Nihonkai Math. J.
Vol.27(2016), 89–97

ON CONVERGENCE OF ORBITS TO A FIXED POINT FOR WIDELY MORE GENERALIZED HYBRID MAPPINGS

TOSHIHARU KAWASAKI

Abstract. The concept of widely more generalized hybrid mapping introduced by Kawasaki and Takahashi in [6] in the case of Hilbert spaces. We also use this definition in the case of Banach spaces. In this paper we discuss some strong convergence theorems of orbits to a fixed point for widely more generalized hybrid mappings.

1. Introduction

Let E be a Banach space and let C be a non-empty subset of E. A mapping T from C into E is said to be widely more generalized hybrid if there exist real numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ and η such that

$$\begin{aligned}\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ +\varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \le 0\end{aligned}$$

for any $x, y \in C$. Such a mapping is called an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping. The definition above introduced by Kawasaki and Takahashi in [6] in the case where E is a Hilbert space. In this paper we also use this definition in the case of Banach spaces. We obtained the following fixed point theorems in the case of Hilbert spaces.

Theorem 1.1 ([6, 7, 2])**.** *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself which satisfies the following condition* (1)*,* (2) *or* (3)*:*

(1) $\alpha + \beta + \gamma + \delta \ge 0$, $\alpha + \gamma + \varepsilon + \eta > 0$ *and* $\zeta + \eta \ge 0$;

(2) $\alpha + \beta + \gamma + \delta \ge 0$, $\alpha + \beta + \zeta + \eta > 0$ *and* $\varepsilon + \eta \ge 0$;

(3) $\alpha + \beta + \gamma + \delta \ge 0$, $2\alpha + \beta + \gamma + \varepsilon + \zeta + 2\eta > 0$ *and* $\varepsilon + \zeta + 2\eta \ge 0$.

2010 *Mathematics Subject Classification.* 47H10.

Key words and phrases. Fixed point theorem, convergence theorem, widely more generalized hybrid mapping, Banach space, Hilbert space.

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. In particular, if $\alpha + \beta + \gamma + \delta > 0$, then a fixed point of T is unique.

Theorem 1.2 ([6, 7, 2])**.** *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself which satisfies the following conditions* (1)*,* (2) *or* (3)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$, *and there exists* $\lambda \in [0, 1)$ *such that* $(\alpha + \beta)\lambda + \zeta + \eta \geq 0$;

(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$, *and there exists* $\lambda \in [0, 1)$ *such that* $(\alpha + \gamma)\lambda + \varepsilon + \eta \geq 0$;

(3) $\alpha + \beta + \gamma + \delta \geq 0$, $2\alpha + \beta + \gamma + \varepsilon + \zeta + 2\eta > 0$, *and there exists* $\lambda \in [0, 1)$ *such that* $(2\alpha + \beta + \gamma)\lambda + \varepsilon + \zeta + 2\eta \geq 0$.

Then T has a fixed point if and only if there exists $z \in C$ such that $\{((1 - \lambda)T + \lambda I)^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded for $\lambda \in [0, 1) \cap \{\lambda \mid (\alpha + \beta)\lambda + \zeta + \eta \geq 0\}$, $\lambda \in [0, 1) \cap \{\lambda \mid (\alpha + \gamma)\lambda + \varepsilon + \eta \geq 0\}$ or $\lambda \in [0, 1) \cap \{\lambda \mid (2\alpha + \beta + \gamma)\lambda + \varepsilon + \zeta + 2\eta \geq 0\}$, respectively. In particular, if $\alpha + \beta + \gamma + \delta > 0$, then a fixed point of T is unique.

Theorem 1.3 ([8])**.** *Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid demicontinuous mapping from C into itself which satisfies the following conditions* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $2\alpha + \beta + \gamma + \varepsilon + \zeta + 2\eta \geq 0$, $\beta + \gamma + \varepsilon + \zeta \geq 0$ *and* $\beta + \gamma + 2(\varepsilon + \zeta + \eta) < 0$;

(2) $\alpha + \beta + \gamma + \delta \geq 0$, $2\alpha + \beta + \gamma + \varepsilon + \zeta + 2\eta \geq 0$, $\beta + \gamma + \varepsilon + \zeta < 0$ *and* $-\beta - \gamma + 2\eta \leq 0$.

Then T has a fixed point. In particular, if $\alpha + \beta + \gamma + \delta > 0$, then a fixed point of T is unique.

Theorem 1.4 ([8])**.** *Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid demicontinuous mapping from C into itself which satisfies the following conditions* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $2\alpha + \beta + \gamma + \varepsilon + \zeta + 2\eta \geq 0$ *and, there exists* $\lambda \in [0, 1)$ *such that* $\beta + \gamma + \varepsilon + \zeta \geq 0$ *and* $\beta + \gamma + 2(\varepsilon + \zeta + \eta) + (2\alpha + \beta + \gamma)\lambda < 0$;

(2) $\alpha + \beta + \gamma + \delta \geq 0$, $2\alpha + \beta + \gamma + \varepsilon + \zeta + 2\eta \geq 0$ *and, there exists* $\lambda \in [0, 1)$ *such that* $\beta + \gamma + \varepsilon + \zeta < 0$ *and* $-\beta - \gamma + 2\eta + (2\alpha + \beta + \gamma)\lambda \leq 0$.

Then T has a fixed point. In particular, if $\alpha + \beta + \gamma + \delta > 0$, then a fixed point of T is unique.

Theorem 1.5 ([3]). *Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid demicontinuous mapping from C into itself which satisfies the following conditions* (1) *or* (2)*:*

(1) $\alpha + 2\min\{\beta,\gamma\} + \delta \geq 0$, $\alpha + \min\{\beta,\gamma\} + \min\{\varepsilon,\zeta\} + \eta \geq 0$, $\min\{\beta,\gamma\} + \min\{\varepsilon,\zeta\} \geq 0$ *and* $\min\{\beta,\gamma\} + 2\min\{\varepsilon,\zeta\} + \eta < 0$*;*

(2) $\alpha + 2\min\{\beta,\gamma\} + \delta \geq 0$, $\alpha + \min\{\beta,\gamma\} + \min\{\varepsilon,\zeta\} + \eta \geq 0$, $\min\{\beta,\gamma\} + \min\{\varepsilon,\zeta\} < 0$ *and* $-\min\{\beta,\gamma\} + \eta \leq 0$.

Then T has a fixed point. In particular, if $\alpha+\beta+\gamma+\delta > 0$, then a fixed point of T is unique.

The results above are necessary conditions for mappings to have a fixed point in a Hilbert space. When a mapping have a fixed point, is the orbit $\{T^n x \mid n \in \mathbb{N} \cup \{0\}\}$ convergent to a fixed point or not? We know that, if T is a contraction, then the orbit is convergent to a fixed point. In this paper we discuss some strong convergence theorems of orbits to a fixed point for widely more generalized hybrid mappings.

2. Strong convergence theorems of orbits to a fixed point

In this section we discuss some strong convergence theorems of orbits to a fixed point.

Theorem 2.1. *Let E be a Banach space, let C be a non-empty subset of E and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,0)$-widely more generalized hybrid mapping from C into itself which has a fixed point. Then the following hold:*

(1) *in the case of $\alpha+\beta+\gamma+\delta+4\min\{\varepsilon,0\} > 0$,*
 - (i) *if $\alpha+\gamma+2\min\{\varepsilon,0\} \geq 0$, then the orbit $\{T^n x\}$ is convergent to a fixed point for any $x \in C$;*
 - (ii) *if $\alpha+\gamma+2\min\{\varepsilon,0\} < 0$, then $\lim_{n\to\infty}\|T^n x\| = \infty$ for any $x \in C$;*

(2) *in the case of $\alpha+\beta+\gamma+\delta+4\min\{\zeta,0\} > 0$,*
 - (i) *if $\alpha+\beta+2\min\{\zeta,0\} \geq 0$, then the orbit $\{T^n x\}$ is convergent to a fixed point for any $x \in C$;*
 - (ii) *if $\alpha+\beta+2\min\{\zeta,0\} < 0$, then $\lim_{n\to\infty}\|T^n x\| = \infty$ for any $x \in C$;*

(3) *in the case of $\alpha+\beta+\gamma+\delta+2\min\{\varepsilon+\zeta,0\} > 0$,*
 - (i) *if $2\alpha+\beta+\gamma+2\min\{\varepsilon+\zeta,0\} \geq 0$, then the orbit $\{T^n x\}$ is convergent to a fixed point for any $x \in C$;*
 - (ii) *if $2\alpha+\beta+\gamma+2\min\{\varepsilon+\zeta,0\} < 0$, then $\lim_{n\to\infty}\|T^n x\| = \infty$ for any $x \in C$.*

Proof. Let z be a fixed point of T. Replacing x and y by $T^n x$ and z, respectively, we obtain

$$(\alpha+\gamma)\|T^{n+1}x - z\|^2 + (\beta+\delta)\|T^n x - z\|^2 + \varepsilon\|T^n x - T^{n+1}x\|^2 \le 0.$$

Since

$$\begin{aligned}
&\|T^{n+1}x - z\|^2 - 2\|T^{n+1}x - z\|\|T^n x - z\| + \|T^n x - z\|^2 \\
&\quad \le \|T^n x - T^{n+1}x\|^2 \\
&\quad \le \|T^{n+1}x - z\|^2 + 2\|T^{n+1}x - z\|\|T^n x - z\| + \|T^n x - z\|^2,
\end{aligned}$$

we obtain

$$\varepsilon\|T^{n+1}x - z\|^2 - 2|\varepsilon|\|T^{n+1}x - z\|\|T^n x - z\| + \varepsilon\|T^n x - z\|^2 \le \varepsilon\|T^n x - T^{n+1}x\|^2.$$

Therefore we obtain

$$\begin{aligned}
&(\alpha+\gamma+\varepsilon)\|T^{n+1}x - z\|^2 + (\beta+\delta+\varepsilon)\|T^n x - z\|^2 \\
&\qquad -2|\varepsilon|\|T^{n+1}x - z\|\|T^n x - z\| \\
&\quad = (\alpha+\gamma+\varepsilon-|\varepsilon|)\|T^{n+1}x - z\|^2 + (\beta+\delta+\varepsilon-|\varepsilon|)\|T^n x - z\|^2 \\
&\qquad +|\varepsilon|(\|T^{n+1}x - z\| - \|T^n x - z\|)^2 \\
&\quad \le 0
\end{aligned}$$

and hence

$$\begin{aligned}
&(\alpha+\gamma+\varepsilon-|\varepsilon|)\|T^{n+1}x - z\|^2 + (\beta+\delta+\varepsilon-|\varepsilon|)\|T^n x - z\|^2 \\
&\quad = (\alpha+\gamma+2\min\{\varepsilon,0\})\|T^{n+1}x - z\|^2 + (\beta+\delta+2\min\{\varepsilon,0\})\|T^n x - z\|^2 \\
&\quad \le 0.
\end{aligned}$$

If $\alpha+\gamma+2\min\{\varepsilon,0\} = 0$, then $\beta+\delta+2\min\{\varepsilon,0\} > 0$. Therefore $\|T^n x - z\| = 0$ for any x and for any n, and hence

$$\lim_{n\to\infty} T^n x = z$$

for any x. If $\alpha+\gamma+2\min\{\varepsilon,0\} > 0$, then

$$\begin{aligned}
\|T^{n+1}x - z\|^2 &\le -\frac{\beta+\delta+2\min\{\varepsilon,0\}}{\alpha+\gamma+2\min\{\varepsilon,0\}}\|T^n x - z\|^2 \\
&\le \max\left\{-\frac{\beta+\delta+2\min\{\varepsilon,0\}}{\alpha+\gamma+2\min\{\varepsilon,0\}}, 0\right\}\|T^n x - z\|^2.
\end{aligned}$$

Since $\alpha+\beta+\gamma+\delta+4\min\{\varepsilon,0\} > 0$, we obtain $0 \le \max\left\{-\frac{\beta+\delta+2\min\{\varepsilon,0\}}{\alpha+\gamma+2\min\{\varepsilon,0\}}, 0\right\} < 1$. Therefore

$$\lim_{n\to\infty} T^n x = z$$

for any x. If $\alpha+\gamma+2\min\{\varepsilon,0\}<0$, then

$$\|T^{n+1}x-z\|^2 \geq -\frac{\beta+\delta+2\min\{\varepsilon,0\}}{\alpha+\gamma+2\min\{\varepsilon,0\}}\|T^n x-z\|^2$$

and $-\frac{\beta+\delta+2\min\{\varepsilon,0\}}{\alpha+\gamma+2\min\{\varepsilon,0\}}>1$. Therefore

$$\lim_{n\to\infty}\|T^n x-z\|=\infty$$

and hence

$$\lim_{n\to\infty}\|T^n x\|=\infty$$

for any x.

In the case of (2) we can obtain the desired result similarly by interchanging the variables x and y.

Interchanging the variables x and y, we obtain

$$\alpha\|Tx-Ty\|^2+\gamma\|x-Ty\|^2+\beta\|Tx-y\|^2+\delta\|x-y\|^2 \\ +\zeta\|x-Tx\|^2+\varepsilon\|y-Ty\|^2\leq 0.$$

Therefore we obtain

$$2\alpha\|Tx-Ty\|^2+(\beta+\gamma)\|x-Ty\|^2+(\beta+\gamma)\|Tx-y\|^2+2\delta\|x-y\|^2 \\ +(\varepsilon+\zeta)\|x-Tx\|^2+(\varepsilon+\zeta)\|y-Ty\|^2\leq 0.$$

In the case of (3) we obtain the desired result similarly. □

Using Theorem 2.1, we obtain the following theorem.

Theorem 2.2. *Let H be a real Hilbert space, let C be a non-empty convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into itself which has a fixed point. Then the following hold:*

(1) *in the case where there exists $\lambda\in[0,1)$ such that*

$$\alpha+\beta+\gamma+\delta+4\min\left\{\frac{(\alpha+\gamma)\lambda+\varepsilon+\eta}{(1-\lambda)^2},0\right\}>0,$$

(i) *if $\alpha+\gamma+2\min\left\{\frac{(\alpha+\gamma)\lambda+\varepsilon+\eta}{1-\lambda},0\right\}\geq 0$, then the orbit $\{((1-\lambda)T+\lambda I)^n x\}$ is convergent to a fixed point for any $x\in C$;*

(ii) *if $\alpha+\gamma+2\min\left\{\frac{(\alpha+\gamma)\lambda+\varepsilon+\eta}{1-\lambda},0\right\}<0$, then $\lim_{n\to\infty}\|((1-\lambda)T+\lambda I)^n x\|=\infty$ for any $x\in C$;*

(2) *in the case where there exists $\lambda\in[0,1)$ such that*

$$\alpha+\beta+\gamma+\delta+4\min\left\{\frac{(\alpha+\beta)\lambda+\zeta+\eta}{(1-\lambda)^2},0\right\}>0,$$

(i) *if* $\alpha+\beta+2\min\left\{\frac{(\alpha+\beta)\lambda+\zeta+\eta}{1-\lambda},0\right\}\geq 0$, *then the orbit* $\{((1-\lambda)T+\lambda I)^n x\}$ *is convergent to a fixed point for any* $x\in C$*;*

(ii) *if* $\alpha+\beta+2\min\left\{\frac{(\alpha+\beta)\lambda+\zeta+\eta}{1-\lambda},0\right\}<0$, *then* $\lim_{n\to\infty}\|((1-\lambda)T+\lambda I)^n x\|=\infty$ *for any* $x\in C$*;*

(3) *in the case where there exists* $\lambda\in[0,1)$ *such that*

$$\alpha+\beta+\gamma+\delta+2\min\left\{\frac{(2\alpha+\beta+\gamma)\lambda+\varepsilon+\zeta+2\eta}{(1-\lambda)^2},0\right\}>0,$$

(i) *if* $2\alpha+\beta+\gamma+2\min\left\{\frac{(2\alpha+\beta+\gamma)\lambda+\varepsilon+\zeta+2\eta}{1-\lambda},0\right\}\geq 0$, *then the orbit* $\{((1-\lambda)T+\lambda I)^n x\}$ *is convergent to a fixed point for any* $x\in C$*;*

(ii) *if* $2\alpha+\beta+\gamma+2\min\left\{\frac{(2\alpha+\beta+\gamma)\lambda+\varepsilon+\zeta+2\eta}{1-\lambda},0\right\}<0$, *then* $\lim_{n\to\infty}\|((1-\lambda)T+\lambda I)^n x\|=\infty$ *for any* $x\in C$.

Proof. Let $S=(1-\lambda)T+\lambda I$. Since C is convex, S is a mapping from C into itself and $F(S)=F(T)$. Moreover

$$\begin{aligned}
&\alpha\left\|\left(\frac{1}{1-\lambda}Sx-\frac{\lambda}{1-\lambda}x\right)-\left(\frac{1}{1-\lambda}Sy-\frac{\lambda}{1-\lambda}y\right)\right\|^2\\
&\quad+\beta\left\|x-\left(\frac{1}{1-\lambda}Sy-\frac{\lambda}{1-\lambda}y\right)\right\|^2+\gamma\left\|\left(\frac{1}{1-\lambda}Sx-\frac{\lambda}{1-\lambda}x\right)-y\right\|^2\\
&\quad+\delta\|x-y\|^2\\
&\quad+\varepsilon\left\|x-\left(\frac{1}{1-\lambda}Sx-\frac{\lambda}{1-\lambda}x\right)\right\|^2+\zeta\left\|y-\left(\frac{1}{1-\lambda}Sy-\frac{\lambda}{1-\lambda}y\right)\right\|^2\\
&\quad+\eta\left\|\left(x-\left(\frac{1}{1-\lambda}Sx-\frac{\lambda}{1-\lambda}x\right)\right)-\left(y-\left(\frac{1}{1-\lambda}Sy-\frac{\lambda}{1-\lambda}y\right)\right)\right\|^2\\
&=\frac{\alpha}{1-\lambda}\|Sx-Sy\|^2+\frac{\beta}{1-\lambda}\|x-Sy\|^2+\frac{\gamma}{1-\lambda}\|Sx-y\|^2\\
&\quad+\left(-\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta\right)\|x-y\|^2\\
&\quad+\frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2}\|x-Sx\|^2+\frac{\zeta+\beta\lambda}{(1-\lambda)^2}\|y-Sy\|^2\\
&\quad+\frac{\eta+\alpha\lambda}{(1-\lambda)^2}\|(x-Sx)-(y-Sy)\|^2\\
&\leq 0.
\end{aligned}$$

Since

$$\begin{aligned}
&\|(x-Sx)-(y-Sy)\|^2\\
&=\|Sx-Sy\|^2-\|x-Sy\|^2-\|Sx-y\|^2+\|x-y\|^2+\|x-Sx\|^2+\|y-Sy\|^2,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\frac{\alpha}{1-\lambda}\|Sx-Sy\|^2+\frac{\beta}{1-\lambda}\|x-Sy\|^2+\frac{\gamma}{1-\lambda}\|Sx-y\|^2\\
&\quad+\left(-\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta\right)\|x-y\|^2\\
&\quad+\frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2}\|x-Sx\|^2+\frac{\zeta+\beta\lambda}{(1-\lambda)^2}\|y-Sy\|^2\\
&\quad+\frac{\eta+\alpha\lambda}{(1-\lambda)^2}\|(x-Sx)-(y-Sy)\|^2\\
&=\frac{\alpha+\eta}{(1-\lambda)^2}\|Sx-Sy\|^2+\frac{\beta(1-\lambda)-\eta-\alpha\lambda}{(1-\lambda)^2}\|x-Sy\|^2\\
&\quad+\frac{\gamma(1-\lambda)-\eta-\alpha\lambda}{(1-\lambda)^2}\|Sx-y\|^2\\
&\quad+\frac{\alpha\lambda^2-(\beta+\gamma)(1-\lambda)\lambda+\delta(1-\lambda)^2+\eta}{(1-\lambda)^2}\|x-y\|^2\\
&\quad+\frac{\varepsilon+\eta+(\alpha+\gamma)\lambda}{(1-\lambda)^2}\|x-Sx\|^2+\frac{\zeta+\eta+(\alpha+\beta)\lambda}{(1-\lambda)^2}\|y-Sy\|^2\\
&\le 0.
\end{aligned}$$

Therefore S is an $\left(\frac{\alpha+\eta}{(1-\lambda)^2}, \frac{\beta(1-\lambda)-\eta-\alpha\lambda}{(1-\lambda)^2}, \frac{\gamma(1-\lambda)-\eta-\alpha\lambda}{(1-\lambda)^2}, \frac{\alpha\lambda^2-(\beta+\gamma)(1-\lambda)\lambda+\delta(1-\lambda)^2+\eta}{(1-\lambda)^2}, \frac{\varepsilon+\eta+(\alpha+\gamma)\lambda}{(1-\lambda)^2}, \frac{\zeta+\eta+(\alpha+\beta)\lambda}{(1-\lambda)^2}, 0\right)$-widely more generalized hybrid mapping. Moreover we obtain

$$\begin{aligned}
&\frac{\alpha+\eta}{(1-\lambda)^2}+\frac{\beta(1-\lambda)-\eta-\alpha\lambda}{(1-\lambda)^2}+\frac{\gamma(1-\lambda)-\eta-\alpha\lambda}{(1-\lambda)^2}\\
&\quad+\frac{\alpha\lambda^2-(\beta+\gamma)(1-\lambda)\lambda+\delta(1-\lambda)^2+\eta}{(1-\lambda)^2}+4\min\left\{\frac{\varepsilon+\eta+(\alpha+\gamma)\lambda}{(1-\lambda)^2},0\right\}\\
&=\alpha+\beta+\gamma+\delta+4\min\left\{\frac{(\alpha+\gamma)\lambda+\varepsilon+\eta}{(1-\lambda)^2},0\right\}\\
&>0.
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{\alpha+\eta}{(1-\lambda)^2}+\frac{\gamma(1-\lambda)-\eta-\alpha\lambda}{(1-\lambda)^2}+2\min\left\{\frac{\varepsilon+\eta+(\alpha+\gamma)\lambda}{(1-\lambda)^2},0\right\}\\
&\quad=\frac{\alpha+\gamma+2\min\left\{\frac{(\alpha+\gamma)\lambda+\varepsilon+\eta}{1-\lambda},0\right\}}{1-\lambda},
\end{aligned}$$

by Theorem 2.1 we obtain that, if $\alpha+\gamma+2\min\left\{\frac{(\alpha+\gamma)\lambda+\varepsilon+\eta}{1-\lambda},0\right\}\ge 0$, then the orbit $\{S^nx\}$ is convergent to a fixed point for any $x\in C$, and if $\alpha+\gamma+2\min\left\{\frac{(\alpha+\gamma)\lambda+\varepsilon+\eta}{1-\lambda},0\right\}<0$, then $\lim_{n\to\infty}\|S^nx\|=\infty$ for any $x\in C$.

In the case of (2) we can obtain the desired result similarly by interchanging the variables x and y.

Interchanging the variables x and y, we obtain

$$\alpha\|Tx-Ty\|^2+\gamma\|x-Ty\|^2+\beta\|Tx-y\|^2+\delta\|x-y\|^2 \\ +\zeta\|x-Tx\|^2+\varepsilon\|y-Ty\|^2+\eta\|(x-Tx)-(y-Ty)\|^2\le 0.$$

Therefore we obtain

$$2\alpha\|Tx-Ty\|^2+(\beta+\gamma)\|x-Ty\|^2+(\beta+\gamma)\|Tx-y\|^2+2\delta\|x-y\|^2 \\ +(\varepsilon+\zeta)\|x-Tx\|^2+(\varepsilon+\zeta)\|y-Ty\|^2 \\ +2\eta\|(x-Tx)-(y-Ty)\|^2\le 0.$$

In the case of (3) we obtain the desired result similarly. $\square$

References

[1] T. Kawasaki, *An extension of existence and mean approximation of fixed points of generalized hybrid non-self mappings in Hilbert spaces*, J. Nonlinear Convex Anal., to appear.

[2] T. Kawasaki, *Fixed points theorems and mean convergence theorems for generalized hybrid self mappings and non-self mappings in Hilbert spaces*, Pac. J. Optim., to appear.

[3] T. Kawasaki, *Fixed point theorem for widely more generalized hybrid demicontinuous mappings in Hilbert spaces*, Proceedings of Nonlinear Analysis and Convex Analysis 2015, Yokohama Publ., Yokohama, to appear.

[4] T. Kawasaki, *Fixed point theorems for widely more generalized hybrid mappings in a Banach space*, submitted.

[5] T. Kawasaki and T. Kobayashi, *Existence and mean approximation of fixed points of generalized hybrid non-self mappings in Hilbert spaces*, Sci. Math. Jpn. **77 (Online Version: e-2014)** (2014), 13–26 (Online Version: 29–42).

[6] T. Kawasaki and W. Takahashi, *Existence and mean approximation of fixed points of generalized hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **14** (2013), 71–87.

[7] T. Kawasaki and W. Takahashi, *Fixed point and nonlinear ergodic theorems for widely more generalized hybrid mappings in Hilbert spaces and applications*, Proceedings of Nonlinear Analysis and Convex Analysis 2013, Yokohama Publ., Yokohama, to appear.

[8] T. Kawasaki and W. Takahashi, *Fixed point theorems for generalized hybrid demicontinuous mappings in Hilbert spaces*, Linear Nonlinear Anal. **1** (2015), 125–138.

[9] W. Takahashi, *Unique fixed point theorems for nonlinear mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **15** (2014), 831–849.

(T. Kawasaki) College of Engineering, Nihon University, Fukushima 963–8642, Japan
E-mail address: toshiharu.kawasaki@nifty.ne.jp

Received January 12, 2016
Revised March 20, 2016