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A CONFIRMATION BY HAND CALCULATION THAT THE MÖBIUS BALL IS A GYROVECTOR SPACE

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ABSTRACT. We give a confirmation that the Möbius ball of any real inner product space is a gyrovector space by using only elementary hand calculation. Some remarks to [2] will also be made.

1. Introduction

Gyrogroups have been intensively studied by A. Ungar since 1988. The first known gyrogroup was the unit ball of Euclidean space $\mathbb{R}^3$ endowed with Einstein's velocity addition associated with the special theory of relativity. Another example of a gyrogroup is the complex unit disc $\mathbb{D} = \{a \in \mathbb{C} : |a| < 1\}$ endowed with the Möbius addition defined by

$$a \oplus b = \frac{a+b}{1+\overline{a}b},$$

which is a non-associative and non-commutative binary operation. Ungar extended in [2] the Möbius addition in the complex disc to the ball of an arbitrary real inner product space, and observed that the ball endowed with the Möbius addition turns out to be a gyrocommutative gyrogroup. Although it was observed by Ungar that the result can be verified by computer algebra, it is natural and desirable to give a solid proof. M. Ferreira and G. Ren [1] achieved such a proof by using Clifford algebra formalism among other advanced studies on Möbius gyrogroups.

Ungar introduced the notion of gyrovector spaces as well. Some gyrocommutative gyrogroups admit scalar multiplication, turning themselves into gyrovector spaces. The latter are analogous to vector spaces just as gyrogroups are analogous to groups.

It would be worthy to present a proof checking directly that the example satisfies the axioms, even if the calculation is somewhat lengthy. In this paper, we will give a confirmation that the Möbius ball of any real inner product space is not only a

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gyrocommutative gyrogroup but also a gyrovector space by using only elementary hand calculation for inner product spaces, along the scheme by Ungar himself. Some remarks related with signatures in some formulae for the Möbius addition of parallel vectors in [2] will also be made.

2. A proof of that the Möbius ball is a gyrocommutative gyrogroup

For definitions and basic properties of gyrogroups and gyrovector spaces, the reader should refer to the monograph [2].

Theorem 2.1. [2, **3.5** pp.78–79] *Let $\mathbb{V} = (\mathbb{V}, +, \cdot)$ be a real inner product space with a binary operation $+$ and a positive definite inner product $\cdot$ and let $\mathbb{V}_s$ be the ball*

$$\mathbb{V}_s = \{\boldsymbol{v} \in \mathbb{V} : \ ||\boldsymbol{v}|| < s\}$$

for any fixed $s > 0$. Let the Möbius addition $\oplus_{\mathrm{M}}$ be a binary operation in $\mathbb{V}_s$ given by the equation

$$\boldsymbol{u} \oplus_{\mathrm{M}} \boldsymbol{v} = \frac{(1 + \frac{2}{s^2}\boldsymbol{u}\cdot\boldsymbol{v} + \frac{1}{s^2}||\boldsymbol{v}||^2)\boldsymbol{u} + (1 - \frac{1}{s^2}||\boldsymbol{u}||^2)\boldsymbol{v}}{1 + \frac{2}{s^2}\boldsymbol{u}\cdot\boldsymbol{v} + \frac{1}{s^4}||\boldsymbol{u}||^2||\boldsymbol{v}||^2} \tag{1}$$

where $\cdot$ and $||\cdot||$ are the inner product and the norm that the ball $\mathbb{V}_s$ inherits from the space $\mathbb{V}$. Then the Möbius ball $(\mathbb{V}_s, \oplus_{\mathrm{M}})$ is a gyrocommutative gyrogroup.

We will give a proof of Theorem 2.1 by showing each proof of a number of propositions and lemmas. It is easy to see that, without loss of generality, we may assume $s = 1$. Ungar [2, **3.6** p.82] provides a scheme which we will make use of it. Throughout the rest of this section, we will denote $\oplus_{\mathrm{M}}$ by $\oplus$ simply.

Definition 2.2. For any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{V}_1$, we put

$$A = -\boldsymbol{u}\cdot\boldsymbol{w}||\boldsymbol{v}||^2 + \boldsymbol{v}\cdot\boldsymbol{w} + 2(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{w})$$

$$B = -\boldsymbol{v}\cdot\boldsymbol{w}||\boldsymbol{u}||^2 - \boldsymbol{u}\cdot\boldsymbol{w}$$

$$D = 1 + 2\boldsymbol{u}\cdot\boldsymbol{v} + ||\boldsymbol{u}||^2||\boldsymbol{v}||^2$$

$$D_1 = 1 + 2\boldsymbol{v}\cdot\boldsymbol{w} + ||\boldsymbol{v}||^2||\boldsymbol{w}||^2$$

$$D_2 = 1 + 2\boldsymbol{u}\cdot(\boldsymbol{v} \oplus \boldsymbol{w}) + ||\boldsymbol{u}||^2||\boldsymbol{v} \oplus \boldsymbol{w}||^2$$

$$D_3 = 1 + 2(\boldsymbol{u} \oplus \boldsymbol{v})\cdot\left(\boldsymbol{w} + 2\frac{A\boldsymbol{u} + B\boldsymbol{v}}{D}\right) + ||\boldsymbol{u} \oplus \boldsymbol{v}||^2 \left|\left|\boldsymbol{w} + 2\frac{A\boldsymbol{u} + B\boldsymbol{v}}{D}\right|\right|^2$$

$$V = 1 + 2\boldsymbol{u}\cdot\boldsymbol{v} + ||\boldsymbol{v}||^2$$

$$V_1 = 1 + 2\boldsymbol{v}\cdot\boldsymbol{w} + ||\boldsymbol{w}||^2$$

$$V_2 = 1 + 2\boldsymbol{u}\cdot(\boldsymbol{v}\oplus\boldsymbol{w}) + ||\boldsymbol{v}\oplus\boldsymbol{w}||^2$$

$$V_3 = 1 + 2(\boldsymbol{u}\oplus\boldsymbol{v})\cdot\left(\boldsymbol{w} + 2\frac{A\boldsymbol{u}+B\boldsymbol{v}}{D}\right) + \left|\left|\boldsymbol{w} + 2\frac{A\boldsymbol{u}+B\boldsymbol{v}}{D}\right|\right|^2.$$

Proposition 2.3. *The identity*

$$||\boldsymbol{u}\oplus\boldsymbol{v}||^2 = \frac{||\boldsymbol{u}||^2 + 2\boldsymbol{u}\cdot\boldsymbol{v} + ||\boldsymbol{v}||^2}{1 + 2\boldsymbol{u}\cdot\boldsymbol{v} + ||\boldsymbol{u}||^2||\boldsymbol{v}||^2} \tag{2}$$

holds for any $\boldsymbol{u},\boldsymbol{v}\in\mathbb{V}_1$, *so that the Möbius addition* $\oplus$ *is a binary operation in* $\mathbb{V}_1$.

Proof. We have $\boldsymbol{u}\oplus\boldsymbol{v} = \dfrac{V\boldsymbol{u} + (1-||\boldsymbol{u}||^2)\boldsymbol{v}}{D}$, where the Cauchy-Schwarz inequality $|\boldsymbol{u}\cdot\boldsymbol{v}| \le ||\boldsymbol{u}||||\boldsymbol{v}||$ shows that the denominator D is positive. An easy calculation shows that

$$\begin{aligned} D^2||\boldsymbol{u}\oplus\boldsymbol{v}||^2 &= \{V\boldsymbol{u} + (1-||\boldsymbol{u}||^2)\boldsymbol{v}\}\cdot\{V\boldsymbol{u} + (1-||\boldsymbol{u}||^2)\boldsymbol{v}\} \\ &= V^2||\boldsymbol{u}||^2 + 2V(1-||\boldsymbol{u}||^2)\boldsymbol{u}\cdot\boldsymbol{v} + (1-||\boldsymbol{u}||^2)^2||\boldsymbol{v}||^2 \\ &= \left(1 + 4(\boldsymbol{u}\cdot\boldsymbol{v})^2 + ||\boldsymbol{v}||^4 + 4\boldsymbol{u}\cdot\boldsymbol{v} + 4\boldsymbol{u}\cdot\boldsymbol{v}||\boldsymbol{v}||^2 + 2||\boldsymbol{v}||^2\right)||\boldsymbol{u}||^2 \\ &\quad + 2\left(1 - ||\boldsymbol{u}||^2 + 2\boldsymbol{u}\cdot\boldsymbol{v} - 2\boldsymbol{u}\cdot\boldsymbol{v}||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 - ||\boldsymbol{u}||^2||\boldsymbol{v}||^2\right)\boldsymbol{u}\cdot\boldsymbol{v} \\ &\quad + \left(1 - 2||\boldsymbol{u}||^2 + ||\boldsymbol{u}||^4\right)||\boldsymbol{v}||^2 \\ &= \left(||\boldsymbol{u}||^2 + 2\boldsymbol{u}\cdot\boldsymbol{v} + ||\boldsymbol{v}||^2\right)\left(1 + 2\boldsymbol{u}\cdot\boldsymbol{v} + ||\boldsymbol{u}||^2||\boldsymbol{v}||^2\right). \end{aligned}$$

Hence we have the identity (2). Moreover, it is immediate to see that

$$D\left(1 - ||\boldsymbol{u}\oplus\boldsymbol{v}||^2\right) = (1-||\boldsymbol{u}||^2)(1-||\boldsymbol{v}||^2), \tag{3}$$

which implies that $||\boldsymbol{u}\oplus\boldsymbol{v}|| < 1$ and completes the proof. □

Proofs of the following two propositions are straightforward, so we omit them.

Proposition 2.4. *The zero vector* $\mathbf{0}$ *is a left identity.*

Proposition 2.5. *For any* $\boldsymbol{u}\in\mathbb{V}_1$, $-\boldsymbol{u}$ *is a left inverse of* $\boldsymbol{u}$.

Proposition 2.6. *The left cancellation law holds in* $(\mathbb{V}_1,\oplus)$, *that is,*

$$(-\boldsymbol{u})\oplus(\boldsymbol{u}\oplus\boldsymbol{v}) = \boldsymbol{v} \tag{4}$$

for any $\boldsymbol{u},\boldsymbol{v}\in\mathbb{V}_1$.

Proof. By the definition of the Möbius addition, we have

$$-\boldsymbol{u}\oplus(\boldsymbol{u}\oplus\boldsymbol{v})=\frac{-\left(1-2\boldsymbol{u}\boldsymbol{\cdot}(\boldsymbol{u}\oplus\boldsymbol{v})+||\boldsymbol{u}\oplus\boldsymbol{v}||^2\right)\boldsymbol{u}+(1-||\boldsymbol{u}||^2)(\boldsymbol{u}\oplus\boldsymbol{v})}{1-2\boldsymbol{u}\boldsymbol{\cdot}(\boldsymbol{u}\oplus\boldsymbol{v})+||\boldsymbol{u}||^2||\boldsymbol{u}\oplus\boldsymbol{v}||^2},$$

so it is sufficient to show that

$$\begin{aligned}&-\left(1-2\boldsymbol{u}\boldsymbol{\cdot}(\boldsymbol{u}\oplus\boldsymbol{v})+||\boldsymbol{u}\oplus\boldsymbol{v}||^2\right)\boldsymbol{u}+(1-||\boldsymbol{u}||^2)(\boldsymbol{u}\oplus\boldsymbol{v})\\&=\left\{1-2\boldsymbol{u}\boldsymbol{\cdot}(\boldsymbol{u}\oplus\boldsymbol{v})+||\boldsymbol{u}||^2||\boldsymbol{u}\oplus\boldsymbol{v}||^2\right\}\boldsymbol{v}.\end{aligned}$$

Multiplying D to both sides, the above equality is equivalent to the following:

$$\begin{aligned}&-\left\{D-2\boldsymbol{u}\boldsymbol{\cdot}(V\boldsymbol{u}+(1-||\boldsymbol{u}||^2)\boldsymbol{v})+(||\boldsymbol{u}||^2+2\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v}+||\boldsymbol{v}||^2)\right\}\boldsymbol{u}\\&\quad+(1-||\boldsymbol{u}||^2)(V\boldsymbol{u}+(1-||\boldsymbol{u}||^2)\boldsymbol{v})\\&=\left\{D-2\boldsymbol{u}\boldsymbol{\cdot}(V\boldsymbol{u}+(1-||\boldsymbol{u}||^2)\boldsymbol{v})+||\boldsymbol{u}||^2(||\boldsymbol{u}||^2+2\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v}+||\boldsymbol{v}||^2)\right\}\boldsymbol{v}.\end{aligned}\tag{5}$$

It is easy to calculate the coefficient of $\boldsymbol{u}$ in the lefthand side as

$$-\left\{D-2\boldsymbol{u}\boldsymbol{\cdot}(V\boldsymbol{u}+(1-||\boldsymbol{u}||^2)\boldsymbol{v})+(||\boldsymbol{u}||^2+2\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v}+||\boldsymbol{v}||^2)\right\}+(1-||\boldsymbol{u}||^2)V=0,$$

while the righthand side one is obviously 0. Since

$$D-2\boldsymbol{u}\boldsymbol{\cdot}(V\boldsymbol{u}+(1-||\boldsymbol{u}||^2)\boldsymbol{v})=-(||\boldsymbol{u}||^2+2\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v}+||\boldsymbol{v}||^2)+(1-||\boldsymbol{u}||^2)V,$$

the coefficient of $\boldsymbol{v}$ in the righthand side of (5) is

$$\begin{aligned}&-(||\boldsymbol{u}||^2+2\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v}+||\boldsymbol{v}||^2)+(1-||\boldsymbol{u}||^2)V+||\boldsymbol{u}||^2(||\boldsymbol{u}||^2+2\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v}+||\boldsymbol{v}||^2)\\&=(1-||\boldsymbol{u}||^2)\{V-(||\boldsymbol{u}||^2+2\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v}+||\boldsymbol{v}||^2)\}=(1-||\boldsymbol{u}||^2)^2,\end{aligned}$$

which coincides with the coefficient of $\boldsymbol{v}$ in the lefthand side of (5). This completes the proof. □

Proposition 2.7. *The identity*

$$\left|\left|\boldsymbol{w}+2\frac{A\boldsymbol{u}+B\boldsymbol{v}}{D}\right|\right|=||\boldsymbol{w}||\tag{6}$$

holds for any $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\mathbb{V}_1$.

Proof. Since $\left|\left|\boldsymbol{w}+2\dfrac{A\boldsymbol{u}+B\boldsymbol{v}}{D}\right|\right|^2$

$$=||\boldsymbol{w}||^2+\frac{4}{D}(A\boldsymbol{u}+B\boldsymbol{v})\boldsymbol{\cdot}\boldsymbol{w}+\frac{4}{D^2}\left(A^2||\boldsymbol{u}||^2+2AB\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v}+B^2||\boldsymbol{v}||^2\right),$$

the identity (6) is equivalent to the next equality:

$$D\,(A\boldsymbol{u}+B\boldsymbol{v})\boldsymbol{\cdot}\boldsymbol{w}+\left(A^2||\boldsymbol{u}||^2+2AB\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v}+B^2||\boldsymbol{v}||^2\right)=0.$$

However, it is easy to see that

$$
\begin{aligned}
&A^2||\boldsymbol{u}||^2 + 2AB\boldsymbol{u}\cdot\boldsymbol{v} + B^2||\boldsymbol{v}||^2 \\
&= \left\{(\boldsymbol{u}\cdot\boldsymbol{w})^2||\boldsymbol{v}||^4 + (\boldsymbol{v}\cdot\boldsymbol{w})^2 + 4(\boldsymbol{u}\cdot\boldsymbol{v})^2(\boldsymbol{v}\cdot\boldsymbol{w})^2 - 2(\boldsymbol{u}\cdot\boldsymbol{w})(\boldsymbol{v}\cdot\boldsymbol{w})||\boldsymbol{v}||^2\right. \\
&\qquad \left. +4(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{w})^2 - 4(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{u}\cdot\boldsymbol{w})(\boldsymbol{v}\cdot\boldsymbol{w})||\boldsymbol{v}||^2\right\}||\boldsymbol{u}||^2 \\
&\quad + 2\left\{(\boldsymbol{u}\cdot\boldsymbol{w})(\boldsymbol{v}\cdot\boldsymbol{w})||\boldsymbol{u}||^2||\boldsymbol{v}||^2 + (\boldsymbol{u}\cdot\boldsymbol{w})^2||\boldsymbol{v}||^2 - (\boldsymbol{v}\cdot\boldsymbol{w})^2||\boldsymbol{u}||^2 - (\boldsymbol{u}\cdot\boldsymbol{w})(\boldsymbol{v}\cdot\boldsymbol{w})\right. \\
&\qquad \left. -2(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{w})^2||\boldsymbol{u}||^2 - 2(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{u}\cdot\boldsymbol{w})(\boldsymbol{v}\cdot\boldsymbol{w})\right\}\boldsymbol{u}\cdot\boldsymbol{v} \\
&\quad + \left\{(\boldsymbol{v}\cdot\boldsymbol{w})^2||\boldsymbol{u}||^4 + 2(\boldsymbol{u}\cdot\boldsymbol{w})(\boldsymbol{v}\cdot\boldsymbol{w})||\boldsymbol{u}||^2 + (\boldsymbol{u}\cdot\boldsymbol{w})^2\right\}||\boldsymbol{v}||^2 \\
&= \left\{(\boldsymbol{u}\cdot\boldsymbol{w})^2||\boldsymbol{v}||^4 + (\boldsymbol{v}\cdot\boldsymbol{w})^2 + 2(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{w})^2 - 2(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{u}\cdot\boldsymbol{w})(\boldsymbol{v}\cdot\boldsymbol{w})||\boldsymbol{v}||^2\right\}||\boldsymbol{u}||^2 \\
&\quad + 2\left\{(\boldsymbol{u}\cdot\boldsymbol{w})^2||\boldsymbol{v}||^2 - (\boldsymbol{u}\cdot\boldsymbol{w})(\boldsymbol{v}\cdot\boldsymbol{w}) - 2(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{u}\cdot\boldsymbol{w})(\boldsymbol{v}\cdot\boldsymbol{w})\right\}\boldsymbol{u}\cdot\boldsymbol{v} \\
&\quad + \left\{(\boldsymbol{v}\cdot\boldsymbol{w})^2||\boldsymbol{u}||^4 + (\boldsymbol{u}\cdot\boldsymbol{w})^2\right\}||\boldsymbol{v}||^2 \\
&= D\left\{(\boldsymbol{u}\cdot\boldsymbol{w})^2||\boldsymbol{v}||^2 - 2(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{u}\cdot\boldsymbol{w})(\boldsymbol{v}\cdot\boldsymbol{w}) + (\boldsymbol{v}\cdot\boldsymbol{w})^2||\boldsymbol{u}||^2\right\} \\
&= -D\,(A\boldsymbol{u} + B\boldsymbol{v})\cdot\boldsymbol{w}.
\end{aligned}
$$

This completes the proof. □

In order to show the left gyroassociative law, we should express each $\boldsymbol{u} \oplus (\boldsymbol{v} \oplus \boldsymbol{w})$ and $(\boldsymbol{u} \oplus \boldsymbol{v}) \oplus \left(\boldsymbol{w} + 2\frac{A\boldsymbol{u} + B\boldsymbol{v}}{D}\right)$ as a linear combination of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, and compare their coefficients.

Lemma 2.8. *For any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{V}_1$, the following identities hold:*

$$
\begin{aligned}
&\boldsymbol{u} \oplus (\boldsymbol{v} \oplus \boldsymbol{w}) \\
&= \frac{1}{D_1 D_2}\left\{D_1 V_2 \boldsymbol{u} + (1 - ||\boldsymbol{u}||^2)V_1\boldsymbol{v} + (1 - ||\boldsymbol{u}||^2)(1 - ||\boldsymbol{v}||^2)\boldsymbol{w}\right\}, \qquad (7)
\end{aligned}
$$

$$
\begin{aligned}
&(\boldsymbol{u} \oplus \boldsymbol{v}) \oplus \left(\boldsymbol{w} + 2\frac{A\boldsymbol{u} + B\boldsymbol{v}}{D}\right) \\
&= \frac{1}{DD_3}\left\{V_3 V\boldsymbol{u} + V_3(1 - ||\boldsymbol{u}||^2)\boldsymbol{v} + (1 - ||\boldsymbol{u} \oplus \boldsymbol{v}||^2)D\boldsymbol{w}\right. \\
&\qquad\qquad \left. +2(1 - ||\boldsymbol{u} \oplus \boldsymbol{v}||^2)(A\boldsymbol{u} + B\boldsymbol{v})\right\}. \qquad (8)
\end{aligned}
$$

Proof. Since $\boldsymbol{v} \oplus \boldsymbol{w} = \dfrac{V_1\boldsymbol{v} + (1 - ||\boldsymbol{v}||^2)\boldsymbol{w}}{D_1}$, it is immediate to see that

$$\begin{aligned}\boldsymbol{u} \oplus (\boldsymbol{v} \oplus \boldsymbol{w}) &= \frac{V_2\boldsymbol{u} + (1 - ||\boldsymbol{u}||^2)(\boldsymbol{v} \oplus \boldsymbol{w})}{D_2} \\ &= \frac{1}{D_1D_2}\left\{D_1V_2\boldsymbol{u} + (1 - ||\boldsymbol{u}||^2)V_1\boldsymbol{v} + (1 - ||\boldsymbol{u}||^2)(1 - ||\boldsymbol{v}||^2)\boldsymbol{w}\right\},\end{aligned}$$

which establishes the identity (7). Secondly, by definitions of D_3 and V_3, we have

$$\begin{aligned}&(\boldsymbol{u} \oplus \boldsymbol{v}) \oplus \left(\boldsymbol{w} + 2\frac{A\boldsymbol{u} + B\boldsymbol{v}}{D}\right) \\ &\quad= \frac{V_3(\boldsymbol{u} \oplus \boldsymbol{v}) + (1 - ||\boldsymbol{u} \oplus \boldsymbol{v}||^2)\left(\boldsymbol{w} + 2\frac{A\boldsymbol{u}+B\boldsymbol{v}}{D}\right)}{D_3} \\ &\quad= \frac{1}{DD_3}\left\{V_3V\boldsymbol{u} + V_3(1 - ||\boldsymbol{u}||^2)\boldsymbol{v} + (1 - ||\boldsymbol{u} \oplus \boldsymbol{v}||^2)D\boldsymbol{w}\right. \\ &\qquad\qquad\left. +2(1 - ||\boldsymbol{u} \oplus \boldsymbol{v}||^2)(A\boldsymbol{u} + B\boldsymbol{v})\right\},\end{aligned}$$

which establishes the identity (8). □

Lemma 2.9. *We have the following two identities:*

$$\left(||\boldsymbol{v}||^2\boldsymbol{u} + \boldsymbol{v}\right)\cdot(A\boldsymbol{u} + B\boldsymbol{v}) = D\left\{-\boldsymbol{u}\cdot\boldsymbol{w}||\boldsymbol{v}||^2 + (\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{w})\right\}, \tag{9}$$

$$\left\{(1 + 2\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{u} - ||\boldsymbol{u}||^2\boldsymbol{v}\right\}\cdot(A\boldsymbol{u} + B\boldsymbol{v}) = D\left\{\boldsymbol{v}\cdot\boldsymbol{w}||\boldsymbol{u}||^2 - (\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{u}\cdot\boldsymbol{w})\right\}. \tag{10}$$

Proof. One can easily expand, collect terms and factorize as follows:

$$\begin{aligned}&\left(||\boldsymbol{v}||^2\boldsymbol{u} + \boldsymbol{v}\right)\cdot(A\boldsymbol{u} + B\boldsymbol{v}) \\ &= A||\boldsymbol{u}||^2||\boldsymbol{v}||^2 + A\boldsymbol{u}\cdot\boldsymbol{v} + B||\boldsymbol{v}||^2\boldsymbol{u}\cdot\boldsymbol{v} + B||\boldsymbol{v}||^2 \\ &= -\boldsymbol{u}\cdot\boldsymbol{w}||\boldsymbol{u}||^2||\boldsymbol{v}||^4 + (\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{w})||\boldsymbol{u}||^2||\boldsymbol{v}||^2 - 2(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{u}\cdot\boldsymbol{w})||\boldsymbol{v}||^2 \\ &\quad + (\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{w}) + 2(\boldsymbol{u}\cdot\boldsymbol{v})^2(\boldsymbol{v}\cdot\boldsymbol{w}) - \boldsymbol{u}\cdot\boldsymbol{w}||\boldsymbol{v}||^2 \\ &= D\left\{-\boldsymbol{u}\cdot\boldsymbol{w}||\boldsymbol{v}||^2 + (\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{w})\right\}.\end{aligned}$$

This completes the proof of the identity (9). Secondly, a simple calculation shows that

$$
\begin{aligned}
&\left\{(1+2\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{u}-||\boldsymbol{u}||^2\boldsymbol{v}\right\}\cdot(A\boldsymbol{u}+B\boldsymbol{v})\\
&=A||\boldsymbol{u}||^2+A\boldsymbol{u}\cdot\boldsymbol{v}||\boldsymbol{u}||^2+B\boldsymbol{u}\cdot\boldsymbol{v}+2B(\boldsymbol{u}\cdot\boldsymbol{v})^2-B||\boldsymbol{u}||^2||\boldsymbol{v}||^2\\
&=-\boldsymbol{u}\cdot\boldsymbol{w}||\boldsymbol{u}||^2||\boldsymbol{v}||^2+\boldsymbol{v}\cdot\boldsymbol{w}||\boldsymbol{u}||^2+2(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{w})||\boldsymbol{u}||^2\\
&\quad-(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{u}\cdot\boldsymbol{w})||\boldsymbol{u}||^2||\boldsymbol{v}||^2+(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{w})||\boldsymbol{u}||^2+2(\boldsymbol{u}\cdot\boldsymbol{v})^2(\boldsymbol{v}\cdot\boldsymbol{w})||\boldsymbol{u}||^2\\
&\quad-(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{w})||\boldsymbol{u}||^2-(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{u}\cdot\boldsymbol{w})\\
&\quad-2(\boldsymbol{u}\cdot\boldsymbol{v})^2(\boldsymbol{v}\cdot\boldsymbol{w})||\boldsymbol{u}||^2-2(\boldsymbol{u}\cdot\boldsymbol{v})^2(\boldsymbol{u}\cdot\boldsymbol{w})\\
&\quad+\boldsymbol{v}\cdot\boldsymbol{w}||\boldsymbol{u}||^4||\boldsymbol{v}||^2+\boldsymbol{u}\cdot\boldsymbol{w}||\boldsymbol{u}||^2||\boldsymbol{v}||^2\\
&=D\left\{\boldsymbol{v}\cdot\boldsymbol{w}||\boldsymbol{u}||^2-(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{u}\cdot\boldsymbol{w})\right\},
\end{aligned}
$$

which completes the proof of the identity (10). □

Proposition 2.10. *The identity*

$$(\boldsymbol{u}\oplus\boldsymbol{v})\cdot(D\boldsymbol{w}+2(A\boldsymbol{u}+B\boldsymbol{v}))=A-B \tag{11}$$

holds.

Proof. By adding both sides of the identities (9) and (10), we have

$$
\begin{aligned}
&\left\{(1+2\boldsymbol{u}\cdot\boldsymbol{v}+||\boldsymbol{v}||^2)\boldsymbol{u}+(1-||\boldsymbol{u}||^2)\boldsymbol{v}\right\}\cdot(A\boldsymbol{u}+B\boldsymbol{v})\\
&=D\left\{-\boldsymbol{u}\cdot\boldsymbol{w}||\boldsymbol{v}||^2+(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{w})+\boldsymbol{v}\cdot\boldsymbol{w}||\boldsymbol{u}||^2-(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{u}\cdot\boldsymbol{w})\right\}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&(\boldsymbol{u}\oplus\boldsymbol{v})\cdot(D\boldsymbol{w}+2(A\boldsymbol{u}+B\boldsymbol{v}))\\
&=\left\{(1+2\boldsymbol{u}\cdot\boldsymbol{v}+||\boldsymbol{v}||^2)\boldsymbol{u}+(1-||\boldsymbol{u}||^2)\boldsymbol{v}\right\}\cdot\boldsymbol{w}\\
&\quad+2\left\{-\boldsymbol{u}\cdot\boldsymbol{w}||\boldsymbol{v}||^2+(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{w})+\boldsymbol{v}\cdot\boldsymbol{w}||\boldsymbol{u}||^2-(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{u}\cdot\boldsymbol{w})\right\}\\
&=\boldsymbol{u}\cdot\boldsymbol{w}-\boldsymbol{u}\cdot\boldsymbol{w}||\boldsymbol{v}||^2+\boldsymbol{v}\cdot\boldsymbol{w}+\boldsymbol{v}\cdot\boldsymbol{w}||\boldsymbol{u}||^2+2(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{w})\\
&=A-B.
\end{aligned}
$$

This completes the proof. □

Lemma 2.11. *The identity $D_1D_2=DD_3$ holds.*

Proof. Since $\boldsymbol{v} \oplus \boldsymbol{w} = \dfrac{V_1\boldsymbol{v} + (1 - ||\boldsymbol{v}||^2)\boldsymbol{w}}{D_1}$, Proposition 2.3 and an easy calculation shows that

$$
\begin{aligned}
D_1D_2 &= D_1 + 2\boldsymbol{u}\cdot\left(V_1\boldsymbol{v} + (1 - ||\boldsymbol{v}||^2)\boldsymbol{w}\right) + ||\boldsymbol{u}||^2\left(||\boldsymbol{v}||^2 + 2\boldsymbol{v}\cdot\boldsymbol{w} + ||\boldsymbol{w}||^2\right)\\
&= 1 + 2\boldsymbol{v}\cdot\boldsymbol{w} + ||\boldsymbol{v}||^2||\boldsymbol{w}||^2\\
&\quad + 2\boldsymbol{u}\cdot\boldsymbol{v} + 4(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{w}) + 2\boldsymbol{u}\cdot\boldsymbol{v}||\boldsymbol{w}||^2 + 2\boldsymbol{u}\cdot\boldsymbol{w} - 2\boldsymbol{u}\cdot\boldsymbol{w}||\boldsymbol{v}||^2\\
&\quad + ||\boldsymbol{u}||^2||\boldsymbol{v}||^2 + 2\boldsymbol{v}\cdot\boldsymbol{w}||\boldsymbol{u}||^2 + ||\boldsymbol{u}||^2||\boldsymbol{w}||^2\\
&= 1 + 2\boldsymbol{u}\cdot\boldsymbol{v} + ||\boldsymbol{u}||^2||\boldsymbol{v}||^2\\
&\quad + 2\left\{\boldsymbol{v}\cdot\boldsymbol{w} + 2(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{w}) + \boldsymbol{u}\cdot\boldsymbol{w} - \boldsymbol{u}\cdot\boldsymbol{w}||\boldsymbol{v}||^2 + \boldsymbol{v}\cdot\boldsymbol{w}||\boldsymbol{u}||^2\right\}\\
&\quad + \left(||\boldsymbol{u}||^2 + 2\boldsymbol{u}\cdot\boldsymbol{v} + ||\boldsymbol{v}||^2\right)||\boldsymbol{w}||^2\\
&= D + 2(A - B) + \left(||\boldsymbol{u}||^2 + 2\boldsymbol{u}\cdot\boldsymbol{v} + ||\boldsymbol{v}||^2\right)||\boldsymbol{w}||^2\\
&= D + 2\left(\boldsymbol{u} \oplus \boldsymbol{v}\right)\cdot(D\boldsymbol{w} + 2(A\boldsymbol{u} + B\boldsymbol{v})) + D||\boldsymbol{u} \oplus \boldsymbol{v}||^2||\boldsymbol{w}||^2\\
&= DD_3. \qquad (12)
\end{aligned}
$$

Here we used Propositions 2.10 and 2.7. This completes the proof. □

Lemma 2.12. *The coefficients of* $\boldsymbol{u}$ *in the formulae* (7) *and* (8) *coincide.*

Proof. We have to show the identity

$$D_1V_2 = V_3V + 2(1 - ||\boldsymbol{u} \oplus \boldsymbol{v}||^2)A.$$

Since $D_1V_2 = D_1\left(D_2 + (1 - ||\boldsymbol{u}||^2)||\boldsymbol{v} \oplus \boldsymbol{w}||^2\right)$ and

$$
\begin{aligned}
&V_3V + 2(1 - ||\boldsymbol{u} \oplus \boldsymbol{v}||^2)A\\
&= \left(D_3 + (1 - ||\boldsymbol{u} \oplus \boldsymbol{v}||^2)||\boldsymbol{w}||^2\right)\left(D + (1 - ||\boldsymbol{u}||^2)||\boldsymbol{v}||^2\right) + 2(1 - ||\boldsymbol{u} \oplus \boldsymbol{v}||^2)A\\
&= D_3D + D_3(1 - ||\boldsymbol{u}||^2)||\boldsymbol{v}||^2 + D(1 - ||\boldsymbol{u} \oplus \boldsymbol{v}||^2)||\boldsymbol{w}||^2\\
&\qquad + (1 - ||\boldsymbol{u} \oplus \boldsymbol{v}||^2)||\boldsymbol{w}||^2(1 - ||\boldsymbol{u}||^2)||\boldsymbol{v}||^2 + 2(1 - ||\boldsymbol{u} \oplus \boldsymbol{v}||^2)A,
\end{aligned}
$$

by the previous lemma, it is sufficient to show that

$$
\begin{aligned}
&D_1(1 - ||\boldsymbol{u}||^2)||\boldsymbol{v} \oplus \boldsymbol{w}||^2\\
&= D_3(1 - ||\boldsymbol{u}||^2)||\boldsymbol{v}||^2 + D(1 - ||\boldsymbol{u} \oplus \boldsymbol{v}||^2)||\boldsymbol{w}||^2\\
&\qquad + (1 - ||\boldsymbol{u} \oplus \boldsymbol{v}||^2)||\boldsymbol{w}||^2(1 - ||\boldsymbol{u}||^2)||\boldsymbol{v}||^2 + 2(1 - ||\boldsymbol{u} \oplus \boldsymbol{v}||^2)A.
\end{aligned}
$$

By multiplying $D/(1-||\boldsymbol{u}||^2)$ to both sides, the previous identity is equivalent to that

$$\begin{aligned}
&D(||\boldsymbol{v}||^2+2\boldsymbol{v}\cdot\boldsymbol{w}+||\boldsymbol{w}||^2)\\
&=DD_3||\boldsymbol{v}||^2+D(1-||\boldsymbol{v}||^2)||\boldsymbol{w}||^2\\
&\qquad+(1-||\boldsymbol{u}||^2)(1-||\boldsymbol{v}||^2)||\boldsymbol{v}||^2||\boldsymbol{w}||^2+2(1-||\boldsymbol{v}||^2)A,
\end{aligned}$$

where we used the identity (3). Owing to the identity (12), the righthand side of the above equality can be computed as follows:

$$\begin{aligned}
&\left\{D+2(A-B)+(||\boldsymbol{u}||^2+2\boldsymbol{u}\cdot\boldsymbol{v}+||\boldsymbol{v}||^2)||\boldsymbol{w}||^2\right\}||\boldsymbol{v}||^2\\
&\quad+(1-||\boldsymbol{v}||^2)\left\{D||\boldsymbol{w}||^2+(1-||\boldsymbol{u}||^2)||\boldsymbol{v}||^2||\boldsymbol{w}||^2+2A\right\}\\
&=D||\boldsymbol{v}||^2+2(A-B||\boldsymbol{v}||^2)+D||\boldsymbol{w}||^2\\
&=D\left(||\boldsymbol{v}||^2+2\boldsymbol{v}\cdot\boldsymbol{w}+||\boldsymbol{w}||^2\right).
\end{aligned}$$

This completes the proof. □

Lemma 2.13. *The coefficients of* $\boldsymbol{v}$ *in the formulae* (7) *and* (8) *coincide.*

Proof. It is necessary to show the identity

$$(1-||\boldsymbol{u}||^2)V_1=V_3(1-||\boldsymbol{u}||^2)+2(1-||\boldsymbol{u}\oplus\boldsymbol{v}||^2)B.$$

By multiplying $D/(1-||\boldsymbol{u}||^2)$ to both sides, the previous identity is equivalent to that

$$\begin{aligned}
&D\left(1+2\boldsymbol{v}\cdot\boldsymbol{w}+||\boldsymbol{w}||^2\right)\\
&=\left\{D+2(\boldsymbol{u}\oplus\boldsymbol{v})\cdot(D\boldsymbol{w}+2(A\boldsymbol{u}+B\boldsymbol{v}))+D||\boldsymbol{w}||^2\right\}+2(1-||\boldsymbol{v}||^2)B,
\end{aligned}$$

where we used Proposition 2.7 and the identity (3). By Proposition 2.10, we can easily compute the righthand side of the above equality:

$$\begin{aligned}
&\left\{D+2(A-B)+D||\boldsymbol{w}||^2\right\}+2(1-||\boldsymbol{v}||^2)B\\
&=D+2(A-||\boldsymbol{v}||^2B)+D||\boldsymbol{w}||^2\\
&=D\left(1+2\boldsymbol{v}\cdot\boldsymbol{w}+||\boldsymbol{w}||^2\right),
\end{aligned}$$

as desired. This completes the proof. □

Proposition 2.14. *The identity*

$$\boldsymbol{u}\oplus(\boldsymbol{v}\oplus\boldsymbol{w})=(\boldsymbol{u}\oplus\boldsymbol{v})\oplus\left(\boldsymbol{w}+2\frac{A\boldsymbol{u}+B\boldsymbol{v}}{D}\right) \tag{13}$$

holds for any $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\mathbb{V}_1$.

Proof. By Lemmas 2.12 and 2.13, it remains to show that the coefficients of $\boldsymbol{w}$ in the formulae (7) and (8) coincide. It immediately follows from Lemma 2.11 and the identity (3). This completes the proof. □

For any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{V}_1$, there exists an element $\boldsymbol{x} \in \mathbb{V}_1$ such that

$$\boldsymbol{u} \oplus (\boldsymbol{v} \oplus \boldsymbol{w}) = (\boldsymbol{u} \oplus \boldsymbol{v}) \oplus \boldsymbol{x}$$

by Propositions 2.14 and 2.7, and such $\boldsymbol{x}$ is uniquely determined by Proposition 2.6.

Definition 2.15. For any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{V}_1$, we define

$$\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{w} = \boldsymbol{w} + 2\frac{A\boldsymbol{u} + B\boldsymbol{v}}{D}. \tag{14}$$

The mapping $\boldsymbol{w} \mapsto \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{w}$ is from $\mathbb{V}_1$ to itself. Since A and B are linear in $\boldsymbol{w}$ and D is determined by only $\boldsymbol{u}$ and $\boldsymbol{v}$, the map $\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]$ is extendible to a linear map on $\mathbb{V}$.

Proposition 2.16. *The identity*

$$\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{a} \cdot \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b} \tag{15}$$

holds for any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a}, \boldsymbol{b} \in \mathbb{V}_1$.

Proof. The result follows from Proposition 2.7 and the polarization identity. □

Proposition 2.17. *For any* $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}_1$,

$$\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]^{-1} = \mathrm{gyr}[\boldsymbol{v}, \boldsymbol{u}], \tag{16}$$

so that $\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]$ *is a bijection on* $\mathbb{V}_1$.

Proof. By Definition 2.15, it is obvious that

$$\begin{aligned} \mathrm{gyr}[\boldsymbol{v}, \boldsymbol{u}]\left(\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{w}\right) &= \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{w} + 2\frac{A_1\boldsymbol{v} + B_1\boldsymbol{u}}{D} \\ &= \boldsymbol{w} + 2\frac{A\boldsymbol{u} + B\boldsymbol{v}}{D} + 2\frac{A_1\boldsymbol{v} + B_1\boldsymbol{u}}{D}, \end{aligned}$$

where we put

$$\begin{aligned} A_1 = &-\boldsymbol{v} \cdot \left(\boldsymbol{w} + 2\frac{A\boldsymbol{u} + B\boldsymbol{v}}{D}\right) ||\boldsymbol{u}||^2 + \boldsymbol{u} \cdot \left(\boldsymbol{w} + 2\frac{A\boldsymbol{u} + B\boldsymbol{v}}{D}\right) \\ &+ 2(\boldsymbol{v} \cdot \boldsymbol{u}) \left(\boldsymbol{u} \cdot \left(\boldsymbol{w} + 2\frac{A\boldsymbol{u} + B\boldsymbol{v}}{D}\right)\right) \\ B_1 = &-\boldsymbol{u} \cdot \left(\boldsymbol{w} + 2\frac{A\boldsymbol{u} + B\boldsymbol{v}}{D}\right) ||\boldsymbol{v}||^2 - \boldsymbol{v} \cdot \left(\boldsymbol{w} + 2\frac{A\boldsymbol{u} + B\boldsymbol{v}}{D}\right). \end{aligned}$$

Then an easy calculation shows that

$$\begin{aligned}
DA_1 &= -\boldsymbol{v}\cdot(D\boldsymbol{w}+2(A\boldsymbol{u}+B\boldsymbol{v}))\,||\boldsymbol{u}||^2+\boldsymbol{u}\cdot(D\boldsymbol{w}+2(A\boldsymbol{u}+B\boldsymbol{v}))\\
&\quad+2(\boldsymbol{v}\cdot\boldsymbol{u})\,(\boldsymbol{u}\cdot(D\boldsymbol{w}+2(A\boldsymbol{u}+B\boldsymbol{v})))\\
&= D\{-\boldsymbol{v}\cdot\boldsymbol{w}||\boldsymbol{u}||^2+\boldsymbol{u}\cdot\boldsymbol{w}+2(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{u}\cdot\boldsymbol{w})\}\\
&\quad+2\left\{(1+2\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{u}-||\boldsymbol{u}||^2\boldsymbol{v}\right\}\cdot(A\boldsymbol{u}+B\boldsymbol{v})\\
&= D\{-\boldsymbol{v}\cdot\boldsymbol{w}||\boldsymbol{u}||^2+\boldsymbol{u}\cdot\boldsymbol{w}+2(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{u}\cdot\boldsymbol{w})\}\\
&\quad+2D\left\{\boldsymbol{v}\cdot\boldsymbol{w}||\boldsymbol{u}||^2-(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{u}\cdot\boldsymbol{w})\right\}\\
&= -DB,
\end{aligned}$$

where we used the identity (10). Similarly, we have

$$\begin{aligned}
-DB_1 &= \boldsymbol{u}\cdot(D\boldsymbol{w}+2(A\boldsymbol{u}+B\boldsymbol{v}))\,||\boldsymbol{v}||^2+\boldsymbol{v}\cdot(D\boldsymbol{w}+2(A\boldsymbol{u}+B\boldsymbol{v}))\\
&= D(\boldsymbol{u}\cdot\boldsymbol{w}||\boldsymbol{v}||^2+\boldsymbol{v}\cdot\boldsymbol{w})+2\left(||\boldsymbol{v}||^2\boldsymbol{u}+\boldsymbol{v}\right)\cdot(A\boldsymbol{u}+B\boldsymbol{v})\\
&= D(\boldsymbol{u}\cdot\boldsymbol{w}||\boldsymbol{v}||^2+\boldsymbol{v}\cdot\boldsymbol{w})+2D\left\{-\boldsymbol{u}\cdot\boldsymbol{w}||\boldsymbol{v}||^2+(\boldsymbol{u}\cdot\boldsymbol{v})(\boldsymbol{v}\cdot\boldsymbol{w})\right\}\\
&= DA,
\end{aligned}$$

where we used the identity (9). Hence $A_1=-B$ and $B_1=-A$, and immediately we have $\mathrm{gyr}[\boldsymbol{v},\boldsymbol{u}]\,(\mathrm{gyr}[\boldsymbol{u},\boldsymbol{v}]\boldsymbol{w})=\boldsymbol{w}$. This completes the proof. □

Proposition 2.18. *For any* $\boldsymbol{u},\boldsymbol{v},\boldsymbol{a},\boldsymbol{b}\in\mathbb{V}_1$*,*

$$\mathrm{gyr}[\boldsymbol{u},\boldsymbol{v}]\,(\boldsymbol{a}\oplus\boldsymbol{b})=\mathrm{gyr}[\boldsymbol{u},\boldsymbol{v}]\boldsymbol{a}\oplus\mathrm{gyr}[\boldsymbol{u},\boldsymbol{v}]\boldsymbol{b},\tag{17}$$

so that $\mathrm{gyr}[\boldsymbol{u},\boldsymbol{v}]$ *is an automorphism on* $(\mathbb{V}_1,\oplus)$*.*

Proof. Since the map $\mathrm{gyr}[\boldsymbol{u},\boldsymbol{v}]$ preserves the norm and the inner product by Proposition 2.16 and $\mathrm{gyr}[\boldsymbol{u},\boldsymbol{v}]$ is extendible to a linear map on $\mathbb{V}$, it is easy to see that

$$\begin{aligned}
&\mathrm{gyr}[\boldsymbol{u},\boldsymbol{v}]\boldsymbol{a}\oplus\mathrm{gyr}[\boldsymbol{u},\boldsymbol{v}]\boldsymbol{b}\\
&=\frac{(1+2\boldsymbol{a}\cdot\boldsymbol{b}+||\boldsymbol{b}||^2)\,\mathrm{gyr}[\boldsymbol{u},\boldsymbol{v}]\boldsymbol{a}+(1-||\boldsymbol{a}||^2)\,\mathrm{gyr}[\boldsymbol{u},\boldsymbol{v}]\boldsymbol{b}}{1+2\boldsymbol{a}\cdot\boldsymbol{b}+||\boldsymbol{a}||^2||\boldsymbol{b}||^2}\\
&=\mathrm{gyr}[\boldsymbol{u},\boldsymbol{v}]\frac{(1+2\boldsymbol{a}\cdot\boldsymbol{b}+||\boldsymbol{b}||^2)\,\boldsymbol{a}+(1-||\boldsymbol{a}||^2)\,\boldsymbol{b}}{1+2\boldsymbol{a}\cdot\boldsymbol{b}+||\boldsymbol{a}||^2||\boldsymbol{b}||^2}\\
&=\mathrm{gyr}[\boldsymbol{u},\boldsymbol{v}]\,(\boldsymbol{a}\oplus\boldsymbol{b})\,.
\end{aligned}$$

This completes the proof. □

Now we define A', B' and D', replacing $\boldsymbol{u}$ by $\boldsymbol{u} \oplus \boldsymbol{v}$ in the definitions A, B and D, respectively:

$$A' = -(\boldsymbol{u} \oplus \boldsymbol{v}) \cdot \boldsymbol{w} ||\boldsymbol{v}||^2 + \boldsymbol{v} \cdot \boldsymbol{w} + 2\left((\boldsymbol{u} \oplus \boldsymbol{v}) \cdot \boldsymbol{v}\right)(\boldsymbol{v} \cdot \boldsymbol{w})$$

$$B' = -\boldsymbol{v} \cdot \boldsymbol{w} ||\boldsymbol{u} \oplus \boldsymbol{v}||^2 - (\boldsymbol{u} \oplus \boldsymbol{v}) \cdot \boldsymbol{w}$$

$$D' = 1 + 2(\boldsymbol{u} \oplus \boldsymbol{v}) \cdot \boldsymbol{v} + ||\boldsymbol{u} \oplus \boldsymbol{v}||^2 ||\boldsymbol{v}||^2.$$

Lemma 2.19. *The identities $DA' = VA$ and $DD' = V^2$ hold.*

Proof. It is easy to see that

$$\begin{aligned} DA' &= -\left\{\left(V\boldsymbol{u} + (1 - ||\boldsymbol{u}||^2)\boldsymbol{v}\right) \cdot \boldsymbol{w}\right\} ||\boldsymbol{v}||^2 + D\boldsymbol{v} \cdot \boldsymbol{w} \\ &\quad + 2\left\{\left(V\boldsymbol{u} + (1 - ||\boldsymbol{u}||^2)\boldsymbol{v}\right) \cdot \boldsymbol{v}\right\} (\boldsymbol{v} \cdot \boldsymbol{w}) \\ &= V\left\{-\boldsymbol{u} \cdot \boldsymbol{w} ||\boldsymbol{v}||^2 + 2(\boldsymbol{u} \cdot \boldsymbol{v})(\boldsymbol{v} \cdot \boldsymbol{w})\right\} + \left\{D + (1 - ||\boldsymbol{u}||^2)||\boldsymbol{v}||^2\right\}(\boldsymbol{v} \cdot \boldsymbol{w}) \\ &= VA, \end{aligned}$$

which establishes the first identity. Further, we have

$$\begin{aligned} DD' &= D + 2\left(V\boldsymbol{u} + (1 - ||\boldsymbol{u}||^2)\boldsymbol{v}\right) \cdot \boldsymbol{v} + \left(||\boldsymbol{u}||^2 + 2\boldsymbol{u} \cdot \boldsymbol{v} + ||\boldsymbol{v}||^2\right) ||\boldsymbol{v}||^2 \\ &= 2\boldsymbol{u} \cdot \boldsymbol{v} V + 1 + 2\boldsymbol{u} \cdot \boldsymbol{v} + 2||\boldsymbol{v}||^2 + 2\boldsymbol{u} \cdot \boldsymbol{v} ||\boldsymbol{v}||^2 + ||\boldsymbol{v}||^4 \\ &= V^2, \end{aligned}$$

which shows the second identity as desired. This completes the proof. □

Proposition 2.20. *The identity*

$$\operatorname{gyr}[\boldsymbol{u}, \boldsymbol{v}] = \operatorname{gyr}[\boldsymbol{u} \oplus \boldsymbol{v}, \boldsymbol{v}] \tag{18}$$

holds for any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}_1$.

Proof. Since

$$\operatorname{gyr}[\boldsymbol{u} \oplus \boldsymbol{v}, \boldsymbol{v}]\boldsymbol{w} = \boldsymbol{w} + 2\frac{A'(\boldsymbol{u} \oplus \boldsymbol{v}) + B'\boldsymbol{v}}{D'},$$

it is sufficient to show that

$$D(A'(\boldsymbol{u} \oplus \boldsymbol{v}) + B'\boldsymbol{v}) = D'(A\boldsymbol{u} + B\boldsymbol{v}),$$

which is equivalent to that

$$A'V = D'A \quad \text{and} \quad A'(1 - ||\boldsymbol{u}||^2) + DB' = D'B.$$

It follows from Lemma 2.19 that the first identity $A'V = D'A$ holds. In order to prove the second identity, it suffices to show that

$$D\left\{A'(1 - ||\boldsymbol{u}||^2) + DB'\right\} = V^2 B.$$

It is easy to see that

$$\begin{aligned}-DB' &= \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{w}\left(||\boldsymbol{u}||^2 + 2\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v} + ||\boldsymbol{v}||^2\right) + \left(V\boldsymbol{u} + (1-||\boldsymbol{u}||^2)\boldsymbol{v}\right)\boldsymbol{\cdot}\boldsymbol{w}\\ &= V\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{w} + \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{w}\left(1 + 2\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v} + ||\boldsymbol{v}||^2\right)\\ &= V(\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{w} + \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{w}).\end{aligned}$$

Therefore, by using $DA' = VA$, we can calculate as follows:

$$\begin{aligned}&D\left\{A'(1-||\boldsymbol{u}||^2) + DB'\right\}\\ &= VA(1-||\boldsymbol{u}||^2) - DV(\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{w} + \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{w})\\ &= V\left\{A(1-||\boldsymbol{u}||^2) - D(\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{w} + \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{w})\right\}\\ &= V\left\{-\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{w}||\boldsymbol{v}||^2 - \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{w}||\boldsymbol{u}||^2 - 2(\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v})(\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{w})||\boldsymbol{u}||^2\right.\\ &\qquad\qquad \left.-\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{w} - 2(\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v})(\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{w}) - \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{w}||\boldsymbol{u}||^2||\boldsymbol{v}||^2\right\}\\ &= V^2B.\end{aligned}$$

This completes the proof. □

Next we define A'' and B'', replacing $\boldsymbol{w}$ by $\boldsymbol{v}\oplus\boldsymbol{u}$ in the definitions A and B:

$$\begin{aligned}A'' &= -\boldsymbol{u}\boldsymbol{\cdot}(\boldsymbol{v}\oplus\boldsymbol{u})||\boldsymbol{v}||^2 + \boldsymbol{v}\boldsymbol{\cdot}(\boldsymbol{v}\oplus\boldsymbol{u}) + 2(\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v})\left(\boldsymbol{v}\boldsymbol{\cdot}(\boldsymbol{v}\oplus\boldsymbol{u})\right)\\ B'' &= -\boldsymbol{v}\boldsymbol{\cdot}(\boldsymbol{v}\oplus\boldsymbol{u})||\boldsymbol{u}||^2 - \boldsymbol{u}\boldsymbol{\cdot}(\boldsymbol{v}\oplus\boldsymbol{u}).\end{aligned}$$

Lemma 2.21. *The identities* $A'' = ||\boldsymbol{v}||^2 + \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v}$ *and* $B'' = -||\boldsymbol{u}||^2 - \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v}$ *hold.*

Proof. Since $\boldsymbol{v}\oplus\boldsymbol{u} = \dfrac{(1 + 2\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v} + ||\boldsymbol{u}||^2)\,\boldsymbol{v} + (1-||\boldsymbol{v}||^2)\boldsymbol{u}}{D}$, a straightforward calculation shows that

$$\begin{aligned}DA'' &= -\boldsymbol{u}\boldsymbol{\cdot}\left\{\left(1 + 2\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v} + ||\boldsymbol{u}||^2\right)\boldsymbol{v} + (1-||\boldsymbol{v}||^2)\boldsymbol{u}\right\}||\boldsymbol{v}||^2\\ &\quad + \boldsymbol{v}\boldsymbol{\cdot}\left\{\left(1 + 2\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v} + ||\boldsymbol{u}||^2\right)\boldsymbol{v} + (1-||\boldsymbol{v}||^2)\boldsymbol{u}\right\}\\ &\quad + 2(\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v})\left(\boldsymbol{v}\boldsymbol{\cdot}\left\{\left(1 + 2\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v} + ||\boldsymbol{u}||^2\right)\boldsymbol{v} + (1-||\boldsymbol{v}||^2)\boldsymbol{u}\right\}\right)\\ &= ||\boldsymbol{u}||^2||\boldsymbol{v}||^4 + ||\boldsymbol{v}||^2 + 2\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v}||\boldsymbol{v}||^2 + \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v} + \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v}||\boldsymbol{u}||^2||\boldsymbol{v}||^2 + 2(\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v})^2\\ &= D\left(||\boldsymbol{v}||^2 + \boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v}\right),\end{aligned}$$

which establishes the first identity. Further, we have

$$\begin{aligned}
-DB'' &= \boldsymbol{v}\cdot\left\{\left(1+2\boldsymbol{u}\cdot\boldsymbol{v}+||\boldsymbol{u}||^2\right)\boldsymbol{v}+(1-||\boldsymbol{v}||^2)\boldsymbol{u}\right\}||\boldsymbol{u}||^2 \\
&\quad +\boldsymbol{u}\cdot\left\{\left(1+2\boldsymbol{u}\cdot\boldsymbol{v}+||\boldsymbol{u}||^2\right)\boldsymbol{v}+(1-||\boldsymbol{v}||^2)\boldsymbol{u}\right\} \\
&= \boldsymbol{u}\cdot\boldsymbol{v}||\boldsymbol{u}||^2||\boldsymbol{v}||^2+||\boldsymbol{u}||^4||\boldsymbol{v}||^2+2\boldsymbol{u}\cdot\boldsymbol{v}||\boldsymbol{u}||^2+\boldsymbol{u}\cdot\boldsymbol{v}+2(\boldsymbol{u}\cdot\boldsymbol{v})^2+||\boldsymbol{u}||^2 \\
&= D\left(||\boldsymbol{u}||^2+\boldsymbol{u}\cdot\boldsymbol{v}\right),
\end{aligned}$$

which shows the second identity. This completes the proof. □

Proposition 2.22. *The identity*

$$\boldsymbol{u}\oplus\boldsymbol{v}=\mathrm{gyr}[\boldsymbol{u},\boldsymbol{v}](\boldsymbol{v}\oplus\boldsymbol{u}) \tag{19}$$

holds for any $\boldsymbol{u},\boldsymbol{v}\in\mathbb{V}_1$.

Proof. Since

$$\begin{aligned}
\mathrm{gyr}[\boldsymbol{u},\boldsymbol{v}](\boldsymbol{v}\oplus\boldsymbol{u}) &= (\boldsymbol{v}\oplus\boldsymbol{u})+2\frac{A''\boldsymbol{u}+B''\boldsymbol{v}}{D} \\
&= \frac{(1+2\boldsymbol{v}\cdot\boldsymbol{u}+||\boldsymbol{u}||^2)\,\boldsymbol{v}+(1-||\boldsymbol{v}||^2)\boldsymbol{u}}{D}+2\frac{A''\boldsymbol{u}+B''\boldsymbol{v}}{D},
\end{aligned}$$

it is sufficient to show that

$$D(\boldsymbol{u}\oplus\boldsymbol{v})=\left(1+2\boldsymbol{v}\cdot\boldsymbol{u}+||\boldsymbol{u}||^2\right)\boldsymbol{v}+(1-||\boldsymbol{v}||^2)\boldsymbol{u}+2(A''\boldsymbol{u}+B''\boldsymbol{v}). \tag{20}$$

By the previous lemma, the righthand side of the equation (20) equals to

$$\begin{aligned}
&\left(1+2\boldsymbol{v}\cdot\boldsymbol{u}+||\boldsymbol{u}||^2\right)\boldsymbol{v}+(1-||\boldsymbol{v}||^2)\boldsymbol{u}+2\left\{\left(||\boldsymbol{v}||^2+\boldsymbol{u}\cdot\boldsymbol{v}\right)\boldsymbol{u}-\left(||\boldsymbol{u}||^2+\boldsymbol{u}\cdot\boldsymbol{v}\right)\boldsymbol{v}\right\} \\
&= (1+2\boldsymbol{u}\cdot\boldsymbol{v}+||\boldsymbol{v}||^2)\boldsymbol{u}+(1-||\boldsymbol{u}||^2)\boldsymbol{v} \\
&= D(\boldsymbol{u}\oplus\boldsymbol{v}).
\end{aligned}$$

Hence the identity (20) holds, and this completes the proof. □

Thus we have shown a certification of Theorem 2.1.

3. Some remarks to the proof of that the Möbius ball is a gyrovector space

We need a slight modification to the identity (3.138) in [2, **3.5** p.80].

Proposition 3.1. *When non-zero vectors* $\boldsymbol{u}$, $\boldsymbol{v} \in \mathbb{V}_s$ *are parallel in* $\mathbb{V}$, *that is,* $\boldsymbol{u} = \lambda\boldsymbol{v}$ *for some* $\lambda \in \mathbb{R}$, *the Möbius addition reduces to the binary operation*

$$\boldsymbol{u} \oplus_{\mathrm{M}} \boldsymbol{v} = \begin{cases} \dfrac{\boldsymbol{u}+\boldsymbol{v}}{1+\frac{1}{s^2}||\boldsymbol{u}||||\boldsymbol{v}||} & (\lambda > 0) \\ \dfrac{\boldsymbol{u}+\boldsymbol{v}}{1-\frac{1}{s^2}||\boldsymbol{u}||||\boldsymbol{v}||} & (\lambda < 0). \end{cases} \tag{21}$$

Proof. We may assume $s = 1$. Then we have

$$\boldsymbol{u} \oplus_{\mathrm{M}} \boldsymbol{v} = \frac{(1+2\lambda\boldsymbol{v}\cdot\boldsymbol{v}+||\boldsymbol{v}||^2)\lambda\boldsymbol{v}+(1-||\lambda\boldsymbol{v}||^2)\boldsymbol{v}}{1+2\lambda\boldsymbol{v}\cdot\boldsymbol{v}+||\lambda\boldsymbol{v}||^2||\boldsymbol{v}||^2} = \frac{1+\lambda}{1+\lambda||\boldsymbol{v}||^2}\boldsymbol{v},$$

which immediately yields the result. This completes the proof. □

Definition 3.2. [2, Definition 6.83] Let $(\mathbb{V}_s, \oplus_{\mathrm{M}})$ be a Möbius gyrogroup. The Möbius scalar multiplication $r \otimes_{\mathrm{M}} \boldsymbol{v}$ in $\mathbb{V}_s$ is given by the equation

$$r \otimes_{\mathrm{M}} \boldsymbol{v} = s\tanh(r\tanh^{-1}\frac{||\boldsymbol{v}||}{s})\frac{\boldsymbol{v}}{||\boldsymbol{v}||} \tag{22}$$

for $r \in \mathbb{R}$, $\boldsymbol{0} \neq \boldsymbol{v} \in \mathbb{V}_s$ and $r \otimes_{\mathrm{M}} \boldsymbol{0} = \boldsymbol{0}$.

Theorem 3.3. [2, Theorem 6.84] *A Möbius gyrogroup* $(\mathbb{V}_s, \oplus_{\mathrm{M}})$ *with the Möbius scalar multiplication* $\otimes_{\mathrm{M}}$ *in* $\mathbb{V}_s$ *forms a gyrovector space* $(\mathbb{V}_s, \oplus_{\mathrm{M}}, \otimes_{\mathrm{M}})$.

While Ungar states in [2, p.79] as "Like the Möbius disc $(\mathbb{D}, \oplus_{\mathrm{M}})$, the Möbius ball $(\mathbb{V}_s, \oplus_{\mathrm{M}})$ turns out to be a gyrocommutative gyrogroup, as one can readily check by computer algebra", he presents in [2, pp.205–209] a proof of the remaining part of Theorem 3.3. We will give some remarks to the proof in the sequel. Throughout the rest of this paper, we denote $\otimes_{\mathrm{M}}$ (resp. $\oplus_{\mathrm{M}}$) by $\otimes$ (resp. $\oplus$) simply.

Remark 3.4. There are typos in the formula [2, (6.257)], that is, the symbol $\oplus$ should be replaced by $\otimes$. It immediately follows from $|\tanh t| = \tanh|t|$ for any $t \in \mathbb{R}$ that

$$||r\otimes\boldsymbol{a}|| = \left|\left|s\tanh(r\tanh^{-1}\frac{||\boldsymbol{a}||}{s})\frac{\boldsymbol{a}}{||\boldsymbol{a}||}\right|\right| = s\tanh(|r|\tanh^{-1}\frac{||\boldsymbol{a}||}{s}).$$

In order to verify the Axiom (V2), by the addition formula of tanh, we can easily see that

$$\begin{aligned}(r_1+r_2)\otimes\boldsymbol{a} &= s\tanh\{(r_1+r_2)\tanh^{-1}\frac{||\boldsymbol{a}||}{s}\}\frac{\boldsymbol{a}}{||\boldsymbol{a}||} \\ &= s\tanh\{r_1\tanh^{-1}\frac{||\boldsymbol{a}||}{s}+r_2\tanh^{-1}\frac{||\boldsymbol{a}||}{s}\}\frac{\boldsymbol{a}}{||\boldsymbol{a}||} \\ &= s\frac{\tanh(r_1\tanh^{-1}\frac{||\boldsymbol{a}||}{s})+\tanh(r_2\tanh^{-1}\frac{||\boldsymbol{a}||}{s})}{1+\tanh(r_1\tanh^{-1}\frac{||\boldsymbol{a}||}{s})\tanh(r_2\tanh^{-1}\frac{||\boldsymbol{a}||}{s})}\frac{\boldsymbol{a}}{||\boldsymbol{a}||}.\end{aligned}$$

It is necessary to modify slightly the formula [2, (6.252)] as follows:

$$
\begin{aligned}
&= \begin{cases} \dfrac{r_1 \otimes \boldsymbol{a} + r_2 \otimes \boldsymbol{a}}{1 + \frac{1}{s^2}||r_1 \otimes \boldsymbol{a}||||r_2 \otimes \boldsymbol{a}||} & (r_1 r_2 > 0) \\[2ex] \dfrac{r_1 \otimes \boldsymbol{a} + r_2 \otimes \boldsymbol{a}}{1 - \frac{1}{s^2}||r_1 \otimes \boldsymbol{a}||||r_2 \otimes \boldsymbol{a}||} & (r_1 r_2 < 0) \end{cases} \\
&= r_1 \otimes \boldsymbol{a} \oplus r_2 \otimes \boldsymbol{a},
\end{aligned}
$$

where we used Proposition 3.1. In the verification of the Axiom (V3), we need a similar slight modification. Let us use the notation

$$
\boldsymbol{b} = r_2 \otimes \boldsymbol{a} = s \tanh(r_2 \tanh^{-1} \frac{||\boldsymbol{a}||}{s}) \frac{\boldsymbol{a}}{||\boldsymbol{a}||},
$$

so that

$$
\frac{||\boldsymbol{b}||}{s} = \tanh(|r_2| \tanh^{-1} \frac{||\boldsymbol{a}||}{s})
$$

and

$$
\frac{\boldsymbol{b}}{||\boldsymbol{b}||} = \begin{cases} \dfrac{\boldsymbol{a}}{||\boldsymbol{a}||} & (r_2 > 0) \\[2ex] -\dfrac{\boldsymbol{a}}{||\boldsymbol{a}||} & (r_2 < 0). \end{cases}
$$

Then we have

$$
\begin{aligned}
r_1 \otimes (r_2 \otimes \boldsymbol{a}) &= r_1 \otimes \boldsymbol{b} \\
&= s \tanh(r_1 \tanh^{-1} \frac{||\boldsymbol{b}||}{s}) \frac{\boldsymbol{b}}{||\boldsymbol{b}||} \\
&= \begin{cases} s \tanh\{r_1 \tanh^{-1} \tanh(|r_2| \tanh^{-1} \frac{||\boldsymbol{a}||}{s})\} \dfrac{\boldsymbol{a}}{||\boldsymbol{a}||} & (r_2 > 0) \\[2ex] -s \tanh\{r_1 \tanh^{-1} \tanh(|r_2| \tanh^{-1} \frac{||\boldsymbol{a}||}{s})\} \dfrac{\boldsymbol{a}}{||\boldsymbol{a}||} & (r_2 < 0) \end{cases} \\
&= s \tanh(r_1 r_2 \tanh^{-1} \frac{||\boldsymbol{a}||}{s}) \frac{\boldsymbol{a}}{||\boldsymbol{a}||} \\
&= (r_1 r_2) \otimes \boldsymbol{a}.
\end{aligned}
$$

We conclude the paper by showing the gamma identity, which is used to verify the Axiom (V8), i.e., the Möbius gyrotriangle inequality [2, Theorem 3.42].

Proposition 3.5. [2, (3.130)] *The Möbius addition satisfies the gamma identity*

$$\gamma_{\boldsymbol{u}\oplus\boldsymbol{v}} = \gamma_{\boldsymbol{u}}\gamma_{\boldsymbol{v}}\sqrt{1 + \frac{2}{s^2}\boldsymbol{u}\cdot\boldsymbol{v} + \frac{1}{s^4}||\boldsymbol{u}||^2||\boldsymbol{v}||^2} \tag{23}$$

for any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}_s$*, where* $\gamma_{\boldsymbol{v}}$ *is the gamma factor is defined by the identity*

$$\gamma_{\boldsymbol{v}} = \frac{1}{\sqrt{1 - \frac{||\boldsymbol{v}||^2}{s^2}}} \tag{24}$$

in $\mathbb{V}_s$*.*

Proof. We may assume $s = 1$. By considering the reciprocal of the square of the equality (23), it is sufficient to show that

$$1 - ||\boldsymbol{u} \oplus \boldsymbol{v}||^2 = \frac{(1 - ||\boldsymbol{u}||^2)(1 - ||\boldsymbol{v}||^2)}{1 + 2\boldsymbol{u}\cdot\boldsymbol{v} + ||\boldsymbol{u}||^2||\boldsymbol{v}||^2},$$

which has been already verified as the identity (3). This completes the proof. □

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