kappa_2 - \kappa_4 = 0$. Consequently, we have $\kappa_i = 0\,(1 \le i \le 4)$. Thus, the proof of (i) is complete.

(ii) Suppose that $\overrightarrow{m}$ contains infinitely many even numbers. Then, for $t \in I_2 \cap S(\overrightarrow{m})$, the function $s \mapsto \varphi(s,t)$ is positive definite on $I_1 \cap S(\overrightarrow{m})$ because $2S(\overrightarrow{m}) = S(\overrightarrow{m})$ and $(s_1 + s_2, t) = (s_1, t/2) + (s_2, t/2)$ for $s_1, s_2, t \in S(\overrightarrow{m})$. Therefore, by Theorem 2.1, there exists a unique measure $\mu_t \in E_+(I_1, \boldsymbol{R})$ such that

$$\varphi(s,t) = \int_{\boldsymbol{R}} e^{sx} d\mu_t(x) \quad \text{for } s \in I_1 \cap S(\overrightarrow{m}).$$

The rest of the proof is similar to that of (i) and is therefore omitted. □

Let $\mathcal{H}$ be a complex Hilbert space, $\langle\cdot,\cdot\rangle$ be the inner product on $\mathcal{H}$, $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$, and $B(\mathcal{H})_+$ be the set of all positive operators on $\mathcal{H}$. We give a condition for a function of $\Omega \cap \Pi_{k=1}^2 S_k(\overrightarrow{m})$ into $B(\mathcal{H})$, where Ω is an open convex subset of $\boldsymbol{R}^2$, to have an integral representation such as (1) or (2) of Proposition 2.2. We denote by $E_+(\Omega, \boldsymbol{R}^2, \mathcal{H})$ the set of all functions $F : \mathcal{B}(\boldsymbol{R}^2) \to B(\mathcal{H})_+$ satisfying $\langle F(\cdot)\xi, \xi\rangle \in E_+(\Omega, \boldsymbol{R}^2)$ for all $\xi \in \mathcal{H}$. In the following, we consider only the case where $\overrightarrow{m}$ contains at most finitely many even numbers. We can also obtain an analogous result for the case where $\overrightarrow{m}$ contains infinitely many even numbers.

Theorem 2.2. *Let Ω be a nonempty open convex set in $\boldsymbol{R}^2$ and let $\overrightarrow{m} = \{m_n\}_{n=1}^\infty$ be a sequence of integers, $m_n \geq 2$, which contains at most finitely many even numbers. For a function $\varphi : \Omega \cap \Pi_{k=1}^2 S_k(\overrightarrow{m}) \to B(\mathcal{H})$, the following conditions are mutually equivalent*:

(i) *φ is of positive type, in the sense that $\sum_{i,j=1}^n \langle \varphi(r_i + r_j)\xi_i, \xi_j\rangle \geq 0$ for all $n \geq 1, r_1, r_2, \ldots, r_n \in \Pi_{k=1}^2 S_k(\overrightarrow{m})$, such that $2r_i \in \Omega \cap \Pi_{k=1}^2 S_k(\overrightarrow{m})$ for $i = 1, 2, \ldots, n$ and $\xi_1, \xi_2, \ldots, \xi_n \in \mathcal{H}$;*

(ii) *φ is positive definite, in the sense that for each $\xi \in \mathcal{H}$, the function $r \mapsto \langle \varphi(r)\xi, \xi\rangle$ is positive definite on $\Omega \cap \Pi_{k=1}^2 S_k(\overrightarrow{m})$;*

(iii) *For any fixed $\alpha \in \Omega \cap \Pi_{k=1}^2 2S_k(\overrightarrow{m})$, there exist functions $F_i : \mathcal{B}(\boldsymbol{R}^2) \to B(\mathcal{H})_+ \, (1 \leq i \leq 4)$ such that $e^{-\alpha\cdot x}F_i \in E_+(\Omega, \boldsymbol{R}^2, \mathcal{H})$ and*

$$\langle \varphi(r)\xi, \eta\rangle = \int_{\boldsymbol{R}^2} e^{(r-\alpha)\cdot x} d\langle F_1(x)\xi, \eta\rangle + \int_{\boldsymbol{R}^2} \chi(s) e^{(r-\alpha)\cdot x} d\langle F_2(x)\xi, \eta\rangle$$
$$+ \int_{\boldsymbol{R}^2} \chi(t) e^{(r-\alpha)\cdot x} d\langle F_3(x)\xi, \eta\rangle + \int_{\boldsymbol{R}^2} \chi(s)\chi(t) e^{(r-\alpha)\cdot x} d\langle F_4(x)\xi, \eta\rangle$$

for $r = (s, t) \in \Omega \cap \Pi_{k=1}^2 S_k(\overrightarrow{m})$, $\xi, \eta \in \mathcal{H}$.

Moreover, the quadruple (F_1, F_2, F_3, F_4) is uniquely determined by φ and α.

Proof. Clearly (i) implies (ii), and by Proposition 1.1 in [5], we see that (iii) implies (i). To prove that (ii) implies (iii), we first suppose that $\dim \mathcal{H} = 1$. Once the case where $\dim \mathcal{H} = 1$ is proved, the proof of the general case is obtained in a manner similar to that used for Theorem 3.1 in [4]. For an arbitrarily fixed $\alpha \in 2S(\overrightarrow{m}) \times 2S(\overrightarrow{m})$, let I_1 and I_2 be open intervals in $\boldsymbol{R}$ such that $\alpha \in I_1 \times I_2 \subset \Omega$. Then the function $\varphi_\alpha : (I_1 \times I_2 - \alpha) \cap \Pi_{k=1}^2 S_k(\overrightarrow{m}) \to \boldsymbol{R}$, defined by $\varphi_\alpha(r) = \varphi(r + \alpha)$, is positive definite because

$$\sum_{i,j=1}^n c_i c_j \varphi_\alpha(r_i + r_j) = \sum_{i,j=1}^n c_i c_j \varphi((r_i + \alpha/2) + (r_j + \alpha/2)) \geq 0$$

for $r_i \in (I_1 \times I_2 - \alpha) \cap \Pi_{k=1}^2 S_k(\overrightarrow{m})$ with $2r_i \in I_1 \times I_2 - \alpha$. By Proposition 2.2 there exist measures $\kappa_i\,(1 \leq i \leq 4)$, such that φ_α has a representation of the form (1) on $(I_1 \times \tilde{I}_2 - \alpha) \cap \Pi_{k=1}^2 S_k(\overrightarrow{m})$ with some $\tilde{I}_2 \subset I_2$. Putting $\Omega_\alpha = I_1 \times \tilde{I}_2$ and $\kappa_i^\alpha = e^{-\alpha \cdot x}\kappa_i\,(1 \leq i \leq 4)$, we have $\kappa_i^\alpha \in E_+(\Omega_\alpha, \boldsymbol{R}^2)$ and

$$\varphi(r) = \int_{\boldsymbol{R}^2} e^{r\cdot x} d\kappa_1^\alpha(x) + \int_{\boldsymbol{R}^2} \chi(s) e^{r\cdot x} d\kappa_2^\alpha(x) + \int_{\boldsymbol{R}^2} \chi(t) e^{r\cdot x} d\kappa_3^\alpha(x) + \int_{\boldsymbol{R}^2} \chi(s)\chi(t) e^{r\cdot x} d\kappa_4^\alpha(x)$$

for $r = (s,t) \in \Omega_\alpha \cap \Pi_{k=1}^2 S_k(\overrightarrow{m})$. We show that each measure $\kappa_i^\alpha\,(1 \leq i \leq 4)$ is independent of the choice of α. Suppose that $\alpha, \alpha' \in \Omega \cap \Pi_{k=1}^2 2S_k(\overrightarrow{m})$ and $\alpha \neq \alpha'$. Let l denote the line segment between α and α'. Then there exist finite points $\alpha = w_0, w_1, \ldots, w_n = \alpha'$ in $l \cap \Pi_{k=1}^2 2S_k(\overrightarrow{m})$ such that $\Omega_{w_p} \cap \Omega_{w_{p+1}} \neq \emptyset\,(0 \leq p \leq n-1)$. By Proposition 2.1, we have $\kappa_i^{r_p} = \kappa_i^{r_{p+1}}$ for each p. Therefore, we have $\kappa_i^\alpha = \kappa_i^{w_0} = \kappa_i^{w_n} = \kappa_i^{\alpha'}\,(1 \leq i \leq 4)$, which completes the proof. □

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Faculty of Liberal Arts and Sciences, Tokyo City University, Tamazutsumi, Setagaya, Tokyo 158-8557, Japan

E-mail address: `kfuruta@tcu.ac.jp`

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