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THE SPLIT COMMON FIXED POINT PROBLEM WITH FAMILIES OF MAPPINGS AND STRONG CONVERGENCE THEOREMS BY HYBRID METHODS IN BANACH SPACES

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Abstract. In this paper, we consider the split common fixed point problem with families of mappings in Banach spaces. Then using the hybrid method and the shrinking projection method, we prove strong convergence theorems for finding a solution of the split common null point problem with families of mappings in Banach spaces.

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces. Given mappings $T : H_1 \to H_1$ and $U : H_2 \to H_2$, respectively, and a bounded linear operator $A : H_1 \to H_2$, the *split common fixed point problem* is to find a point $z \in H_1$ such that $z \in F(T) \cap A^{-1}F(U)$, where $F(T)$ and $F(U)$ are fixed point sets of T and U, respectively. Let D and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility problem* [8] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Defining $T = P_D$ and $U = P_Q$, where P_D and P_Q are the metric projections of H_1 onto D and H_2 onto Q, respectively, we have that $z \in D \cap A^{-1}Q$ is equivalent to $z \in F(T) \cap A^{-1}F(U)$. Furthermore, given set-valued mappings $G : H_1 \to 2^{H_1}$ and $B : H_2 \to 2^{H_2}$, respectively, and a bounded linear operator $A : H_1 \to H_2$, the *split common null point problem* [7] is to find a point $z \in H_1$ such that $z \in G^{-1}0 \cap A^{-1}(B^{-1}0)$, where $G^{-1}0$ and $B^{-1}0$ are null point sets of G and B, respectively. Defining $T = J_\lambda$ and $U = Q_\mu$, where J_λ and Q_μ are the resolvents of G for $\lambda > 0$ and B for $\mu > 0$, respectively, we have that $z \in G^{-1}0 \cap A^{-1}(B^{-1}0)$ is equivalent to $z \in F(T) \cap A^{-1}F(U)$. Thus, the split common fixed point problem generalizes the split feasibility problem and the split common null point problem. If $U = A^*(I - P_Q)A$ in the split feasibility problem, then we

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have that $U : H_1 \to H_1$ is an inverse strongly monotone operator [2], where A^* is the adjoint operator of A and P_Q is the metric projection of H_2 onto Q. Furthermore, if $D \cap A^{-1}Q$ is nonempty, then $z \in D \cap A^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda A^*(I - P_Q)A)z, \tag{1.1}$$

where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. Using such results regarding nonlinear operators and fixed points, many authors have studied the split feasibility problem, the split common null point problem and the split common fixed point problem in Hilbert spaces; see, for instance, [2, 7, 9, 19, 35]. Recently, Takahashi [30] and [32] extended such an equivalent relation (1.1) in Hilbert spaces to Banach spaces and then he obtained strong convergence theorems for finding a solution of the split feasibility problem in Banach spaces. Very recently, using the hybrid method [21, 22, 24], Alsulami, Latif and Takahashi [1] prove a strong convergence theorem for finding a solution of the split feasibility problem in Banach spaces; see also [31].

Theorem 1 ([1])**.** *Let H be a Hilbert space and let F be a strictly convex, reflexive and smooth Banach space. Let J_F be the duality mapping on F. Let C and D be nonempty, closed and convex subsets of H and F, respectively. Let P_C and P_D be the metric projections of H onto C and F onto D, respectively. Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $C \cap A^{-1}D \neq \emptyset$. Let $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = P_C\Big(x_n - rA^*J_F(Ax_n - P_DAx_n)\Big), \\ y_n = \alpha_n x_n + (1-\alpha_n)z_n, \\ C_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ for some $a \in \mathbb{R}$ and $0 < r\|A\|^2 < 2$. Then $\{x_n\}$ converges strongly to a point $z_0 \in C \cap A^{-1}D$, where $z_0 = P_{C \cap A^{-1}D}x_1$.

Takahashi [32] also obtained the following result from the idea of the shrinking projection method by Takahashi, Takeuchi and Kubota [34].

Theorem 2 ([32])**.** *Let H be a Hilbert space and let F be a uniformly convex Banach space whose norm is Fréchet differentiable. Let J_F be the duality mapping on F. Let C and D be nonempty, closed and convex subsets of H and F, respectively. Let P_C and P_D be the metric projections of H onto C and F onto D, respectively. Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint* *operator of A. Suppose that $C \cap A^{-1}D \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. Let $x_1 \in H$, $C_1 = H$, and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = P_C\Big(x_n - rA^*J_F(Ax_n - P_DAx_n)\Big), \\ C_{n+1} = \{z \in H : \|z_n - z\| \leq \|x_n - z\|\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}}u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < r\|A\|^2 < 2$. Then $\{x_n\}$ converges strongly to a point $z_0 \in C \cap A^{-1}D$, where $z_0 = P_{C\cap A^{-1}D}u$.

In this paper, motivated by these problems and results, we consider the split common fixed point problem with families of mappings in Banach spaces. Then using the hybrid method and the shrinking projection method, we prove two strong convergence theorems for finding a solution of the split common fixed point problem with families of mappings in Banach spaces. We also apply these results to obtain new results for the split common fixed point problem with families of mappings in Banach spaces.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot\,,\ \cdot\rangle$ and norm $\|\cdot\|$, respectively. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have from [28] that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle; \tag{2.1}$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2. \tag{2.2}$$

Furthermore we have that for $x, y, u, v \in H$,

$$2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.3}$$

Let C be a nonempty, closed and convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_Cx\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such a mapping P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_Cx - P_Cy\|^2 \leq \langle P_Cx - P_Cy, x - y\rangle \tag{2.4}$$

for all $x, y \in H$. Furthermore $\langle x - P_Cx, y - P_Cx\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [26].

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon \right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. It is known that a Banach space E is uniformly convex if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. A uniformly convex Banach space is strictly convex and reflexive. We also know that a uniformly convex Banach space has the Kadec-Klee property, i.e., $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$; see [10, 23].

The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.5}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [26] and [27]. We know the following result.

Lemma 3 ([26])**.** *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x-y, Jx-Jy \rangle \ge 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x-y, Jx-Jy \rangle = 0$, then $x = y$.*

Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x-z\| \le \|x-y\|$ for all $y \in C$. Putting $z = P_C x$, we call such a mapping P_C the metric projection of E onto C.

Lemma 4 ([26])**.** *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x_1 \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = P_C x_1$;

(2) $\langle z - y, J(x_1 - z)\rangle \geq 0, \quad \forall y \in C.$

Let E be a Banach space and let B be a mapping of E into 2^{E^*}. The effective domain of B is denoted by $\mathrm{dom}(B)$, that is, $\mathrm{dom}(B) = \{x \in E : Bx \neq \emptyset\}$. A multi-valued mapping B on E is said to be monotone if $\langle x - y, u^* - v^*\rangle \geq 0$ for all $x, y \in \mathrm{dom}(B)$, $u^* \in Bx$, and $v^* \in By$. A monotone operator B on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to Browder [5]; see also [27, Theorem 3.5.4].

Theorem 5 ([5])**.** *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let B be a monotone operator of E into 2^{E^*}. Then B is maximal if and only if for any $r > 0$,*

$$R(J + rB) = E^*,$$

where $R(J + rB)$ is the range of $J + rB$.

Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let B be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation

$$0 \in J(x_r - x) + rBx_r.$$

This equation has a unique solution x_r. We define J_r by $x_r = J_r x$. Such $J_r, r > 0$ are called the metric resolvents of B. The set of null points of B is defined by $B^{-1}0 = \{z \in E : 0 \in Bz\}$. We know that $B^{-1}0$ is closed and convex; see [27].

Let B be a maximal monotone operator on a Hilbert space H. In a Hilbert space H, the metric resolvent J_r of B is simply called the resolvent of B. It is known that the resolvent J_r of B for $r > 0$ is firmly nonexpansive, i.e.,

$$\|J_r x - J_r y\|^2 \leq \langle x - y, J_r x - J_r y\rangle, \quad \forall x, y \in H.$$

It is also known that $\|J_\lambda x - J_\mu x\| \leq (|\lambda - \mu| / \lambda) \|x - J_\lambda x\|$ holds for all $\lambda, \mu > 0$ and $x \in H$; see [26, 11] for more details. As a matter of fact, we know the following lemma from Takahashi, Takahashi and Toyoda [25].

Lemma 6 ([25])**.** *Let H be a Hilbert space and let B be a maximal monotone operator on H. For $r > 0$ and $x \in H$, define the resolvent $J_r x$. Then the following holds:*

$$\frac{s - t}{s}\langle J_s x - J_t x, J_s x - x\rangle \geq \|J_s x - J_t x\|^2$$

for all $s, t > 0$ and $x \in H$.

Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty subset of E and let η be a real number with $\eta \in (-\infty, 1)$. Then a mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called η-demimetric [33] if, for any $x \in C$ and $q \in F(U)$,

$$\langle x - q, J(x - Ux)\rangle \geq \frac{1-\eta}{2}\|x - Ux\|^2,$$

where $F(U)$ is the set of fixed points of U.

Example. (1) Let H be a Hilbert space, let C be a nonempty subset of H and let k be a real number with $0 \leq k < 1$. A mapping $U : C \to H$ is called a k-strict pseudo-contraction [6] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + k\|x - Ux - (y - Uy)\|^2$$

for all $x, y \in C$. If U is a k-strict pseudo-contraction and $F(U) \neq \emptyset$, then U is k-demimetric; see [33].

(2) Let H be a Hilbert space and let C be a nonempty subset of H. A mapping $U : C \to H$ is called generalized hybrid [14] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Ux - Uy\|^2 + (1-\alpha)\|x - Uy\|^2 \leq \beta\|Ux - y\|^2 + (1-\beta)\|x - y\|^2 \tag{2.6}$$

for all $x, y \in C$. Such a mapping U is called $(\alpha,\ \beta)$-generalized hybrid. If U is generalized hybrid and $F(U) \neq \emptyset$, then U is 0-demimetric. In fact, setting $x = u \in F(U)$ and $y = x \in C$ in (2.6), we have that

$$\alpha\|u - Ux\|^2 + (1-\alpha)\|u - Ux\|^2 \leq \beta\|u - x\|^2 + (1-\beta)\|u - x\|^2$$

and hence

$$\|Ux - u\|^2 \leq \|x - u\|^2.$$

From $\|Ux - x + x - u\|^2 \leq \|x - u\|^2$, we have that

$$2\langle x - u, x - Ux\rangle \geq \|x - Ux\|^2$$

for all $x \in C$ and $u \in F(U)$. This means that U is 0-demimetric. Notice that the class of generalized hybrid mappings covers several well-known classes of mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is nonspreading [15, 16] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also hybrid [29] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [13].

(3) Let E be a strictly convex, reflexive and smooth Banach space and let D be a nonempty, closed and convex subset of E. Let P_D be the metric projection of E onto D. Then P_D is (-1)-demimetric; see [33].

(4) Let E be a uniformly convex and smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Let $\lambda > 0$. Then the metric resolvent J_λ is (-1)-demimetric; see [33].

The following lemma is crucial in the proofs of our main theorems.

Lemma 7 ([33])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let η be a real number with $\eta \in (-\infty, 1)$. Let U be an η-demimetric mapping of E into itself. Then $F(U)$ is closed and convex.*

For a sequence $\{C_n\}$ of nonempty, closed and convex subsets of a Banach space E, define $\text{s-Li}_n C_n$ and $\text{w-Ls}_n C_n$ as follows: $x \in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in \text{w-Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies

$$C_0 = \underset{n}{\text{s-Li}}\, C_n = \underset{n}{\text{w-Ls}}\, C_n, \tag{2.7}$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [18] and we write $C_0 = \text{M-lim}_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [18]. The following lemma was proved by Tsukada [38].

Lemma 8 ([38])**.** *Let E be a uniformly convex Banach space. Let $\{C_n\}$ be a sequence of nonempty, closed and convex subsets of E. If $C_0 = M\text{-lim}_{n\to\infty} C_n$ exists and nonempty, then for each $x \in E$, $\{P_{C_n}x\}$ converges strongly to $P_{C_0}x$, where P_{C_n} and P_{C_0} are the mertic projections of E onto C_n and C_0, respectively.*

3. Main results

In this section, using the hybrid method by Nakajo and Takahashi [21], we first prove a strong convergence theorem for finding a solution of the split common fixed point problem with families of mappings in Banach spaces. Let E be a Banach space and let $\{U_n\}$ be a sequence of mappings of E into itself such that $\cap_{n=1}^{\infty} F(U_n) \neq \emptyset$. The sequence $\{U_n\}$ is said to satisfy the condition (I) [3] if for any bounded sequence $\{z_n\}$ of F such that $\lim_{n\to\infty} \|z_n - U_n z_n\| = 0$, every weak cluster point of $\{z_n\}$ belongs to $\cap_{n=1}^{\infty} F(U_n) \neq \emptyset$.

Theorem 9. *Let H be a Hilbert space and let F be a smooth, strictly convex and reflexive Banach space. Let J_F be the duality mapping on F and let $\{\eta_n\}$ be a sequence of real numbers with $\eta_n \in (-\infty, 1)$. Let $\{T_n\}$ be a sequence of nonexpansive mappings of H to H satisfying the condition (I) and let $\{U_n\}$ be a sequence of η_n-demimetric mappings of F to F satisfying the condition (I). Let $A : H \to F$ be a* *bounded linear operator such that* $A \neq 0$ *and let* A^* *be the adjoint operator of* A. *Suppose that* $\mathbf{G} := \cap_{n=1}^{\infty} F(T_n) \cap A^{-1} \cap_{n=1}^{\infty} F(U_n) \neq \emptyset$. *Let* $x_1 \in H$ *and let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} z_n = T_n\Big(x_n - \lambda_n A^* J_F(Ax_n - U_n Ax_n)\Big), \\ y_n = \alpha_n x_n + (1-\alpha_n) z_n, \\ C_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ D_n = \{z \in H : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\eta_n\} \subset (-\infty, 1)$, $\{\alpha_n\} \subset [0,1]$ *and* $\{\lambda_n\} \subset (0,\infty)$ *satisfy the conditions:*

$$0 \leq \alpha_n \leq a < 1 \quad \text{and} \quad 0 < b \leq \lambda_n \|A\|^2 \leq c < d \leq 1 - \eta_n$$

for some $a, b, c, d \in \mathbb{R}$. *Then* $\{x_n\}$ *converges strongly to* $z_0 \in \mathbf{G}$, *where* $z_0 = P_{\mathbf{G}} x_1$.

Proof. Since

$$\begin{aligned} \|y_n - z\| \leq \|x_n - z\| &\iff \|y_n - z\|^2 \leq \|x_n - z\|^2 \\ &\iff \|y_n\|^2 - \|x_n\|^2 - 2\langle y_n - x_n, z \rangle \leq 0, \end{aligned}$$

it follows that C_n are closed and convex for all $n \in \mathbb{N}$. It is obvious that D_n are closed and convex. Then $C_n \cap D_n$ are closed and convex for all $n \in \mathbb{N}$. Let us show that $\mathbf{G} \subset C_n$ for all $n \in \mathbb{N}$. Let $z \in \mathbf{G}$. Then $z = T_n z$ and $Az = U_n Az$. Since T_n is nonexpansive, we have that for $z \in \mathbf{G}$,

$$\begin{aligned} \|z_n - z\|^2 &= \|T_n\Big(x_n - \lambda_n A^* J_F(Ax_n - U_n Ax_n)\Big) - T_n z\|^2 \\ &\leq \|x_n - \lambda_n A^* J_F(Ax_n - U_n Ax_n) - z\|^2 \\ &= \|x_n - z - \lambda_n A^* J_F(Ax_n - U_n Ax_n)\|^2 \\ &= \|x_n - z\|^2 - 2\langle x_n - z, \lambda_n A^* J_F(Ax_n - U_n Ax_n)\rangle \\ &\quad + \|\lambda_n A^* J_F(Ax_n - U_n Ax_n)\|^2 \\ &\leq \|x_n - z\|^2 - 2\lambda_n \langle Ax_n - Az, J_F(Ax_n - U_n Ax_n)\rangle \\ &\quad + \lambda_n^2 \|A\|^2 \|J_F(Ax_n - U_n Ax_n)\|^2 \qquad (3.1) \\ &\leq \|x_n - z\|^2 - \lambda_n(1-\eta_n)\|Ax_n - U_n Ax_n\|^2 + \lambda_n^2\|A\|^2\|Ax_n - U_n Ax_n\|^2 \\ &= \|x_n - z\|^2 + \lambda_n(\lambda_n\|A\|^2 - (1-\eta_n))\|Ax_n - U_n Ax_n\|^2 \\ &\leq \|x_n - z\|^2 \end{aligned}$$

and hence

$$\begin{aligned} \|y_n - z\| &= \|\alpha_n x_n + (1-\alpha_n) z_n - z\| \\ &\leq \alpha_n \|x_n - z\| + (1-\alpha_n)\|z_n - z\| \end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n\|x_n - z\| + (1-\alpha_n)\|x_n - z\| \\
&= \|x_n - z\|.
\end{aligned}
$$

Therefore, $\mathbf{G} \subset C_n$ for all $n \in \mathbb{N}$. Let us show that $\mathbf{G} \subset D_n$ for all $n \in \mathbb{N}$. It is obvious that $\mathbf{G} \subset D_1$. Suppose that $\mathbf{G} \subset D_k$ for some $k \in \mathbb{N}$. Then $\mathbf{G} \subset C_k \cap D_k$. From $x_{k+1} = P_{C_k \cap D_k} x_1$, we have that

$$\langle x_{k+1} - z, x_1 - x_{k+1} \rangle \geq 0, \quad \forall z \in C_k \cap D_k$$

and hence

$$\langle x_{k+1} - z, x_1 - x_{k+1} \rangle \geq 0, \quad \forall z \in \mathbf{G}.$$

Then $\mathbf{G} \subset D_{k+1}$. We have by induction that $\mathbf{G} \subset D_n$ for all $n \in \mathbb{N}$. Thus, we have that $\mathbf{G} \subset C_n \cap D_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Since $F(U_n)$ is nonempty, closed and convex from Lemma 7, $\mathbf{G}$ is also nonempty, closed and convex. Thus, there exists $z_0 \in \mathbf{G}$ such that $z_0 = P_{\mathbf{G}} x_1$. From $x_{n+1} = P_{C_n \cap D_n} x_1$, we have that

$$\|x_1 - x_{n+1}\| \leq \|x_1 - y\|, \quad \forall y \in C_n \cap D_n.$$

Since $z_0 \in \mathbf{G} \subset C_n \cap D_n$, we have that

$$\|x_1 - x_{n+1}\| \leq \|x_1 - z_0\|. \tag{3.2}$$

This means that $\{x_n\}$ is bounded.

Next we show that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. From the definition of D_n, we have that $x_n = P_{D_n} x_1$. From $x_{n+1} = P_{C_n \cap D_n} x_1$ we have $x_{n+1} \in D_n$. Thus

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|$$

for all $n \in \mathbb{N}$. This implies that $\{\|x_1 - x_n\|\}$ is bounded and nondecreasing. Then there exists the limit of $\{\|x_1 - x_n\|\}$. From $x_{n+1} \in D_n$ we have that

$$\langle x_n - x_{n+1}, x_1 - x_n \rangle \geq 0.$$

This implies from (2.3) that

$$0 \leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 - \|x_{n+1} - x_n\|^2$$

and hence

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2.$$

Since there exists the limit of $\{\|x_1 - x_n\|\}$, we have that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.3}$$

From $x_{n+1} \in C_n$, we also have that $\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$. Then we get from (3.3) that $\|y_n - x_{n+1}\| \to 0$. Using this, we have that

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{3.4}$$

We have from (3.1) that for any $z \in \mathbf{G}$,

$$
\begin{aligned}
\|y_n - z\|^2 &= \|\alpha_n x_n + (1-\alpha_n) z_n - z\|^2 \\
&\le \alpha_n \|x_n - z\|^2 + (1-\alpha_n) \|z_n - z\|^2 \\
&\le \alpha_n \|x_n - z\|^2 + (1-\alpha_n) \|x_n - z\|^2 \\
&\qquad + (1-\alpha_n)\lambda_n(\lambda_n \|A\|^2 - (1-\eta_n))\|Ax_n - U_n Ax_n\|^2 \\
&= \|x_n - z\|^2 + (1-\alpha_n)\lambda_n(\lambda_n \|A\|^2 - (1-\eta_n))\|Ax_n - U_n A_n\|^2.
\end{aligned}
$$

Thus we have that

$$
\begin{aligned}
(1-\alpha_n)\lambda_n(1-\eta_n - \lambda_n \|A\|^2)\|Ax_n - U_n Ax_n\|^2 &\le \|x_n - z\|^2 - \|y_n - z\|^2 \\
&= (\|x_n - z\| + \|y_n - z\|)(\|x_n - z\| - \|y_n - z\|) \\
&\le (\|x_n - z\| + \|y_n - z\|) \|x_n - y_n\| .
\end{aligned}
$$

From $\|y_n - x_n\| \to 0$, $0 \le \alpha_n \le a < 1$ and $0 < b \le \lambda_n \|A\|^2 \le c < d \le (1-\eta_n)$, we have that

$$
\lim_{n\to\infty} \|Ax_n - U_n Ax_n\|^2 = 0. \tag{3.5}
$$

We also have that $\|y_n - x_n\| = \|\alpha_n x_n + (1-\alpha_n) z_n - x_n\| = (1-\alpha_n)\|z_n - x_n\|$. From $\|y_n - x_n\| \to 0$ and $0 \le \alpha_n \le a < 1$, we have that

$$
\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.6}
$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to w. Since A is bounded and linear, we also have that $\{Ax_{n_i}\}$ converges weakly to Aw. Since $\lim_{n\to\infty} \|Ax_n - U_n Ax_n\| = 0$ and the family $\{U_n\}$ satisfies the condition (I), we have that $Aw \in \cap_{n=1}^{\infty} F(U_n)$ and hence $w \in A^{-1} \cap_{n=1}^{\infty} F(U_n)$. We also have that

$$
\begin{aligned}
\|x_n - T_n x_n\| &= \|x_n - z_n + z_n - T_n x_n\| \\
&= \|x_n - z_n + T_n\big(x_n - \lambda_n A^* J_F(Ax_n - U_n Ax_n)\big) - T_n x_n\| \\
&\le \|x_n - z_n\| + \|x_n - \lambda_n A^* J_F(Ax_n - U_n Ax_n) - x_n\| \\
&= \|x_n - z_n\| + \|\lambda_n A^* J_F(Ax_n - U_n Ax_n)\| \\
&\le \|x_n - z_n\| + \lambda_n \|A\| \|Ax_n - U_n Ax_n\| \to 0.
\end{aligned}
$$

Since $x_{n_i} \rightharpoonup w$ and $\{T_n\}$ satisfies the condition (I), we have $w \in \cap_{n=1}^{\infty} F(T_n)$. This implies that $w \in \mathbf{G}$. From $z_0 = P_{\mathbf{G}} x_1$, $w \in \mathbf{G}$ and (3.2), we have that

$$
\begin{aligned}
\|x_1 - z_0\| \le \|x_1 - w\| &\le \liminf_{i\to\infty} \|x_1 - x_{n_i}\| \\
&\le \limsup_{i\to\infty} \|x_1 - x_{n_i}\| \le \|x_1 - z_0\|.
\end{aligned}
$$

Then we get that

$$\lim_{i\to\infty} \|x_1 - x_{n_i}\| = \|x_1 - w\| = \|x_1 - z_0\|.$$

Since H satisfies the Kadec-Klee property, we have that $x_1 - x_{n_i} \to x_1 - w$ and hence $x_{n_i} \to w = z_0$. Therefore, we have $x_n \to w = z_0$. This completes the proof. $\square$

Next, using the shrinking projection method introduced by Takahashi, Takeuchi and Kubota [34], we prove a strong convergence theorem for finding a solution of the split common fixed point problem with families of mappings in Banach spaces.

Theorem 10. *Let H be a Hilbert space and let F be a smooth, strictly convex and reflexive Banach space. Let J_F be the duality mapping on F and let $\{\eta_n\}$ be a sequence of real numbers with $\eta_n \in (-\infty, 1)$. Let $\{T_n\}$ be a sequence of nonexpansive mappings of H to H satisfying the condition (I) and let $\{U_n\}$ be a sequence of η_n-demimetric mappings of F to F satisfying the condition (I). Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $\mathbf{G} := \cap_{n=1}^{\infty} F(T_n) \cap A^{-1} \cap_{n=1}^{\infty} F(U_n) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. Let $x_1 \in H$ and $C_1 = H$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = T_n\Big(x_n - \lambda_n A^* J_F(Ax_n - U_n Ax_n)\Big), \\ y_n = \alpha_n x_n + (1-\alpha_n) z_n, \\ C_{n+1} = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\eta_n\} \subset (-\infty, 1)$, $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, \infty)$ satisfy the conditions:

$$0 \leq \alpha_n \leq a < 1 \quad \text{and} \quad 0 < b \leq \lambda_n \|A\|^2 \leq c < d \leq 1 - \eta_n$$

for some $a, b, c, d \in \mathbb{R}$. Then the sequence $\{x_n\}$ converges strongly to a point $x_0 \in \mathbf{G}$, where $x_0 = P_{\mathbf{G}} u$.

Proof. We first show that the sequence $\{x_n\}$ is well defined. It is obvious that $\mathbf{G} \subset C_1$. Suppose that $\mathbf{G} \subset C_k$ for some $k \in \mathbb{N}$. To show $\mathbf{G} \subset C_{k+1}$, let us show that $\|y_k - z\| \leq \|x_k - z\|$ for all $z \in \mathbf{G}$. From $0 < b \leq \lambda_k \|A\|^2 \leq c < d \leq (1 - \eta_k)$, as in the proof of Theorem 9, we have that for $z \in \mathbf{G}$,

$$\begin{aligned} \|z_k - z\|^2 &= \|T_k\Big(x_k - \lambda_k A^* J_F(Ax_k - U_k Ax_k)\Big) - T_k z\|^2 \\ &\leq \|x_k - \lambda_k A^* J_F(Ax_k - U_k Ax_k) - z\|^2 \\ &\leq \|x_k - z\|^2 - \lambda_k (1-\eta_k) \|Ax_k - U_k Ax_k\|^2 + \lambda_k^2 \|A\|^2 \|Ax_k - U_k Ax_k\|^2 \qquad (3.7) \\ &= \|x_k - z\|^2 + \lambda_k (\lambda_k \|A\|^2 - (1-\eta_k)) \|Ax_k - U_k Ax_k\|^2 \\ &\leq \|x_k - z\|^2 \end{aligned}$$

and hence

$$\begin{aligned}\|y_k - z\| &= \|\alpha_k x_k + (1-\alpha_k) z_k - z\| \\ &\le \alpha_k \|x_k - z\| + (1-\alpha_k)\|z_k - z\| \\ &\le \alpha_k \|x_k - z\| + (1-\alpha_k)\|x_k - z\| \\ &= \|x_k - z\|.\end{aligned}$$

Then $\mathbf{G} \subset C_{k+1}$. We have by mathematical induction that $\mathbf{G} \subset C_n$ for all $n \in \mathbb{N}$. Moreover, since

$$\begin{aligned}\{z \in H : \|y_n - z\| \le \|x_n - z\|\} &= \{z \in H : \|y_n - z\|^2 \le \|x_n - z\|^2\} \\ &= \{z \in H : \|y_n\|^2 - \|x_n\|^2 \le 2\langle y_n - x_n, z\rangle\},\end{aligned}$$

it is closed and convex. Applying these facts inductively, we obtain that C_n are nonempty, closed, and convex for all $n \in \mathbb{N}$, and hence $\{x_n\}$ is well defined.

Let $C_0 = \bigcap_{n=1}^{\infty} C_n$. Then since $C_0 \supset \mathbf{G} \neq \emptyset$, C_0 is nonempty. Let $w_n = P_{C_n} u$ for every $n \in \mathbb{N}$. Then, by Lemma 8, we have $w_n \to w_0 = P_{C_0} u$. Since a metric projection on H is nonexpansive, it follows that

$$\begin{aligned}\|x_n - w_0\| &\le \|x_n - w_n\| + \|w_n - w_0\| \\ &= \|P_{C_n} u_n - P_{C_n} u\| + \|w_n - w_0\| \\ &\le \|u_n - u\| + \|w_n - w_0\|\end{aligned}$$

and hence $x_n \to w_0$.

Since $w_0 \in C_0 \subset C_{n+1}$, we have $\|y_n - w_0\| \le \|x_n - w_0\|$ for all $n \in \mathbb{N}$. Tending $n \to \infty$, we get that $y_n \to w_0$. Then we have that

$$\|x_n - y_n\| \le \|x_n - w_0\| + \|w_0 - y_n\| \to 0. \tag{3.8}$$

From $y_n - x_n = \alpha_n x_n + (1-\alpha_n) z_n - x_n = (1-\alpha_n)(z_n - x_n)$, we also have that

$$\|y_n - x_n\| = (1-\alpha_n)\|z_n - x_n\| \ge (1 - a_n)\|z_n - x_n\|$$

and hence

$$\|z_n - x_n\| \to 0. \tag{3.9}$$

On the other hand, from (3.7) we know that for $z \in \mathbf{G}$,

$$\|z_n - z\|^2 \le \|x_n - z\|^2 + \lambda_n(\lambda_n \|A\|^2 - (1-\eta_n))\|Ax_n - U_n A x_n\|^2.$$

Then we get that

$$\begin{aligned}\lambda_n(1 - \eta_n - \lambda_n\|A\|^2)\|Ax_n - U_n A x_n\|^2 &\le \|x_n - z\|^2 - \|z_n - z\|^2 \\ &= (\|x_n - z\| - \|z_n - z\|)(\|x_n - z\| + \|z_n - z\|) \\ &\le \|x_n - z_n\|(\|x_n - z\| + \|z_n - z\|).\end{aligned}$$

Since $0 < b \leq \lambda_n \|A\|^2 \leq c < d \leq (1-\eta_n)$ and $\|x_n - z_n\| \to 0$, we have that

$$\lim_{n\to\infty} \|Ax_n - U_n Ax_n\| = 0. \tag{3.10}$$

Since $x_n \to w_0$ and A is bounded and linear, we have that $\{Ax_n\}$ converges strongly to Aw_0 and hence $\{Ax_n\}$ converges weakly to Aw_0. Since a family $\{U_n\}$ satisfies the condition (I) and $\lim_{n\to\infty} \|Ax_n - UAx_n\| = 0$, we have that $Aw \in \cap_{n=1}^{\infty} F(U_n)$ and hence $w \in A^{-1} \cap_{n=1}^{\infty} F(U_n)$. We also have that

$$\begin{aligned}
\|x_n - T_n x_n\| &= \|x_n - z_n + z_n - T_n x_n\| \\
&= \|x_n - z_n + T_n\Big(x_n - \lambda_n A^* J_F(Ax_n - U_n Ax_n)\Big) - T_n x_n\| \\
&\leq \|x_n - z_n\| + \|x_n - \lambda_n A^* J_F(Ax_n - U_n Ax_n) - x_n\| \\
&= \|x_n - z_n\| + \|\lambda_n A^* J_F(Ax_n - U_n Ax_n)\| \to 0.
\end{aligned}$$

Since $x_n \to w_0$ and $\{T_n\}$ satisfies the condition (I), we have $w_0 \in \cap_{n=1}^{\infty} F(T_n)$. This implies that $w_0 \in \mathbf{G}$.

Since $F(U_n)$ is nonempty, closed and convex from Lemma 7, $\mathbf{G}$ is also nonempty, closed and convex. Then there exists $z_0 \in \mathbf{G}$ such that $z_0 = P_{\mathbf{G}} u$. From $x_{n+1} = P_{C_{n+1}} u_{n+1}$, we have that

$$\|u_{n+1} - x_{n+1}\| \leq \|u_{n+1} - y\|$$

for all $y \in C_{n+1}$. Since $z_0 \in \mathbf{G} \subset C_{n+1}$, we have that

$$\|u_{n+1} - x_{n+1}\| \leq \|u_{n+1} - z_0\|. \tag{3.11}$$

From $z_0 = P_{\mathbf{G}} u$, $w_0 \in \mathbf{G}$ and (3.11), we have that

$$\begin{aligned}
\|u - z_0\| \leq \|u - w_0\| &= \lim_{n\to\infty} \|u_{n+1} - x_{n+1}\| \\
&\leq \lim_{n\to\infty} \|u_{n+1} - z_0\| = \|u - z_0\|.
\end{aligned}$$

Then we get that $\|u - w_0\| = \|u - z_0\|$ and hence $z_0 = w_0$. Therefore, we have $x_n \to w_0 = z_0$. This completes the proof. ☐

We do not know whether a Hilbert space H in Theorems 9 and 10 is replaced by a Banach space E or not.

4. Applications

In this section, using Theorems 9 and 10, we get new strong convergence theorems which are connected with the split common fixed point problem with families of mappings in Banach spaces. We know the following result obtained by Marino and Xu [17]; see also [36].

Lemma 11 ([17]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $0 \leq k < 1$ and let $U : C \to H$ be a k-strict pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

We also know the following result from Kocourek, Takahashi and Yao [14].

Lemma 12 ([14]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $U : C \to H$ be generalized hybrid. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

Theorem 13. *Let H_1 and H_2 be Hilbert spaces. Let k be a real number with $k \in [0, 1)$. Let $T : H_1 \to H_1$ be a nonexpansive mapping and let $U : H_2 \to H_2$ be a k-strict pseudo-contraction such that $F(U) \neq \emptyset$. Define $T_n = \frac{1}{n}\sum_{i=0}^{n-1} T^i$ and $U_n = \gamma_n I + (1 - \gamma_n)U$ for all $n \in \mathbb{N}$. Assume that $0 \leq \gamma_n < 1$ and $\sup_{n\in\mathbb{N}} \gamma_n < 1$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $F(T) \cap A^{-1}F(U) \neq \emptyset$. Let $x_1 \in H_1$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = T_n\Big(x_n - \lambda_n A^*(Ax_n - U_n Ax_n)\Big), \\ y_n = \alpha_n x_n + (1 - \alpha_n)z_n, \\ C_n = \{z \in H_1 : \|y_n - z\| \leq \|x_n - z\|\}, \\ D_n = \{z \in H_1 : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, \infty)$ satisfy the conditions:

$$0 \leq \alpha_n \leq a < 1 \quad \text{and} \quad 0 < b \leq \lambda_n \|A\|^2 \leq c < (1 - k)$$

for some $a, b, c \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $z_0 \in F(T) \cap A^{-1}F(U)$, where $z_0 = P_{F(T)\cap A^{-1}F(U)} x_1$.

Proof. Since T is nonexpansive, T_n is nonexpansive and $\cap_{n=1}^{\infty} F(T_n) = F(T)$. We also have from [20, Lemma 3.10] that $\{T_n\}$ satisfies the condition (I). On the other hand, since U is a k-strict pseudo-contraction of H_2 into itself such that $F(U) \neq \emptyset$, from (1) in Examples, U is k-demimetric. We also have that for $p \in F(U_n) = F(U)$ and $x \in H_2$,

$$\begin{aligned} \langle x - p, x - U_n x\rangle &= \langle x - p, x - (\gamma_n x + (1 - \gamma_n)Ux)\rangle \\ &= (1 - \gamma_n)\langle x - p, x - Ux\rangle \\ &\geq (1 - \gamma_n)\frac{1 - k}{2}\|x - Ux\|^2 \\ &= (1 - \gamma_n)^2 \frac{1 - k}{2(1 - \gamma_n)}\|x - Ux\|^2 \end{aligned}$$

$$
\begin{aligned}
&= \frac{1-k}{2(1-\gamma_n)}\|x - (\gamma_n x + (1-\gamma_n)Ux)\|^2 \\
&\geq \frac{1-k}{2}\|x - (\gamma_n x + (1-\gamma_n)Ux)\|^2 \\
&= \frac{1-k}{2}\|x - U_n x\|^2
\end{aligned}
$$

and hence $\{U_n\}$ is a family of k-demimetric mappings of H_2 into H_2 such that $F(U) = \cap_{n=1}^{\infty} F(U_n)$. Furthermore, let $\{u_n\}$ be a bounded sequence of H_2 such that $u_n - U_n u_n \to 0$. Then we have

$$(1-\gamma_n)(u_n - Uu_n) = u_n - U_n u_n \to 0$$

and hence $u_n - Uu_n \to 0$ from $\sup_{n\in\mathbb{N}} \gamma_n < 1$. It follows from Lemma 11 that every weak cluster point of $\{u_n\}$ belongs to $F(U) = \cap_{n=1}^{\infty} F(U_n)$. This means that the family $\{U_n\}$ satisfies the condition (I). Therefore, we have the desired result from Theorem 9. □

Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A family $\mathbf{S} = \{T(t) : t \in [0,\infty)\}$ of mappings of C into itself satisfying the following conditions is said to be a one-parameter nonexpansive semigroup on C:

(1) For each $t \in [0,\infty)$, $T(t)$ is nonexpansive;
(2) $T(0) = I$;
(3) $T(t+s) = T(t)T(s)$ for every $t, s \in [0,\infty)$;
(4) for each $x \in C$, $t \mapsto T(t)x$ is continuous.

Theorem 14. *Let H_1 and H_2 be Hilbert spaces. Let $\mathbf{S} = \{T(t) : t \in [0,\infty)\}$ be a one-parameter nonexpansive semigroup on H_1 with the common fixed point set $F(\mathbf{S}) = \cap_{t\in[0,\infty)} F(T(t)) \neq \emptyset$ and let $U : H_2 \to H_2$ be a generalized hybrid mapping. Define $T_n x = \frac{1}{t_n}\int_0^{t_n} T(s)x ds$ for all $x \in H_1$ and $n \in \mathbb{N}$ with $t_n \to \infty$. Define $U_n x = \gamma_n x + (1-\gamma_n)Ux$ for all $x \in H_2$ and $n \in \mathbb{N}$ such that $0 \leq \gamma_n < 1$ and $\sup_{n\in\mathbb{N}} \gamma_n < 1$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $F(\mathbf{S}) \cap A^{-1}F(U) \neq \emptyset$. Let $x_1 \in H_1$ and let $\{x_n\}$ be a sequence generated by*

$$
\begin{cases}
z_n = T_n\Big(x_n - \lambda_n A^*(Ax_n - U_n Ax_n)\Big), \\
y_n = \alpha_n x_n + (1-\alpha_n) z_n, \\
C_n = \{z \in H_1 : \|y_n - z\| \leq \|x_n - z\|\}, \\
D_n = \{z \in H_1 : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\
x_{n+1} = P_{C_n \cap D_n} x_1, \quad \forall n \in \mathbb{N},
\end{cases}
$$

where $\{\alpha_n\} \subset [0,1]$ *and* $\{\lambda_n\} \subset (0,\infty)$ *satisfy the conditions:*

$$0 \leq \alpha_n \leq a < 1 \quad \textit{and} \quad 0 < b \leq \lambda_n \|A\|^2 \leq c < 1$$

for some $a, b, c \in \mathbb{R}$. *Then* $\{x_n\}$ *converges strongly to a point* $z_0 \in F(\mathbf{S}) \cap A^{-1}F(U)$, *where* $z_0 = P_{F(\mathbf{S})\cap A^{-1}F(U)}x_1$.

Proof. Since T_n is a nonexpansive mapping of H_1 into itself, from (1) in Examples, T_n is 0-demimetric. We also know from [26] that $\cap_{n=1}^{\infty} F(T_n) = F(\mathbf{S})$. Furthermore, let $\{u_n\}$ be a bounded sequence of H_1 such that $u_n - T_n u_n \to 0$. Then we have from [20] that $u_n - T(s)u_n \to 0$ for all $s \in [0,\infty)$. Sinve $T(s)$ is nonexpansive, every weak cluster point u_0 of $\{u_n\}$ belongs to $F(T(s))$; see [28]. Then, we have $u_0 \in \cap_{n=1}^{\infty} F(T_n) = F(\mathbf{S})$. This means that the family $\{T_n\}$ satisfies the condition (I). On the other hand, since U is a generalized hybrid mapping of H_2 into itself such that $F(U) \neq \emptyset$, from (2) in Examples, U is 0-demimetric. As in the proof of Theorem 13, we have that for $x \in H_2$ and $p \in F(U_n) = F(U)$,

$$\begin{aligned}
\langle x-p, x-U_n x\rangle &= \langle x-p, x-(\gamma_n x + (1-\gamma_n)Ux)\rangle \\
&= (1-\gamma_n)\langle x-p, x-Ux\rangle \\
&\geq (1-\gamma_n)\frac{1}{2}\|x-Ux\|^2 \\
&= \frac{1}{2(1-\gamma_n)}\|x-(\gamma_n x + (1-\gamma_n)Ux)\|^2 \\
&\geq \frac{1}{2}\|x-U_n x\|^2
\end{aligned}$$

and hence $\{U_n\}$ is a family of 0-demimetric mappings of H_2 into H_2 such that $F(U) = \cap_{n=1}^{\infty} F(U_n)$. Furthermore, let $\{u_n\}$ be a bounded sequence of H_2 such that $u_n - U_n u_n \to 0$. Then we have

$$(1-\gamma_n)(u_n - Uu_n) = u_n - U_n u_n \to 0$$

and hence $u_n - Uu_n \to 0$ from $\sup_{n\in\mathbb{N}} \gamma_n < 1$. It follows from Lemma 12 that every weak cluster point of $\{u_n\}$ belongs to $F(U) = \cap_{n=1}^{\infty} F(U_n)$. This means that the family $\{U_n\}$ satisfies the condition (I). Therefore, we have the desired result from Theorem 9. □

Using Theorem 9, we have the following strong convergence theorem for the split common null point problem in Banach spaces; see also Takahashi and Yao [37].

Theorem 15. *Let* H *be a Hilbert space and let* F *be a uniformly convex and smooth Banach space. Let* J_F *be the duality mapping on* F. *Let* G *and* B *be maximal monotone operators of* H *and* F, *respectively. Let* J_r *and* Q_s *be the metric resolvents of* G *for* $r > 0$ *and* B *for* $s > 0$, *respectively. Let* $A : H \to F$ *be a bounded linear* *operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $G^{-1}0 \cap A^{-1}(B^{-1}0) \neq \emptyset$. Let $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_{r_n}\big(x_n - \lambda_n A^* J_F(Ax_n - Q_{s_n}Ax_n)\big), \\ C_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ D_n = \{z \in H : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < a \leq \lambda_n \|A\|^2 \leq b < 1$, $r_n \geq c > 0$ and $s_n \geq d > 0$ for some $a, b, c, d \in \mathbb{R}$. Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in G^{-1}0 \cap A^{-1}(B^{-1}0)$, where $z_0 = P_{G^{-1}0 \cap A^{-1}(B^{-1}0)} x_1$.

Proof. Since Q_{s_n} is the metric resolvent of B for $s_n > 0$, from (4) in Examples, Q_{s_n} is (-1)-demimetric. We also have that if $\{u_n\}$ is a bounded sequence in F such that $u_n - Q_{s_n}u_n \to 0$, then every weak cluster point of $\{u_n\}$ belongs to $B^{-1}0 = \cap_{n=1}^{\infty} F(Q_{s_n})$. In fact, suppose that $\{u_{n_i}\}$ is a subsequence of $\{u_n\}$ such that $u_{n_i} \rightharpoonup p$. Since Q_{s_n} is the metric resolvent of B, we have that

$$J_F(u_n - Q_{s_n}u_n)/s_n \in BQ_{s_n}u_n$$

for all $n \in \mathbb{N}$; see [4, 27]. From the monotonicity of B, we have

$$0 \leq \Big\langle u - Q_{s_{n_i}}u_{n_i}, v^* - \frac{J_F(u_{n_i} - Q_{s_{n_i}}u_{n_i})}{s_{n_i}} \Big\rangle$$

for all $(u, v^*) \in B$ and $i \in \mathbb{N}$. Taking $i \to \infty$, we get that $\langle u - p, v^* \rangle \geq 0$ for all $(u, v^*) \in B$. Since B is a maximal monotone operator, we have

$$p \in B^{-1}0 = \cap_{n=1}^{\infty} F(Q_{s_n}).$$

This means that the family $\{Q_{s_n}\}$ satisfies the condition (I). On the other hand, since J_{r_n} is the metric resolvent (the resolvent) of G on a Hilbert space H, it is nonexpansive. Furthermore, as in the proof of $\{Q_{s_n}\}$, $\{J_{r_n}\}$ satisfies the condition (I). Therefore, we have the desired result from Theorem 9. $\square$

Similarly, using Theorem 10, we have the following results.

Theorem 16. *Let H_1 and H_2 be Hilbert spaces. Let k be a real number with $k \in [0, 1)$. Let $T : H_1 \to H_1$ be a nonexpansive mapping and let $U : H_2 \to H_2$ be a k-strict pseudo-contraction such that $F(U) \neq \emptyset$. Define $T_n = \frac{1}{n}\sum_{i=0}^{n-1} T^i$ and $U_n = \gamma_n I + (1 - \gamma_n)U$ for all $n \in \mathbb{N}$. Assume that $0 \leq \gamma_n < 1$ and $\sup_{n \in \mathbb{N}} \gamma_n < 1$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $F(T) \cap A^{-1}F(U) \neq \emptyset$. Let $\{u_n\}$ be a sequence* *in H such that $u_n \to u$. For $x_1 \in H_1$ and $C_1 = H_1$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = T_n\Big(x_n - \lambda_n A^*(Ax_n - U_n Ax_n)\Big), \\ y_n = \alpha_n x_n + (1-\alpha_n) z_n, \\ C_{n+1} = \{z \in H_1 : \|y_n - z\| \le \|x_n - z\|\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$ satisfy the conditions:

$$0 \le \alpha_n \le a < 1 \quad \text{and} \quad 0 < b \le \lambda_n \|A\|^2 \le c < (1-k)$$

for some $a, b, c \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $z_0 \in F(T) \cap A^{-1}F(U)$, where $z_0 = P_{F(T)\cap A^{-1}F(U)} x_1$.

Theorem 17. *Let H_1 and H_2 be Hilbert spaces. Let $\mathbf{S} = \{T(t) : t \in [0,\infty)\}$ be a one-parameter nonexpansive semigroup on H_1 with the common fixed point set $F(\mathbf{S}) = \cap_{t\in[0,\infty)} F(T(t)) \neq \emptyset$ and let $U : H_2 \to H_2$ be a generalized hybrid mapping Define $T_n x = \frac{1}{t_n}\int_0^{t_n} T(s)x ds$ for all $x \in H_1$ and $n \in \mathbb{N}$ with $t_n \to \infty$. Define $U_n x = \gamma_n x + (1-\gamma_n)Ux$ for all $x \in H_2$ and $n \in \mathbb{N}$ such that $0 \le \gamma_n < 1$ and $\sup_{n\in\mathbb{N}} \gamma_n < 1$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $F(\mathbf{S}) \cap A^{-1}F(U) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H_1 such that $u_n \to u$. For $x_1 \in H_1$ and $C_1 = H_1$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = T_n\Big(x_n - \lambda_n A^*(Ax_n - U_n Ax_n)\Big), \\ y_n = \alpha_n x_n + (1-\alpha_n) z_n, \\ C_{n+1} = \{z \in H_1 : \|y_n - z\| \le \|x_n - z\|\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$ satisfy the conditions:

$$0 \le \alpha_n \le a < 1 \quad \text{and} \quad 0 < b \le \lambda_n \|A\|^2 \le c < 1$$

for some $a, b, c \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in F(\mathbf{S}) \cap A^{-1}F(U)$, where $z_0 = P_{F(\mathbf{S})\cap A^{-1}F(U)} x_1$.

Using Theorem 10, we also have the following theorem for the split common null point problem in Banach spaces; see also Hojo and Takahashi [12].

Theorem 18. *Let H be a Hilbert space and let F be a uniformly convex and smooth Banach space. Let J_F be the duality mapping on F. Let G and B be maximal monotone operators of H and F, respectively. Let J_r and Q_s be the metric resolvents of G for $r > 0$ and B for $s > 0$, respectively. Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that*

$G^{-1}0 \cap A^{-1}(B^{-1}0) \neq \emptyset$. *Let* $\{u_n\}$ *be a sequence in* H *such that* $u_n \to u$. *For* $x_1 \in H$ *and* $C_1 = H$, *let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} z_n = J_{r_n}\big(x_n - \lambda_n A^* J_F(Ax_n - Q_{s_n}Ax_n)\big), \\ y_n = \alpha_n x_n + (1-\alpha_n)z_n, \\ C_{n+1} = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < a \leq \lambda_n \|A\|^2 \leq b < 1$, $r_n \geq c > 0$ *and* $s_n \geq d > 0$ *for some* $a, b, c, d \in \mathbb{R}$. *Then the sequence* $\{x_n\}$ *converges strongly to a point* $z_0 \in F(G) \cap A^{-1}(B^{-1}0)$, *where* $z_0 = P_{F(G)\cap A^{-1}(B^{-1}0)} x_1$.

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