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SOME REMARKS ON OPERATOR EQUATION $C_\varphi = C_\psi X$

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ABSTRACT. We discuss linear equations whose coefficients are bounded composition operators on the Hardy space over the unit disk. Some connections between those equations, Pick interpolation and de Branges-Rovnyak spaces are studied in detail.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$, and let H^2 be the Hardy space over $\mathbb{D}$. $\mathcal{S}$ will denote the set of all holomorphic functions which map $\mathbb{D}$ into itself. For every φ in $\mathcal{S}$, a linear operator C_φ is defined as $C_\varphi f = f \circ \varphi$ for any f in H^2. It is well known that the Littlewood subordination theorem implies that C_φ is bounded. It should be mentioned that Jury gave another proof of boundedness of composition operators with symbols in $\mathcal{S}$ as an application of de Branges-Rovnyak space theory in Jury [1]. The purpose of this paper is to study the following operator linear equation:

$$C_\varphi = C_\psi X, \tag{1.1}$$

where φ and ψ are fixed in $\mathcal{S}$. In this paper, the equation (1.1) is said to have a solution if there exists a linear operator A such that $\mathbb{C}[z]$ is contained in $\operatorname{dom} A$, the domain of A, and $C_\varphi f = C_\psi A f$ for any f in $\operatorname{dom} A$.

This paper is organized as follows. Section 2 is the preliminaries. In Section 3, we study solutions in the set of bounded analytic functions H^∞. A necessary and sufficient condition that the equation $C_\varphi = C_\psi X$ is solvable in H^∞ is given. In Section 4, we will see a certain connection between that condition and the de

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Branges-Rovnyak theory. In Section 5, we deal with a similar problem in the Drury-Arveson space.

2. Preliminaries

We shall begin with some trivial cases. If φ is a constant function valued at a point a in $\mathbb{D}$, then C_φ is the point evaluation P_a at a. It is not difficult to see the following:

(i) Suppose that φ is not constant and $\psi \equiv b \in \mathbb{D}$. Then the equation $C_\varphi = C_\psi X$ has no solution.

(ii) Suppose that $\varphi(z) \equiv a \in \mathbb{D}$ and ψ is not constant. Then the equation $C_\varphi = C_\psi X$ has the unique solution $X = P_a$.

(iii) Suppose that $\varphi(z) \equiv a \in \mathbb{D}$ and $\psi(z) \equiv b \in \mathbb{D}$. Then the equation $C_\varphi = C_\psi X$ has infinitely many solutions.

(iv) If C_ψ is invertible, then $X = C_{\varphi \circ \psi^{-1}}$.

(v) If C_φ is invertible, then C_ψ is also invertible and $X = C_{\varphi \circ \psi^{-1}}$.

To avoid these cases, in the rest of this paper we assume that φ and ψ are not either constant or automorphisms of $\mathbb{D}$.

Proposition 2.1. *Suppose that the equation $C_\varphi = C_\psi X$ has a solution. Then there exists a function u in $\bigcap_{p \geq 1} H^p$ such that $\varphi = u \circ \psi$, where H^p denotes the Hardy space for $1 \leq p < \infty$.*

Proof. Let A be a solution of $C_\varphi = C_\psi X$. We set $u_n = Az^n$ and $u = u_1$. Trivially, it follows that $\varphi = u \circ \psi$. Moreover, we have that

$$u^n \circ \psi = (u \circ \psi)^n = \varphi^n = C_\varphi z^n = C_\psi A z^n = C_\psi u_n = u_n \circ \psi.$$

By the unicity theorem, we have that $u^n = u_n$. Hence u belongs to H^{2n} for every $n \geq 0$. This concludes the proof. $\square$

Remark 2.1. The conclusion of Proposition 2.1 implies that $\varphi(\lambda) = \varphi(\mu)$ if $\psi(\lambda) = \psi(\mu)$. Hence, it is easy to find pairs of functions φ and ψ in $\mathcal{S}$ such that $C_\varphi = C_\psi X$ has no solution.

Let k_λ be the reproducing kernel of H^2 for $\lambda \in \mathbb{D}$. Then it is well known that $C_\varphi^* k_\lambda = k_{\varphi(\lambda)}$. Let T_u be the Toeplitz operator for $u \in H^\infty$. Then we also have that $T_u^* k_\lambda = \overline{u(\lambda)} k_\lambda$.

3. Solutions in H^∞

Let $0 < r \leq 1$. We set

$$Q_r(z, \lambda) = \frac{r^2 - \overline{\varphi(\lambda)}\varphi(z)}{1 - \overline{\psi(\lambda)}\psi(z)}.$$

In particular, Q_1 will be abbreviated as Q.

Lemma 3.1. *Suppose that equation $C_\varphi = C_\psi X$ has a solution A. If $u = Az$ belongs to H^∞ and $\|u\|_\infty \le r$ then Q_r is positive semi-definite.*

Proof. Let $\{\lambda_1, \dots, \lambda_n\}$ be a set of distinct n points in $\mathbb{D}$. Then we have that

$$T_u^* k_{\psi(\lambda)} = \overline{\varphi(\lambda)} k_{\psi(\lambda)} \quad (\lambda \in \{\lambda_1, \dots, \lambda_n\})$$

by Proposition 2.1. Since $\|u\|_\infty \le r$ implies that $r^2 I_{H^2} - T_u T_u^* \ge 0$, we have that

$$0 \le \langle (r^2 I - T_u T_u^*) \sum_{j=1}^n c_j k_{\psi(\lambda_j)}, \sum_{k=1}^n c_k k_{\psi(\lambda_k)} \rangle_{H^2} = \sum_{j,k=1}^n \frac{r^2 - \overline{\varphi(\lambda_j)}\varphi(\lambda_k)}{1 - \overline{\psi(\lambda_j)}\psi(\lambda_k)} c_j \overline{c_k}$$

for any $c_1, \dots, c_n \in \mathbb{C}$. This concludes the proof. □

Lemma 3.2. *If Q_r is positive semi-definite then $\psi(\lambda) = \psi(\mu)$ implies that $\varphi(\lambda) = \varphi(\mu)$.*

Proof. Suppose that $\psi(\lambda) = \psi(\mu)$. Then we have that

$$\det \begin{pmatrix} \frac{r^2 - \overline{\varphi(\lambda)}\varphi(\lambda)}{1 - \overline{\psi(\lambda)}\psi(\lambda)} & \frac{r^2 - \overline{\varphi(\mu)}\varphi(\lambda)}{1 - \overline{\psi(\mu)}\psi(\lambda)} \\ \frac{r^2 - \overline{\varphi(\lambda)}\varphi(\mu)}{1 - \overline{\psi(\lambda)}\psi(\mu)} & \frac{r^2 - \overline{\varphi(\mu)}\varphi(\mu)}{1 - \overline{\psi(\mu)}\psi(\mu)} \end{pmatrix} = -r^2 \frac{|\varphi(\lambda) - \varphi(\mu)|^2}{(1 - |\psi(\lambda)|^2)^2}.$$

Since Q_r is positive semi-definite, we have that $\varphi(\lambda) = \varphi(\mu)$. □

Theorem 3.1. *Let φ and ψ be functions in $\mathcal{S}$. Then Q_r is positive semi-definite if and only if there exists a function u in H^∞ such that $\|u\|_\infty \le r$ and $C_\varphi = C_\psi C_u$.*

Proof. The if part is trivial by Lemma 3.1. We shall show the only if part. The following is a standard argument in the theory of Pick interpolation. We define a densely defined linear operator T^* as follows:

$$T^* k_{\psi(\lambda)} = \overline{\varphi(\lambda)} k_{\psi(\lambda)} \quad (\lambda \in \mathbb{D}).$$

Note that T^* is well defined by Lemma 3.2. By the assumption, we have

$$0 \le \sum_{j,k=1}^n \frac{r^2 - \overline{\varphi(\lambda_j)}\varphi(\lambda_k)}{1 - \overline{\psi(\lambda_j)}\psi(\lambda_k)} c_j \overline{c_k} = r^2 \| \sum_{j=1}^n c_j k_{\psi(\lambda_j)} \|^2 - \|T^* \sum_{j=1}^n c_j k_{\psi(\lambda_j)} \|^2.$$

Therefore T^* can be extended to a bounded linear operator and $\|T^*\| \le r$. Furthermore, it is easy to see that T^* commutes with T_z^*. Hence there exists u in H^∞ such that $T^* = T_u^*$ and $\|u\|_\infty \le r$. It follows that

$$\overline{\varphi(\lambda)} k_{\psi(\lambda)} = T_u^* k_{\psi(\lambda)} = \overline{u(\psi(\lambda))} k_{\psi(\lambda)} \quad (\lambda \in \mathbb{D}),$$

which implies that $\varphi = u \circ \psi$. This concludes the proof. □

Corollary 3.1. *Let φ and ψ be functions in $\mathcal{S}$. Then Q is positive semi-definite if and only if there exists u in $\mathcal{S}$ such that $C_\varphi = C_\psi C_u$.*

Remark 3.1. In the proof of Theorem 3.1, we have seen that the infimum of r's such that Q_r is positive semi-definite is equal to the H^∞-norm of u. Further, the conclusion of Lemma 3.2 valid under the following slightly mild condition: for any λ and μ in $\mathbb{D}$, there exists a positive number $r(\lambda,\mu)$ such that $Q_{r(\lambda,\mu)}$ is positive semi-definite. Then the correspondence $u : \psi(\lambda) \mapsto \varphi(\lambda)$ defines a holomorphic function on $\Omega = \psi(\mathbb{D} \setminus \{\lambda \in \mathbb{D} : \psi'(\lambda) = 0\})$. Indeed, setting $z = \psi(\lambda)$, for sufficiently small h, we can choose λ' as $z + h = \psi(\lambda')$ and

$$\frac{u(z+h)-u(z)}{h} = \frac{\varphi(\lambda')-\varphi(\lambda)}{\psi(\lambda')-\psi(\lambda)} = \frac{\frac{\varphi(\lambda')-\varphi(\lambda)}{\lambda'-\lambda}}{\frac{\psi(\lambda')-\psi(\lambda)}{\lambda'-\lambda}} \to \frac{\varphi'(\lambda)}{\psi'(\lambda)} \quad (h \to 0).$$

Therefore, our problem is closely related to finding the holomorphic extension of u from Ω to $\mathbb{D}$.

If Q is positive semi-definite, then a reproducing kernel Hilbert space corresponds to Q, which will be denoted by $\mathcal{H}_Q$. In the next section, we study the structure of $\mathcal{H}_Q$.

4. Structure of $\mathcal{H}_Q$

Let $\mathcal{H}(\varphi)$ denote the de Branges-Rovnyak complement induced by the Toeplitz operator T_φ for φ in $\mathcal{S}$, that is, $\mathcal{H}(\varphi)$ is equal to the range of $(I - T_\varphi T_\varphi^*)^{1/2}$ as vector subspaces in H^2 and equipped with the range norm (see Sarason [3] for details). Then $\mathcal{H}(\varphi)$ is a reproducing kernel Hilbert space and its kernel is

$$K^\varphi(z,\lambda) = k_\lambda^\varphi(z) = \frac{1-\overline{\varphi(\lambda)}\varphi(z)}{1-\overline{\lambda}z}.$$

By Corollary 3.1, the equation $C_\varphi = C_\psi X$ is solvable in the set of bounded composition operators if and only if the two variable function $Q(z,\lambda) = k_\lambda^\varphi(z)/k_\lambda^\psi(z)$ is positive semi-definite. In this section, we will study the reproducing kernel Hilbert space induced by the kernel function Q. In the following argument, we assume that $Q(z,\lambda) = k_\lambda^\varphi(z)/k_\lambda^\psi(z)$ is positive semi-definite. We denote $\lambda \sim_\psi \mu$ if $\psi(\lambda) = \psi(\mu)$. Then $\sim_\psi$ is an equivalence relation on $\mathbb{D}$, and we set $\Omega = \mathbb{D}/\sim_\psi$. An equivalence class in Ω will be denoted by $[\lambda]$ for λ in $\mathbb{D}$. First, it is trivial that

$$H^\psi([z],[\lambda]) = \frac{1}{1-\overline{\psi(\lambda)}\psi(z)}$$

is a positive semi-definite function on $\Omega \times \Omega$. Hence there exists a Hilbert space $\mathcal{H}^\psi$ and functions $h_{[\lambda]}^\psi$ on Ω such that $\{h_{[\lambda]}^\psi\}_{[\lambda]\in\Omega}$ is a dense subset of $\mathcal{H}^\psi$ and H^ψ can be represented as follows:

$$H^\psi([z],[\lambda]) = \langle h_{[\lambda]}^\psi, h_{[z]}^\psi \rangle_{\mathcal{H}^\psi}.$$

Theorem 4.1. *Suppose that Q is positive semi-definite. Then $T^* : h^\psi_{[\lambda]} \mapsto \overline{\varphi(\lambda)} h^\psi_{[\lambda]}$ ($[\lambda] \in \Omega$) defines a bounded linear operator acting on $\mathcal{H}^\psi$, and the de Branges-Rovnyak complement in $\mathcal{H}^\psi$ induced by T is the reproducing kernel Hilbert space with the kernel function Q.*

Proof. We assume that Q is positive semi-definite. Then, by Lemma 3.2, $T^* : h^\psi_{[\lambda]} \mapsto \overline{\varphi(\lambda)} h^\psi_{[\lambda]}$ is well defined as a densely defined linear operator. Further, we have that

$$0 \le \sum_{j,k=1}^{n} \frac{1 - \overline{\varphi(\lambda_j)}\varphi(\lambda_k)}{1 - \overline{\psi(\lambda_j)}\psi(\lambda_k)} c_j \overline{c_k} = \| \sum_{j=1}^{n} c_j h^\psi_{[\lambda_j]} \|^2_{\mathcal{H}^\psi} - \| T^* \sum_{j=1}^{n} c_j h^\psi_{[\lambda_j]} \|^2_{\mathcal{H}^\psi}.$$

Therefore T^* can be extended to a bounded linear operator and $\|T^*\| \le 1$. Let $\mathcal{M}(T)$ be the de Branges-Rovnyak space induced by T, and $\mathcal{H}(T)$ be the de Branges-Rovnyak complement of $\mathcal{M}(T)$ in $\mathcal{H}^\psi$, that is, $\mathcal{H}(T)$ is equal to the range of $(I - TT^*)^{1/2}$ as vector spaces and is equipped with the range norm

$$\|(I - TT^*)^{1/2} f\|_{\mathcal{H}(T)} = \|Pf\|_{\mathcal{H}^\psi},$$

where P is the orthogonal projection onto the orthogonal complement of $\ker(I - TT^*)^{1/2}$. Then we have that

$$\langle v, (I - TT^*) h^\psi_{[\lambda]} \rangle_{\mathcal{H}(T)} = \langle v, h^\psi_{[\lambda]} \rangle_{\mathcal{H}^\psi} = v([\lambda]) \quad (v \in \mathcal{H}(T)).$$

Hence $(I - TT^*) h^\psi_{[\lambda]}$ is the reproducing kernel of $\mathcal{H}(T)$ at $[\lambda]$. Furthermore, we have that

$$\begin{aligned}
\langle (I - TT^*) h^\psi_{[\lambda]}, (I - TT^*) h^\psi_{[z]} \rangle_{\mathcal{H}(T)} &= \langle (I - TT^*) h^\psi_{[\lambda]}, h^\psi_{[z]} \rangle_{\mathcal{H}^\psi} \\
&= \langle h^\psi_{[\lambda]}, h^\psi_{[z]} \rangle_{\mathcal{H}^\psi} - \langle T^* h^\psi_{[\lambda]}, T^* h^\psi_{[z]} \rangle_{\mathcal{H}^\psi} \\
&= \langle h^\psi_{[\lambda]}, h^\psi_{[z]} \rangle_{\mathcal{H}^\psi} - \langle \overline{\varphi(\lambda)} h^\psi_{[\lambda]}, \overline{\varphi(z)} h^\psi_{[z]} \rangle_{\mathcal{H}^\psi} \\
&= \frac{1 - \overline{\varphi(\lambda)}\varphi(z)}{1 - \overline{\psi(\lambda)}\psi(z)} \\
&= Q(z, \lambda).
\end{aligned}$$

This concludes the proof. □

Remark 4.1. $\mathcal{H}_Q$ can also be described by compositions and pull-backs of reproducing kernel Hilbert spaces. Details of these two operations are given in Paulsen [2]. The composition of $\mathcal{H}(u)$ by ψ will be denoted by $\mathcal{H}(u) \circ \psi$, where u is the unique function in $\mathcal{S}$ such that $C_\varphi = C_\psi C_u$. It is easy to see that its kernel function is $K^u(\psi(z), \psi(\lambda)) = k^\varphi_\lambda(z)/k^\psi_\lambda(z) = Q(z, \lambda)$. Hence $\mathcal{H}(u) \circ \psi$ is isomorphic to $\mathcal{H}_Q$. Next, let K be the kernel function of the tensor product reproducing kernel Hilbert space $\mathcal{H}(\psi) \otimes \mathcal{H}_Q$, and let Δ denote the diagonal map from $\mathbb{D}^2$ into $\mathbb{D}^4$ defined as $\Delta(z, \lambda) = ((z, \lambda), (z, \lambda))$. Then we have that

$$K \circ \Delta(z, \lambda) = K((z, \lambda), (z, \lambda)) = K^{\psi}(z, \lambda) Q(z, \lambda) = K^{\varphi}(z, \lambda),$$

that is, $\mathcal{H}(\varphi)$ is the pull-back of $\mathcal{H}(\psi) \otimes \mathcal{H}_Q$ along the diagonal map Δ.

5. A multivariable case

Let $\mathbb{B}_d$ be the unit ball in $\mathbb{C}^d$ and let H_d^2 be the Drury-Arveson space. H_d^2 is the reproducing kernel Hilbert space consisting of holomorphic functions on $\mathbb{B}_d$ with the following reproducing kernel:

$$k_{\lambda}(z) = \frac{1}{1 - \langle z, \lambda \rangle_{\mathbb{C}^d}} \quad (z, \lambda \in \mathbb{B}_d).$$

$\mathrm{Hol}(\mathbb{B}_d)$ will denote the set of all holomorphic maps acting on $\mathbb{B}_d$, and $\mathcal{B}(H_d^2)$ will denote the set of all bounded linear operators acting on H_d^2. We define two subsets of $\mathrm{Hol}(\mathbb{B}_d)$ as follows:

$$\mathcal{S}_d = \{\varphi \in \mathrm{Hol}(\mathbb{B}_d) : \frac{1 - \langle \varphi(z), \varphi(\lambda) \rangle}{1 - \langle z, \lambda \rangle} \text{ is positive semi-definite}\},$$
$$\mathcal{C}_d = \{\varphi \in \mathrm{Hol}(\mathbb{B}_d) : C_{\varphi} \in \mathcal{B}(H_d^2)\}.$$

$\mathcal{S}_d$ is called the Schur-Agler class for H_d^2. In the case $d = 1$, trivially, $\mathcal{S}_1$ coincides with $\mathcal{C}_1$. For general $d \geq 2$, Jury proved that $\mathcal{S}_d$ is contained in $\mathcal{C}_d$ in Theorem 5 of [1]. In the following argument, we assume that φ and ψ belong to $\mathcal{C}_d$, and we set $\varphi = (\varphi_1, \ldots, \varphi_d)$.

Proposition 5.1. *Let φ and ψ be in $\mathcal{C}_d$. If ψ is an open mapping and the equation $C_{\varphi} = C_{\psi} X$ has a solution in $\mathcal{B}(H_d^2)$, then there exists a function $u_j \in H_d^2$ such that $\varphi_j = u_j \circ \psi$ and u_j^n is in H_d^2 for any $n \geq 1$.*

Proof. The proof is the same as Proposition 2.1. □

Definition 5.1. For φ and ψ in $\mathcal{C}_d$, we define

$$Q(z, \lambda) = \frac{1 - \langle \varphi(z), \varphi(\lambda) \rangle}{1 - \langle \psi(z), \psi(\lambda) \rangle}.$$

In multivariable cases, Lemma 3.2 is false in general. Therefore, in order to obtain any result corresponding to Theorem 3.1, we will need some additional condition.

Theorem 5.1. *Let φ and ψ be in $\mathcal{C}_d$. If ψ is an injective open mapping and Q is positive semi-definite, then there exists u in $\mathcal{S}_d$ such that $C_{\varphi} = C_{\psi} C_u$.*

Proof. We define a column operator whose entries are the densely defined linear operators defined as follows:

$$T_j^* k_{\psi(\lambda)} = \overline{\varphi_j(\lambda)} k_{\psi(\lambda)} \quad (\lambda \in \mathbb{B}_d,\ j = 1, \ldots, d).$$

We note that T_j^* is well defined by the assumption that ψ is injective. By the same argument as that in the proof of Theorem 3.1, $T^* = {}^t(T_1^*, \ldots, T_d^*)$ can be extended to a bounded linear operator from H_d^2 into $\oplus_{j=1}^d H_d^2$ and $\|TT^*\| \leq 1$. Furthermore, it is easy to see that every T_j^* is bounded and commutes with the adjoint of the d-shift. Hence there exists u_j in the multiplier algebra of H_d^2 such that $T = (M_{u_1}, \ldots, M_{u_d})$, where M_u denotes the multiplication operator defined by a multiplier u. Since $\|TT^*\| \leq 1$ and $u = (u_1, \ldots, u_d)$ belongs to $\mathcal{S}_d$, by Jury's theorem, C_u is bounded. This concludes the proof. □

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