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# A NEW CONSTRUCTION OF THE REAL NUMBERS BY ALTERNATING SERIES

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Abstract. We present a new method of constructing the complete ordered field of real numbers from the ordered field of rational numbers. Our method is a generalization of that of A. Knopfmacher and J. Knopfmacher. Our result implies that there exist infinitely many ways of constructing the complete ordered field of real numbers. As an application of our results, we prove the irrationality of certain numbers.

## 1. Introduction

The purpose of this paper is to present a new method of constructing the complete ordered field of real numbers from the ordered field of rational numbers. Our method is similar to the method which was established by A. Knopfmacher and J. Knopfmacher in [6], but our method is more general. Moreover our result gives infinitely many ways of constructing the complete ordered field of real numbers. As an application of our results, we prove the irrationality of certain series.

A. Knopfmacher and J. Knopfmacher constructed the complete ordered field of real numbers by the Sylvester expansion and the Engel expansion in [5] and by the alternating-Sylvester expansion and the alternating-Engel expansion in [6]. The advantages of these constructions are the fact that those are concrete and do not depend on the notion of equivalence classes. The alternating-Sylvester expansion and the alternating-Engel expansion are generalizations of Oppenheim's expansion (see [7]) and special cases of the alternating Balkema-Oppenheim's expansion (see [2]), which were introduced by A. Knopfmacher and J. Knopfmacher in [6]. The definition of the alternating-Sylvester expansion and the alternating-Engel expansion are the following.

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(i) **Alternating-Sylvester expansion**. Let $\alpha \in \mathbb{R}$, $a_0 = [\alpha]$ and $A_1 = \{\alpha\}$, where $\{x\} = x - [x]$. We define, for $n \in \mathbb{N}$ and $A_n > 0$,

$$a_n = \left[\frac{1}{A_n}\right]$$

and

$$A_{n+1} = \frac{1}{a_n} - A_n.$$

Then

$$\alpha = a_0 + \frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_3} - \dots, \tag{1.1}$$

where $a_1 \geq 1$ and $a_{n+1} \geq a_n(a_n + 1)$ for $n \in \mathbb{N}$.

(ii) **Alternating-Engel expansion**. Let $\alpha \in \mathbb{R}$, $a_0 = [\alpha]$ and $A_1 = \{\alpha\}$. We define, for $n \in \mathbb{N}$ and $A_n > 0$,

$$a_n = \left[\frac{1}{A_n}\right]$$

and

$$A_{n+1} = 1 - a_n A_n.$$

Then

$$\alpha = a_0 + \frac{1}{a_1} - \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} - \dots, \tag{1.2}$$

where $a_1 \geq 1$ and $a_{n+1} \geq a_i + 1$ for $n \in \mathbb{N}$.

The relation

$$\frac{1}{d+1} < \alpha \leq \frac{1}{d} \qquad (\alpha \in (0,1],\ d = [\alpha^{-1}]) \tag{1.3}$$

is used in these expansions. We introduce a new series expansion for every real numbers by using a more general relation

$$\frac{c}{d+1} < \alpha \leq \frac{c}{d} \qquad (\alpha \in (0,1],\ c \in \mathbb{N},\ d = [c\alpha^{-1}]).$$

**Definition 1.1** (Generalized alternating-Sylvester expansion)**.** Let $\alpha \in \mathbb{R}$, $q_0 = [\alpha]$ and $A_1 = \{\alpha\}$. Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of positive integers. We define, for $n \in \mathbb{N}$,

$$a_n = \left[\frac{c_n}{A_n}\right] \qquad (\text{for } A_n \neq 0),$$

$$q_n = \begin{cases} \frac{c_n}{a_n} & (A_n \neq 0) \\ 0 & (A_n = 0) \end{cases}$$

and

$$A_{n+1} = q_n - A_n.$$

Then

$$\alpha = q_0 + \sum_{n=1}^{\infty} (-1)^{n-1} q_n. \tag{1.4}$$

If we regard the alternating-Sylvester series (1.1) as an analogue of the regular continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

the generalized alternating-Sylvester series (1.4) is an analogue of the continued fraction

$$a_0 + \cfrac{c_1}{a_1 + \cfrac{c_2}{a_2 + \cfrac{c_3}{a_3 + \cdots}}}.$$

By taking some appropriate $\{c_n\}$, we can get a simple continued fraction representation for some real numbers. For example, we have (see (2.1.22) in [3])

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \cdots}}}}.$$

On the other hand, the regular continued fraction representation of $\pi$ is complicated. Therefore we can expect that if we take some appropriate $\{c_n\}$, then we can get a simple series representation for some real numbers. In fact we can prove irrationality of certain numbers by using such a representation.

The outline of this paper is the following. In Section 2 we study some fundamental properties of the generalized alternating-Sylvester series. In Section 3 we take an arbitrary sequence of positive integers $\{c_n\}_{n=1}^{\infty}$ such that $c_n \mid c_{n+1}$ for all $n \in \mathbb{N}$, and we prove that the set

$$S(\{c_n\}) = \{\{q_n\}_{n=0}^{\infty} \mid \{q_n\} \text{ appears in } (1.4)\} \tag{1.5}$$

can be identified with the complete ordered field of real numbers $\mathbb{R}$ by introducing the relation $<$ and the operator $+$ and $\cdot$. In other words we prove that $S(\{c_n\})$ becomes an ordered field which is isomorphic to $\mathbb{R}$. Since there exist infinitely many $\{c_n\}$ such that $c_n \mid c_{n+1}$, this implies that there exist infinitely many ways of constructing the complete ordered field of real numbers. Our construction is similar to that in [6]. Therefore our construction is also concrete and does not use the notion of equivalence classes. When we prove that $S(\{c_n\})$ becomes an ordered field, we use a general lemma (see Lemma 3.4). It seems that this lemma can be used in [4], [5] and [6]. In section 4, we prove the irrationality of certain series by Proposition 2.3 and Proposition 3.1.

*Remark* 1.1. At first glance, it seems that we can define generalized alternating-Engel series as follows:

Let $\alpha \in \mathbb{R}$, $A_1 = \alpha - a_0$ with $0 < A_1 \leq 1$, $a_0 \in \mathbb{Z}$. Let $\{c_n\}$ be a sequence of positive integers. We define, for $n \in \mathbb{N}$ and $A_n \neq 0$,

$$a_n = \left[\frac{c_n}{A_n}\right]$$

and

$$A_{n+1} = c_n - a_n A_n.$$

Then

$$\alpha = a_0 + \frac{c_1}{a_1} - \frac{c_2}{a_1 a_2} + \frac{c_3}{a_1 a_2 a_3} - \cdots.$$

However, $a_{n+1} \geq a_n$ does not hold in this series. For example, if we set $A_1 = \alpha = 5/7$, $c_1 = 2$ and $c_2 = 1$, then $a_1 = 2$, $A_2 = 4/7$ and $a_2 = 1$. This is a trouble. In order to simplify the argument we do not argue on this series.

## 2. Fundamental properties of the generalized alternating Sylvester series

In this section, we take an arbitrary sequence of positive integers $\{c_n\}_{n=1}^{\infty}$ and fix it.

**Proposition 2.1.** *The generalized alternating-Sylvester series has the following properties for $n \in \mathbb{N}$.*

(1) *If $A_n \neq 0$, then we have*
$$\frac{c_n}{a_n + 1} < A_n \leq \frac{c_n}{a_n}.$$
(2) *If $A_{n+1} \neq 0$, then we have*
$$a_{n+1} + 1 > \frac{c_{n+1}}{c_n} a_n (a_n + 1).$$
(3) *The inequality $A_n \geq A_{n+1}$ holds. If $A_n \neq 0$, then we even have $A_n > A_{n+1}$.*
(4) *The inequality $q_n \leq 1$ holds.*
(5) *If $A_{n+1} \neq 0$, then we have $a_{n+1} > a_n$.*
(6) *If $A_n \neq 0$, then we have $A_{n+1} < \frac{1}{a_n+1}$.*
(7) *The inequality $q_n \geq q_{n+1}$ holds. If $q_{n+1} \neq 0$, then we even have $q_n > q_{n+1}$.*

*Proof.* (1) This trivially follows from the definition of the generalized alternating-Sylvester expansion.

(2) From (1) and the definition, we have

$$\begin{aligned} a_{n+1}+1 &> \frac{c_{n+1}}{A_{n+1}} \\ &= \frac{c_{n+1}}{\frac{c_n}{a_n}-A_n} \\ &> \frac{c_{n+1}}{\frac{c_n}{a_n}-\frac{c_n}{a_n+1}} \\ &= \frac{c_{n+1}}{c_n}a_n(a_n+1). \end{aligned}$$

(3) In the case $A_n = 0$, we have $A_n \geq A_{n+1}$. For $A_n \neq 0$, we have

$$A_{n+1} < \frac{c_n}{a_n} - \frac{c_n}{a_n+1} \leq \frac{c_n}{a_n+1} < A_n.$$

(4) By (3), we have $A_n < 1$ for all $n$. Hence,

$$a_n = \left[\frac{c_n}{A_n}\right] \geq c_n$$

holds. This implies (4).

(5) From (2), we have (5) by using (4).

(6) By (4), we have

$$A_{n+1} < \frac{c_n}{a_n} - \frac{c_n}{a_n+1} = \frac{c_n}{a_n(a_n+1)} \leq \frac{1}{a_n+1}.$$

(7) In the case $q_{n+1} = 0$, we have $q_n \geq q_{n+1}$. For $q_{n+1} \neq 0$, we have

$$q_{n+1} < \frac{c_{n+1}}{c_{n+1}{q_n}^{-1}(a_n+1)-1} \leq \frac{c_{n+1}}{c_{n+1}a_n + c_{n+1} - 1} \leq \frac{1}{a_n} \leq q_n$$

by (2) and (4). □

*Remark* 2.1. Since we have

$$\sum_{k=1}^{n}(-1)^{k-1}q_k = A_1 + (-1)^{n-1}A_{n+1} \qquad \text{(for all } n \in \mathbb{N}),$$

the series in (1.4) converges by Proposition 2.1 (5), (6). Hence,

$$(-1)^{n-1}\sum_{k=n}^{\infty}(-1)^{k-1}q_k = (-1)^{n-1}\sum_{k=n}^{\infty}(-1)^{k-1}(A_{k+1}+A_k) = A_n \tag{2.1}$$

holds for all $n \in \mathbb{N}$.

In order to prove Proposition 2.2 we require some lemmas.

We can easily see that the following lemma holds.

**Lemma 2.1.** *Let $c, d \in \mathbb{N}$ and $\alpha \in (0, 1]$. Then*

(1) *there does not exist* $d' \in \mathbb{Z}$ *such that*

$$\frac{c}{d+1} < \frac{c}{d'} < \frac{c}{d},$$

(2) $d = [c\alpha^{-1}]$ *is equivalent to*

$$\frac{c}{d+1} < \alpha \leq \frac{c}{d}.$$

**Lemma 2.2.** *Let* $\alpha, \alpha' \in (0,1]$, $c \in \mathbb{N}$, $d = [c/\alpha]$ *and* $d' = [c/\alpha']$. *If* $c/d \neq c/d'$ *then* $\alpha < \alpha'$ *is equivalent to* $c/d < c/d'$.

*Proof.* First, we assume $\alpha < \alpha'$. Since $c/(d+1) < \alpha \leq c/d$ and $c/(d'+1) < \alpha' \leq c/d'$ hold by Lemma 2.1 (2), it is sufficient that we consider the following cases.

(1) $\alpha < \alpha' \leq c/d$.
(2) $c/(d'+1) < \alpha \leq c/d < \alpha'$.
(3) $\alpha \leq c/(d'+1) < \alpha'$.

If (1) holds, then we have

$$\frac{c}{d+1} < \alpha < \alpha' \leq \frac{c}{d}.$$

This implies that $c/d = c/d'$ by Lemma 2.1 (2), which is impossible.

If (2) holds, then we have

$$\frac{c}{d'+1} < \frac{c}{d} < \alpha' \leq \frac{c}{d'},$$

which is impossible by Lemma 2.1 (1).

If (3) holds, then we have

$$\alpha \leq \frac{c}{d} \leq \frac{c}{d'+1} < \alpha' < \frac{c}{d'}$$

by Lemma 2.1 (1).

Next, we assume $c/d < c/d'$. Since $c/(d'+1) < c/d$ is impossible by Lemma 2.1 (1), we have

$$\alpha \leq \frac{c}{d} \leq \frac{c}{d'+1} < \alpha'.$$

□

**Proposition 2.2.** *Let* $\alpha, \alpha' \in \mathbb{R}$ *with* $\alpha \neq \alpha'$. *We define* ${a_n}'$, ${A_n}'$ *and* $q_n'$ *as* $a_n$, $A_n$ *and* $q_n$ *which appear in the generalized alternating Sylvester expansion of* $\alpha'$, *respectively. Let*

$$i = \min\{j \in \mathbb{N} \cup \{0\} \mid q_j \neq q_j'\}.$$

*Then* $\alpha < \alpha'$ *is equivalent to*

$$\begin{cases} q_i < q_i' & (i = 0 \text{ or } 2 \nmid i), \\ q_i > q_i' & (2 \mid i \text{ and } i \geq 2). \end{cases}$$

*Proof.* First, we consider the case $i = 0$. If $\alpha < \alpha'$, then we have $q_0 = [\alpha] \leq [\alpha'] = q_0'$. Therefore we obtain $q_0 < q_0'$. On the other hand, if $q_0 < q_0'$, then we have $[\alpha] < [\alpha']$. Therefore we obtain $\alpha < \alpha'$.

Next, we assume $i \neq 0$. Then we can write

$$\alpha = q_0 + \sum_{k=1}^{i-1} (-1)^{k-1} q_k + (-1)^{i-1} A_i, \quad \alpha' = q_0 + \sum_{k=1}^{i-1} (-1)^{k-1} q_k + (-1)^{i-1} A_i' \tag{2.2}$$

by Remark 2.1. These relations imply that $\alpha < \alpha'$ is equivalent to

$$\begin{cases} A_i < A_i' & (2 \nmid i), \\ -A_i < -A_i' & (2 \mid i \text{ and } i \geq 2). \end{cases}$$

By Proposition 2.1 (1) and Lemma 2.2, this is equivalent to

$$\begin{cases} q_i < q_i' & (2 \nmid i), \\ q_i > q_i' & (2 \mid i \text{ and } i \geq 2). \end{cases}$$

This implies the proposition. □

In order to consider the case $\alpha \in \mathbb{Q}$ we prove the next lemma.

**Lemma 2.3.** *Let $c \in \mathbb{N}$ and $p/q \in \mathbb{Q} \cap (0, 1]$ with $p, q \in \mathbb{N}$. Let $d = [cq/p]$. Then the numerator of $c/d - p/q$ is less than $p$. In other words, $cq - dp < p$.*

*Proof.* We have

$$cq - dp = cq - \left( \frac{cq}{p} - \left\{ \frac{cq}{p} \right\} \right) p \leq \frac{p-1}{p} p = p - 1.$$

□

**Proposition 2.3.** *The real number $\alpha$ is rational if and only if there exists an $m \in \mathbb{N}$ such that $q_m = 0$.*

*Proof.* If there exists an $m \in \mathbb{N}$ such that $q_m = 0$, then $\alpha$ is rational. We assume $\alpha = p/q$, where $p, q \in \mathbb{Z}$ and $q \neq 0$. Without loss of generality, we may assume that $q_0 = 0$, $A_1 = p/q$ and $p, q > 0$. By the definition of $a_n, A_n$ and Lemma 2.3, the numerator of $A_n$ is strictly monotonically decreasing. This implies the proposition. □

Propositions 2.1, 2.2 and 2.3 imply that the generalized alternating-Sylvester series is similar to alternating-Sylvester series.

## 3. Construction of the real numbers

In this section we take an arbitrary sequence of positive integers $\{c_n\}_{n=1}^{\infty}$ which satisfies the condition $c_n \mid c_{n+1}$ for all $n \in \mathbb{N}$ and fix it. Moreover we identify $\{q_n\}_{n=0}^{\infty} \in S(\{c_n\})$ with $(q_0, q_1, q_2, \dots)$.

*Remark* 3.1. By the condition $c_n \mid c_{n+1}$ for any $n \in \mathbb{N}$, the inequality in Proposition 2.1 (2) becomes

$$a_{n+1} \geq \frac{c_{n+1}}{c_n} a_n(a_n + 1).$$

If the equality holds in the above and $q_{n+2} = 0$, then we have

$$A_n = q_n - q_{n+1} = \frac{c_n}{a_n + 1}.$$

This contradicts the definition of $q_n$. Hence, $q_{n+2} \neq 0$ or

$$a_{n+1} > \frac{c_{n+1}}{c_n} a_n(a_n + 1)$$

holds.

In Section 1, we assumed the existence of the real numbers, and we defined $S(\{c_n\})$ in (1.5). In order to use $S(\{c_n\})$ for the construction of the real numbers, here we remove that assumption.

First we will define a set of sequences of rational numbers $T(\{c_n\})$. We will prove $S(\{c_n\}) = T(\{c_n\})$ in Proposition 3.1. For the sake of simplicity, we define a set of sequences of positive integers

$$U(\{c_n\}) := \left\{ \{a_n\} \subset \mathbb{N} \mid \forall n \in \mathbb{N} \left[ a_{n+1} \geq \frac{c_{n+1}}{c_n} a_n(a_n + 1) \right] \right\}.$$

**Definition 3.1.** Let $\{q_n\}_{n=0}^{\infty}$ be a sequence of rational numbers. We define $\{q_n\} \in T(\{c_n\})$ if and only if

(1) $q_0 \in \mathbb{Z}$,
(2) $q_n \leq 1$ for all $n \in \mathbb{N}$,
(3) if $q_1 = 1$, then $q_2 \neq 0$,
(4) if $q_m = 0$ for $m \in \mathbb{N}$, then $q_n = 0$ for all $n \geq m$,
(5) there exists a $\{a_n\} \in U(\{c_n\})$ such that $q_n = c_n / a_n$ for all $n \in \mathbb{N}$ if $q_n \neq 0$, and
(6) if $q_{n+1} \neq 0$, then $q_{n+2} \neq 0$ or

$$a_{n+1} > \frac{c_{n+1}}{c_n} a_n(a_n + 1)$$

holds.

We can easily see that the following lemma holds.

**Lemma 3.1.** *Let $\{q_n\} \in T(\{c_n\})$ and $n \in \mathbb{N}$.*

(1) $a_{n+1} > a_n$.
(2) $q_{n+1} \le \frac{1}{a_n+1}$.
(3) $q_{n+1} \le q_n$. *If* $q_{n+1} \neq 0$, *then* $q_{n+1} < q_n$.
(4) *The series*

$$\sum_{k=1}^{\infty}(-1)^{k-1}q_k$$

*converges.*

**Proposition 3.1.** $S(\{c_n\}) = T(\{c_n\})$.

*Proof.* $S(\{c_n\}) \subset T(\{c_n\})$ trivially follows by Proposition 2.1 and Remark 3.1. In order to prove $S(\{c_n\}) \supset T(\{c_n\})$, we take $\{q'_n\} \in T(\{c_n\})$ and assume that $q'_0 \in \mathbb{Z}$ and $q'_n = 0$ or $q'_n = c_n/a'_n$ for all $n \in \mathbb{N}$. Since we can set

$$\alpha = q'_0 + \sum_{k=1}^{\infty}(-1)^{k-1}q'_k$$

by Lemma 3.1 (4), we have

$$\alpha = q_0 + \sum_{k=1}^{\infty}(-1)^{k-1}q_k$$

by the generalized alternating-Sylvester expansion. It is sufficient to prove that $q_n = q'_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since the case $q'_1 = 0$ is trivial, we may assume $q'_1 \neq 0$. By considering $[\alpha]$, we have $q_0 = q'_0$ and $\alpha - q_0 = A_1 \le q'_1 = c_1/a'_1$. If $q'_1 = A_1$, then $q_1 = q'_1$ by Lemma 2.1 (2). If $q'_1 \neq A_1$, then we have $A_1 \ge q'_1 - q'_2 \ge c_1/(a'_1+1)$ by $\{a'_n\} \in U(\{c_n\})$ and Definition 3.1 (5). However, $A_1 = q'_1 - q'_2 = c_1/(a'_1+1)$ is impossible because of Definition 3.1 (6). Thus we obtain $c_1/(a'_1+1) < A_1 < c_1/a'_1$. This implies $q_1 = q'_1$ by Lemma 2.1 (2).

Next we suppose that $q_{n-1} = q'_{n-1}$ holds for $n > 1$. Then we have

$$(-1)^{n-1}A_n = \alpha - q_0 - \sum_{k=1}^{n-1}(-1)^{k-1}q_k = \sum_{k=n}^{\infty}(-1)^{k-1}q'_k$$

by Remark 2.1. Hence, we obtain $A_n \le q'_n = c_n/a'_n$. If $q'_n = A_n$, then $q'_n = q_n$ by Lemma 2.1 (2). If $q'_n \neq A_n$, then we have $A_n \ge q'_n - q'_{n+1} \ge c_n/(a'_n+1)$ by $\{a'_n\} \in U(\{c_n\})$ and Definition 3.1 (5). By Definition 3.1 (6) we obtain $A_n > c_n/(a'_n+1)$. Since this implies $q_n = q'_n$, we obtain the assertion of the proposition inductively. $\square$

In the rest of this section, we set $S = S(\{c_n\})$ for simplicity, and we introduce a relation $<$ and operators $+$, $\cdot$ for $S$.

First we define the binary relation $<$ on $S$.

**Definition 3.2.** Let $\{p_n\}, \{q_n\} \in S$ with $\{p_n\} \neq \{q_n\}$ and

$$i = \min\{j \in \mathbb{N} \cup \{0\} \mid p_j \neq q_j\}.$$

We define $\{p_n\} < \{q_n\}$ if and only if

$$\begin{cases} p_i < q_i & (i = 0 \text{ or } 2 \nmid i), \\ p_i > q_i & (2 \mid i \text{ and } i \geq 2). \end{cases}$$

**Proposition 3.2.** *For any $\{p_n\}, \{q_n\}, \{r_n\} \in S$, we have*

(1) *$\{p_n\} < \{p_n\}$ does not hold (irreflexive law),*
(2) *$\{p_n\} < \{q_n\}$ or $\{p_n\} = \{q_n\}$ or $\{q_n\} < \{p_n\}$ (trichotomy),*
(3) *if $\{p_n\} < \{q_n\}$ and $\{q_n\} < \{r_n\}$ then $\{p_n\} < \{r_n\}$ (transitive law).*

*In other words, $<$ is a linear order in the strict sense on $S$.*

*Proof.* We can easily see that (1) and (2) hold. In order to prove (3), we define

$$i_1 = \min\{j \in \mathbb{N} \cup \{0\} \mid p_j \neq q_j\}, \qquad i_2 = \min\{j \in \mathbb{N} \cup \{0\} \mid q_j \neq r_j\}$$

and $i = \min\{i_1, i_2\}$. Then

$$p_k = q_k = r_k \quad (\text{for any } k \in \{0, 1, \ldots, i-1\})$$

and

$$p_i \neq q_i \quad \text{or} \quad q_i \neq r_i$$

hold. If $i$ is odd, then we have

$$\begin{cases} p_i < q_i \text{ and } q_i < r_i & (i = i_1 = i_2), \\ p_i = q_i \text{ and } q_i < r_i & (i = i_2 \neq i_1), \\ p_i < q_i \text{ and } q_i = r_i & (i = i_1 \neq i_2). \end{cases}$$

Therefore we obtain $p_i < r_i$. The other cases can be proved by the same argument. □

If we define

$$Q_S = \{\{q_n\} \in S \mid \text{there exists an } m \in \mathbb{N} \text{ such that } q_m = 0\},$$

we can identify $Q_S$ with $\mathbb{Q}$ by Proposition 2.2 and 2.3. In short, the map

$$\mathbb{Q} \ni \left( q_0 + \sum_{n=1}^{\infty} (-1)^{n-1} q_n \right) \mapsto \{q_n\} \in Q_S$$

is an order-isomorphism. Hence, we may conclude that $\mathbb{Q} \subset S$.

**Theorem 3.1.** *Let $M$ be a non-empty subset of $S$. If $M$ is bounded from above (below), then there exists a supremum (an infimum).*

*Proof.* Since $M$ is bounded from above, there exists a $d_0$ such that

$$d_0 = \max\{q_0 \in \mathbb{Z} \mid \text{there exists a } (q_0, q_1, \dots) \in M\}.$$

If there does not exist an upper bound for $M$ such that $(d_0, q_1, \dots) \in S$, then $(d_0 + 1, 0, \dots)$ is a supremum for $M$. We assume that there exists an upper bound for $M$ such that $(d_0, q_1, \dots) \in S$. Since there exists a $(q_0, q_1, \dots) \in M$ such that $q_0 = d_0$, we can define

$$d_1 = \max\{q_1 \in \mathbb{Q} \mid \text{there exists a } (d_0, q_1, \dots) \in M\}$$

from the definition of $S$ and $<$. By the same argument, we can define

$$d_2 = \min\{q_2 \in \mathbb{Q} \mid \text{there exists a } (d_0, d_1, q_2, \dots) \in M\}.$$

In general, if we have defined $d_{k-1}$ for $k > 1$, then we define

$$d_k = \begin{cases} \max\{q_k \in \mathbb{Q} \mid \exists (d_0, d_1, \dots, d_{k-1}, q_k, \dots) \in M\} & (k-1 \text{ is even}), \\ \min\{q_k \in \mathbb{Q} \mid \exists (d_0, d_1, \dots, d_{k-1}, q_k, \dots) \in M\} & (k-1 \text{ is odd}). \end{cases}$$

By the definition of $<$ and $\{d_n\}$, $\{d_n\}$ is the supremum for $M$. We can prove this as follows. If $\{d_n\}$ is not an upper bound for $M$, then there exists a $\{q_n\} \in M$ such that $\{d_n\} < \{q_n\}$. By setting $i = \min\{n \in \mathbb{N} \mid d_n \neq q_n\}$, we have $d_i < q_i$ for odd $i$ or $d_i > q_i$ for even $i$. This contradicts the definition of $\{d_n\}$. On the other hand, if $\{d_n\}$ is not minimum upper bound for $M$, then there exists an upper bound for $M$ $\{r_n\}$ such that $\{r_n\} < \{d_n\}$. We set $j = \min\{n \in \mathbb{N} \mid d_n \neq r_n\}$. By the definition of $\{d_n\}$, there exists an $X = (x_0, x_1, \dots) \in M$ such that $x_k = d_k$ for $0 \leq k \leq j$. Then we have $\{r_n\} < X \leq \{d_n\}$. This is impossible.

The case of the infimum can be proved by the same argument. □

In order to introduce the algebraic structure for $S$, we require some preparations.

**Definition 3.3.** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of rational numbers. We say that $L(a_n)$ holds if and only if, for all $m \in \mathbb{N}$, there exists an $N \in \mathbb{N}$ such that $|a_n| < 1/m$ holds for all $n \geq N$.

Note that in the usual sense $L(a_n)$ means $\lim_{n \to \infty} a_n = 0$.

The following definition and lemma are the same as in [6, p. 611].

**Definition 3.4.** Let $X \in S$ with $X = (x_0, x_1, \dots)$. We define

$$X_n = (x_0, x_1, \dots, x_n, 0, \dots),$$

where $n \in \mathbb{N}$.

We can easily see that the next lemma holds.

**Lemma 3.2.** *Let $X \in S$ with $X = (x_0, x_1, \dots)$. Then we have*

(1) $X_{2n} \leq X_{2n+2} \leq X \leq X_{2n+1} \leq X_{2n-1}$,
(2) $L(X_{2n-1} - X_{2n})$,
(3) $\sup X_{2n} = \inf X_{2n-1} = X$.

In order to prove Lemma 3.4, we also require the next lemma.

**Lemma 3.3.** *Let $\{a_n\}$ be a monotonically increasing sequence of rational numbers which is bounded from above. Let $X = \sup a_n$. Then we have $L(X_{2n-1} - a_n)$.*

*Proof.* By contradiction. Assume that there exists an $m$ such that

$$\forall N \in \mathbb{N}, \exists n \in \mathbb{N}[n \geq N \text{ and } |X_{2n-1} - a_n| = X_{2n-1} - a_n \geq 1/m]$$

holds. Since we have $X_{2n-1} - a_n \geq X_{2n+1} - a_{n+1}$ by the assumption of the lemma, we have $X_{2n-1} - a_n \geq 1/m$ for all $n \in \mathbb{N}$. On the other hand, by Lemma 3.2, there exists an $N \in \mathbb{N}$ such that

$$X_{2n-1} - X_{2n} < 1/2m$$

holds for all $n \geq N$. Hence, we have

$$\begin{aligned} a_n &\leq X_{2n-1} - \frac{1}{m} \\ &\leq X_{2N-1} - \frac{1}{m} \\ &< X_{2N} - \frac{1}{2m} \end{aligned}$$

for $n \geq N$. This implies that $X_{2N} - (1/2)m$ is an upper bound for $\{a_n\}$. Therefore we obtain

$$\sup a_n \leq X_{2N} - \frac{1}{2m} < X_{2N} \leq \sup X_{2n} = X.$$

This contradicts the definition of $X$. $\square$

The following lemma is important in the proofs of algebraic properties of $S$. It seems that this lemma can be used in the work of A. Knopfmacher and J. Knopfmacher [4], [5], [6].

**Lemma 3.4.** *Let $\{a_n\}, \{b_n\}$ be monotonically increasing sequence of rational numbers which are bounded from above. Then $\sup a_n = \sup b_n$ is equivalent to $L(a_n - b_n)$.*

*Proof.* First we assume $\sup a_n = \sup b_n$. We set $X = \sup a_n = \sup b_n$. Since

$$|a_n - b_n| \leq |a_n - X_{2n-1}| + |X_{2n-1} - b_n|,$$

we have $L(a_n - b_n)$ by Lemma 3.3.

Next we assume $L(a_n - b_n)$. By contradiction. Assume that $\sup a_n \neq \sup b_n$. Without loss of generality, we may assume $\sup a_n < \sup b_n$. We set $X = \sup a_n$.

Then there exists an $N \in \mathbb{N}$ such that $X_{2n-1} < b_n$ holds for all $n \geq N$. Since $b_n - X_{2n-1} \leq b_{n+1} - X_{2n+1}$ for $n \geq N$, we have

$$|b_n - a_n| = (b_n - X_{2n-1}) + (X_{2n-1} - a_n) \geq b_N - X_{2N-1} > 0$$

for $n \geq N$. This contradicts $L(a_n - b_n)$. $\square$

Now we define the operators on $S$, and prove that $S$ is an ordered field. (These definitions are the same as in [6].)

**Definition 3.5.** Let $X, Y \in S$. We define the following symbols and operators.

(1) $0 = (0, 0, \ldots)(= 0 \in \mathbb{Q})$.
(2) $X + Y = \sup(X_{2n} + Y_{2n})$.
(3) $-X = \sup(-X_{2n-1})$.
(4) $1 = (1, 0, \ldots)(= 1 \in \mathbb{Q})$.
(5)

$$X \cdot Y = \begin{cases} \sup(X_{2n} \cdot Y_{2n}) & (X, Y \geq 0), \\ (-X) \cdot (-Y) & (X, Y \leq 0), \\ -((-X) \cdot Y) & (X \leq 0, Y \geq 0), \\ -(X \cdot (-Y)) & (X \geq 0, Y \leq 0). \end{cases}$$

(6)

$$X^{-1} = \begin{cases} \sup((X_{2n-1})^{-1}) & (X > 0), \\ -((-X)^{-1}) & (X < 0). \end{cases}$$

Since $X_{2n} + Y_{2n} \leq X_1 + Y_1$, $-X_{2n-1} \leq -X_2$, $X_{2n} \cdot Y_{2n} \leq X_1 \cdot Y_1 \quad (X, Y \geq 0)$ and $(X_{2n-1})^{-1} \leq X_2^{-1} \quad (X > 0)$, these definitions are possible.

Now we prove that $+$ (resp. $\cdot$) shares the same properties with the usual addition (resp. multiplication).

**Proposition 3.3.** *Let $X, Y, Z \in S$. We have*

(1) $X + Y = Y + X$,
(2) $X + 0 = X$,
(3) $(X + Y) + Z = X + (Y + Z)$,
(4) $X + (-X) = 0$,
(5) *if* $X < Y$, *then* $X + Z < Y + Z$.

*Proof.* (1), (2) These trivially follow from the definition of $+$.

(3) We set $A = X + Y$, which means $L(A_{2n} - (X_{2n} + Y_{2n}))$ by Lemma 3.4. Since

$$|(A_{2n} + Z_{2n}) - (X_{2n} + Y_{2n} + Z_{2n})| = |A_{2n} - (X_{2n} + Y_{2n})|,$$

we have $L((A_{2n}+Z_{2n})-(X_{2n}+Y_{2n}+Z_{2n}))$. By Lemma 3.4, this implies $\sup(A_{2n}+Z_{2n})=\sup(X_{2n}+Y_{2n}+Z_{2n})$. Hence, we obtain $(X+Y)+Z=\sup(X_{2n}+Y_{2n}+Z_{2n})$. By the same argument, we can also prove that $X+(Y+Z)=\sup(X_{2n}+Y_{2n}+Z_{2n})$.

(4) We set $A=-X$, which means $L(A_{2n}+X_{2n-1})$ by Lemma 3.4. Since

$$|X_{2n}+A_{2n}| \le |X_{2n}-X_{2n-1}|+|X_{2n-1}+A_{2n}|,$$

we have $L((X_{2n}+A_{2n})-0)$ from Lemma 3.2. This implies $\sup(X_{2n}+A_{2n})=\sup 0$ by Lemma 3.4. Hence, we obtain (4).

(5) Since $X_{2n}+Z_{2n}<Y_{2n}+Z_{2n}$ holds for sufficiently large $n$, we have $X+Z\le Y+Z$. If $X+Z=Y+Z$, then we have $L((X_{2n}+Z_{2n})-(Y_{2n}+Z_{2n}))$ by Lemma 3.4. In short we have $L(X_{2n}-Y_{2n})$. However, this is impossible by Lemma 3.4. □

From Proposition 3.3 (1), (2), (3) and (4), it follows that $S$ is an abelian group on $+$. Hence, we can use $-(-X)=X$, $-(X+Y)=(-X)+(-Y)$, etc. Moreover we obtain $0<X \Leftrightarrow 0+(-X)<X+(-X) \Leftrightarrow -X<0$, $X<0 \Leftrightarrow 0<-X$ and $X<Y \Leftrightarrow -X>-Y$ by Proposition 3.3 (5).

**Proposition 3.4.** *Let $X,Y,Z\in S$. We have*

(1) $X\cdot Y=Y\cdot X$,
(2) $X\cdot 1=X$,
(3) $X\cdot Y=-((-X)\cdot Y)=-(X\cdot(-Y))$,
(4) $X\cdot X^{-1}=1 \quad (X\neq 0)$,
(5) $(X\cdot Y)\cdot Z=X\cdot(Y\cdot Z)$,
(6) *if* $X<Y$ *and* $Z>0$, *then* $XZ<YZ$.

*Proof.* (1), (2) These trivially follow from the definition of $\cdot$.

(3) In the case $X,Y\ge 0$, by setting $Z=-X$ and $W=-Y$, we have

$$-((-X)\cdot Y)=-(Z\cdot Y)=-(-((-Z)\cdot Y))=X\cdot Y,$$
$$-(X\cdot(-Y))=-(X\cdot W)=-(-(X\cdot(-W)))=X\cdot Y.$$

From this case, we can prove the other cases. For example, in the case $X\le 0$, $Y\ge 0$, we have

$$-(X\cdot(-Y))=-((-X)\cdot(-(-Y)))=-((-X)\cdot Y)=X\cdot Y.$$

(4) For $X>0$, we set $A=X^{-1}$, which means $L(A_{2n}-(X_{2n-1})^{-1})$ by Lemma 3.4. Since

$$|X_{2n}A_{2n}-X_{2n}(X_{2n-1})^{-1}| \le |X_1|\cdot|A_{2n}-(X_{2n-1})^{-1}|,$$

we obtain $L(X_{2n}A_{2n}-X_{2n}(X_{2n-1})^{-1})$. This implies $X\cdot X^{-1}=\sup(X_{2n}(X_{2n-1})^{-1})$ by Lemma 3.4. On the other hand, since

$$|X_{2n}(X_{2n-1})^{-1}-1|=|(X_{2n-1})^{-1}|\cdot|X_{2n}-X_{2n-1}|\le|X_2^{-1}|\cdot|X_{2n}-X_{2n-1}|,$$

we obtain $L(X_{2n}(X_{2n-1})^{-1}-1)$. This implies $X \cdot X^{-1} = 1$. In the case $X < 0$, by (3), we have

$$X \cdot X^{-1} = X \cdot (-((-X)^{-1})) = (-X) \cdot (-X)^{-1} = 1.$$

(5) For $X, Y, Z \geq 0$, we can prove (5) by the same argument as in the proof of Proposition 3.3 (3). By using (3), we can prove the other cases from this case. For example, in the case $X, Z \geq 0$ and $Y \leq 0$, we have

$$\begin{aligned}
(X \cdot Y) \cdot Z &= (-(X \cdot (-Y))) \cdot Z \\
&= -((X \cdot (-Y)) \cdot Z) \\
&= -(X \cdot ((-Y) \cdot Z)) \qquad (-Y > 0) \\
&= X \cdot (-((-Y) \cdot Z)) \\
&= X \cdot (Y \cdot Z).
\end{aligned}$$

(6) For $X, Y \geq 0$, we can prove (6) by the same argument as in the proof of Proposition 3.3 (5). From this case, we can also prove the other cases easily. For example, in the case $X < Y \leq 0$, by (3), we have

$$-(X \cdot Z) = (-X) \cdot Z > (-Y) \cdot Z = -(Y \cdot Z).$$

This implies $X \cdot Z < Y \cdot Z$. □

**Proposition 3.5.** *Let $X, Y, Z \in S$. We have $X \cdot (Y + Z) = X \cdot Y + X \cdot Z$.*

*Proof.* First we assume $X, Y, Z \geq 0$. Let $A = Y + Z$, $B = X \cdot Y$ and $C = X \cdot Z$. Then $L(A_{2n} - (Y_{2n} + Z_{2n}))$, $L(B_{2n} - X_{2n}Y_{2n})$ and $L(C_{2n} - X_{2n}Z_{2n})$ holds by Lemma 3.4. Since we have

$$\begin{aligned}
&|X_{2n}A_{2n} - (B_{2n} + C_{2n})| \\
&\qquad = |X_{2n}(A_{2n} - (Y_{2n} + Z_{2n})) + (X_{2n}Y_{2n} - B_{2n}) + (X_{2n}Z_{2n} - C_{2n})| \\
&\qquad \leq |X_1| \cdot |A_{2n} - (Y_{2n} + Z_{2n})| + |X_{2n}Y_{2n} - B_{2n}| + |X_{2n}Z_{2n} - C_{2n}|,
\end{aligned}$$

we obtain $L(X_{2n}A_{2n} - (B_{2n} + C_{2n}))$. This implies $\sup(X_{2n}A_{2n}) = \sup(B_{2n} + C_{2n})$ from Lemma 3.4. This implies $X \cdot (Y + Z) = X \cdot Y + X \cdot Z$.

Next we consider the case $X \geq 0$ and $Y + Z \geq 0$. Since $Y \geq 0$ or $Z \geq 0$ holds by Proposition 3.3 (5), we may assume $Z \leq 0$. Since $-Z \geq 0$, we obtain

$$X \cdot (Y + Z) + X \cdot (-Z) = X \cdot (Y + Z + (-Z)) = X \cdot Y.$$

This is equivalent to $X \cdot (Y + Z) = X \cdot Y + X \cdot Z$ by Proposition 3.4 (3).

By Proposition 3.4 (3), we can easily prove the other cases from these cases. For example, in the case $X \geq 0$ and $Y + Z \leq 0$, we have

$$X \cdot (Y + Z) = -(X \cdot ((-Y) + (-Z))) = -(X \cdot (-Y) + X \cdot (-Z)) = X \cdot Y + X \cdot Z.$$

□

By Propositions 3.2, 3.3, 3.4 and 3.5, $S$ is an ordered field. Since any ordered field which satisfies Theorem 3.1 is isomorphic to $\mathbb{R}$ (see [1]), we obtain the following theorem.

**Theorem 3.2.** *The set $S$ can be identified with the complete ordered field of real numbers.*

## 4. An application

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers. For $K \geq 1$, we define a set of sequences of positive integers

$$P(K) := \{\{a_n\} \subset \mathbb{N} \mid \exists N \in \mathbb{N}, \forall n \in \mathbb{N}[n \geq N \Rightarrow a_{n+1} \geq K a_n(a_n+1)]\}.$$

For each $\{a_n\} \in P(K)$ we define

$$f(z;\{a_n\}) = \sum_{n=1}^{\infty} \frac{z^n}{a_n},$$

which is an entire function.

The purpose of this section is to prove the following theorem by using some properties of the generalized alternating-Sylvester expansion.

**Theorem 4.1.** *Let $\{p_n\} \in P(K)$ and $l \in \{1,2,3,\ldots,[K]\}$. Then*

$$f(-l;\{p_n\}) = \sum_{n=1}^{\infty} \frac{(-1)^n}{p_n} l^n$$

*is an irrational number.*

*Proof.* We assume that $p_{n+1} \geq K p_n(p_n+1)$ for all $n \geq N$ ($N \in \mathbb{N}$). We define $a_n = p_{n+2N-1}$ and $c_n = l^{n+2N-1}$. Then we have

$$\begin{aligned} f(-l;\{p_n\}) &= \sum_{n=1}^{2N-1} \frac{(-1)^n}{p_n} l^n + \sum_{n=2N}^{\infty} \frac{(-1)^n}{p_n} l^n \\ &= \sum_{n=1}^{2N-1} \frac{(-1)^n}{p_n} l^n + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} c_n}{a_n} = A_1 + A_2, \end{aligned}$$

say. Note that $A_1 \in \mathbb{Q}$. Since we have

$$\begin{aligned} a_{n+1} = p_{n+2N} &\geq K p_{n+2N-1}(p_{n+2N-1}+1) \\ &\geq [K] a_n(a_n+1) \geq \frac{c_{n+1}}{c_n} a_n(a_n+1), \end{aligned}$$

by Definition 3.1 and Proposition 3.1, the series $A_2$ is the generalized alternating-Sylvester expansion of the number $A_2$. By Proposition 2.3 we obtain the theorem. $\square$

*Remark* 4.1. We cannot obtain Theorem 4.1 by using the alternating-Sylvester series. For example, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^n}{3^{3^n}}$$

is the generalized alternating-Sylvester series, but is not the alternating Sylvester series.

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