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A STRONGER NONCOMMUTATIVE EGOROFF'S THEOREM

CHARLES A. AKEMANN AND G. A. BAGHERI-BARDI

ABSTRACT. We prove a stronger version of Egoroff's theorem in the non-commutative setting.

Egoroff's theorem in abstract measure theory plays a fundamental role. It says

Theorem 1. *Let $(\mathcal{X}, \mu)$ be a measure space with $\mu(\mathcal{X}) < \infty$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of complex-valued measurable functions which converges at almost every point of $\mathcal{X}$ to a complex-valued function f. If $\epsilon > 0$, there is a measurable set $E \subseteq \mathcal{X}$ with $\mu(\mathcal{X} - E) \leq \epsilon$ such that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on E.*

Ignoring a set of measure $< \epsilon/2$, we can assume that $\{f_n\}_{n=1}^{\infty}$ are all bounded, so without lost of generality they are bounded by 1 and $f = 0$. This is routine measure theory, but it uses strongly the fact that we have sequenatial convergence. Using these reductions, we now have the following version of Egoroff's Theorem.

Theorem 2. *Let $(\mathcal{X}, \mu)$ be a measure space with $\mu(\mathcal{X}) < \infty$. Let A be a uniformly bounded set of measurable functions which contains 0 in its L^1 norm closure. If $\epsilon > 0$, there is a measurable set $E \subseteq \mathcal{X}$ with $\mu(\mathcal{X} - E) \leq \epsilon$ and a sequence $\{f_n\}_{n=1}^{\infty} \subset A$ that converges uniformly to 0 on E.*

Let $\mathcal{H}$ be a Hilbert space and $\mathbf{B}(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. A noncommutative version of this theorem was proved by Saito in [3] as follows.

Theorem 3. *Let $\mathcal{M}$ be a von Neumann algebra in $\mathbf{B}(\mathcal{H})$. Let A be a bounded subset of $\mathcal{M}$ such that 0 lies in its strong closure $\overline{A}$. Then, for any positive $\mu \in \mathcal{M}_*$ and any $\epsilon > 0$, there exist a projection e_0 in $\mathcal{M}$ and a sequence $\{a_n\}_{n=1}^{\infty}$ in A such that*

$$\lim_{n\to\infty} \| a_n e_0 \| = 0, \ \mu(1 - e_0) \leq \epsilon.$$

A part of Saito's proof (for which he was forced to use the boundedness assumption) is based on this point: Let $\{a_i\}$ be a uniformly bounded net in $\mathbf{B}(\mathcal{H})$ which strongly converges to zero then $\{a_i^* a_i\}$ strongly converges to zero too. We show by an example that the boundedness assumption is needed for this point.

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Example 4. Let $\{e_n\}_1^\infty$ be an orthonormal set in $\mathcal{H}$ and consider the rank one projection p_n onto $\mathbb{C}e_n$. Then 0 is in the strong closure of $S = \{\sqrt{n}p_n : n \in \mathbb{N}\}$ (see Example C.10 of [5]). We list below two points concerning this example:

(1) Every norm bounded net in S has many finite distinct elements. Therefore there is no norm bounded net in S which converges to 0 with related to the strong operator topology. We now apply the principle of uniform boundedness to conclude there is no sequence in S which converges to 0 with related to the strong operator topology too.

(2) There is a net $\{a_i\}$ in S converging to 0 in the strong operator topology. The squares of elements of S are all the form np_n. Let $h = \sum_n \frac{1}{n}e_n$ then $np_n(h) = e_n$. So there is no subnet of $\{np_n\}$ strongly convergent to 0. Since $\{a_i^2\}$ does converge to zero in the weak operator topology then $\{a_i^2\}$ can not strongly converge to anything.

This example makes sense we cannot apply Saito's theorem for $S \subseteq \mathcal{M} = B(H)$. We make a change in Saito's proof to show the bounded assumption is redundant. This is an important benefit since the strong operator topology is metric on bounded subsets of $\mathcal{M}$. Our proof makes it clear that this part of Egoroff's Theorem is not about sequences.

Theorem 5. *Let $\mathcal{M}$ be a von Neumann algebra in $\mathbf{B}(\mathcal{H})$. Let A be an arbitrary subset of $\mathcal{M}$ and $\overline{A}$ be its strong closure. Take an arbitrary element $a \in \overline{A}$. Then, for any positive $\mu \in \mathcal{M}_*$, any projection $e \in \mathcal{M}$ and any $\epsilon > 0$, there exist a projection $e_0 \leq e$ in $\mathcal{M}$ and a sequence $\{a_n\}_{n=1}^\infty$ in A such that*

$$\lim_{n\to\infty} \| (a_n - a)e_0 \| = 0, \ \mu(e - e_0) \leq \epsilon.$$

Proof. Let $\{a_i\}_{i\in I}$ be a net in $\mathcal{A}$ which is strongly convergent to a.

Step 1. Let us consider projections $p_{1,i} = \chi_{[0,\frac{1}{2}]}(|(a - a_i)e|)$ in $e\mathcal{M}e$.

- We have $\| (a - a_i)p_{1,i} \| \leq \frac{1}{2}$ for every $i \in I$. To prove it

$$\begin{aligned} \| (a - a_i)p_{1,i} \|^2 &= \| p_{1,i}(a - a_i)^*(a - a_i)p_{1,i} \| \\ &= \| p_{1,i}e(a - a_i)^*(a - a_i)ep_{1,i} \| \\ &= \| p_{1,i} \ |(a - a_i)e|^2 \ p_{1,i} \| \\ &= \| \ |(a - a_i)e| \ p_{1,i} \|^2 \leq (\frac{1}{2})^2. \end{aligned}$$

- We show the net $\{p_{1,i}\}_{i\in I}$ strongly converges to e and conclude there is $k_1 \in I$ with $\mu(e - p_{1,k_1}) \leq \frac{\epsilon}{2}$ (since μ is normal). Based on definition of projections $p_{1,i}$'s, we have

$$\frac{1}{2}(e - p_{1,i}) \leq |(a - a_i)e|.$$

But $\{|(a-a_i)e|\}$ is strongly convergent to 0. This point is obtained by the fact that $\{a-a_i\}$ strongly goes to zero and the following equality.

$$\| \, |(a-a_i)e| \, \zeta \, \| = \| \, (a-a_i)e\zeta \, \| \qquad (\zeta \in \mathcal{H}).$$

Step 2. We now consider projections $p_{2,i} = \chi_{[0,\frac{1}{4}]}(|(a-a_i)p_{1,k_1}|)$ in $p_{1,k_1}\mathcal{M}p_{1,k_1}$. Similar to the step 1,

- We have $\| \, (a-a_i)p_{2,i} \, \| \leq \frac{1}{2^2}$ for every $i \in I$. To prove it

$$\begin{aligned}
\| \, (a-a_i)p_{2,i} \, \|^2 &= \| \, p_{2,i}(a-a_i)^*(a-a_i)p_{2,i} \, \| \\
&= \| \, p_{2,i}p_{1,k_1}(a-a_i)^*(a-a_i)p_{1,k_1}p_{2,i} \, \| \\
&= \| \, p_{2,i} \, |(a-a_i)p_{1,k_1}|^2 \, p_{2,i} \, \| \\
&= \| \, |(a-a_i)p_{1,k_1}| \, p_{2,i} \, \|^2 \leq (\frac{1}{4})^2.
\end{aligned}$$

- There is $k_2 \in I$ with $\mu(p_{1,k_1} - p_{2,k_2}) \leq \frac{\epsilon}{2^2}$.

By induction we obtain a decreasing sequence $\{p_{n,k_n}\}_{n=1}^{\infty}$ in $\mathcal{M}$ which should be strongly convergent to a projection $e_0 \in \mathcal{M}$. Then we get

$$\mu(e-e_0) \leq \epsilon \quad \text{and} \quad \lim_{n\to\infty} \| \, (a-a_{k_n})e_0 \, \| \leq \lim_{n\to\infty} \| \, (a-a_{k_n})p_{n,k_n} \, \| = 0.$$

□

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(C. A. Akemann) Department of Mathematics, University of California, Santa Barbara, California 93106 USA.
E-mail address: `Akemann@math.ucsb.edu`

(G. A. Bagheri-Bardi) Department of Mathematics, Persian Gulf University, Boushehr 75168, Iran.
E-mail address: `alihoular@gmail.com , bagheri@mailpgu.ac.ir`

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