Nihonkai Math. J.
Vol.25(2014), 1–26

REMARKS ON SOME ALMOST HERMITIAN STRUCTURE ON THE TANGENT BUNDLE, II

TAKASHI OGURO AND YOSHITSUGU TAKASHIMA

Abstract. Tahara, Vanhecke and Watanabe constructed a family of almost Hermitian structure (J_1, G_1) and two almost complex structures J_2, J_3 on the tangent bundle TM over an almost Hermitian manifold M. In this paper, we define Riemannian metrics G_2 and G_3 on TM which make TM an almost Hermitian manifold and determine the conditions for (J_i, G_i) $(i = 1, 2, 3)$ so that (TM, J_i, G_i) belongs to each of the sixteen classes established by Gray and Hervella.

1. Introduction

Let $\pi : TM \to M$ be the tangent bundle over a Riemannian manifold M endowed with a Riemannian metric g. The tangent space T_uTM of TM at each point $u \in TM$ has a direct sum decomposition of the vertical subspace $V_u = \ker(d\pi)_u$ and the horizontal subspace H_u with respect to the Riemannian connection of g. The vertical subspace V_u can be naturally identified with the tangent space $T_{\pi(u)}M$ of M at $\pi(u) \in M$. For each tangent vector $X \in T_pM$ and a point $u \in TM$ with $\pi(u) = p$, there exists a unique tangent vector $X^H_u \in H_u$ (resp. $X^V_u \in V_u$), called the horizontal lift (resp. the vertical lift) of X, such that $d\pi(X^H_u) = X$ (resp. $X^V_u = X$ under the natural identification). Tangent bundle TM admits a natural almost Hermitian structure (actually, an almost Kähler structure) (J_0, G_0), that is,

$$G_0(X^H_u, Y^H_u) = G_0(X^V_u, Y^V_u) = g(X, Y), \quad G_0(X^H_u, Y^V_u) = 0,$$
$$J_0 X^H_u = X^V_u, \quad J_0 X^V_u = -X^H_u,$$

for $X, Y \in T_{\pi(u)}M$. This is perhaps the most natural almost Hermitian structure on TM. It is well-known that (TM, J_0, G_0) is a Kähler manifold if and only if M is locally flat ([1]). Therefore, this natural structure seems extremely rigid and many almost Hermitian structures on TM have been constructed from various points of view by many authors.

2010 *Mathematics Subject Classification.* 53C15, 53C55, 53C56.
Key words and phrases. Almost Hermitian manifold, Tangent bundle.

In [4], Tahara, Vanhecke and Watanabe constructed a family of almost Hermitian structure (J_1, G_1) and two almost complex structures J_2, J_3 on the tangent bundle TM over an almost Hermitian manifold. The almost Hermitian structure (J_1, G_1) is an extension of (J, G) argued in [3] and [5]. In this paper, we define Riemannian metrics G_2 and G_3 on TM which make TM an almost Hermitian manifold and determine the conditions for (J_i, G_i) $(i = 1, 2, 3)$ so that (TM, J_i, G_i) belongs to each of the well-known sixteen classes by Gray-Hervella ([2]).

2. Preliminaries

In this paper, we will use the same symbols used in [3]. Let $M = (M, J, g)$ be an almost Hermitian manifold of dimension $2n$. We assume $\dim M \geq 4$. Let X, Y be any vector fields of M. At each point $u \in TM$, we have

$$[X^V, Y^V]_u = 0, \tag{2.1}$$

$$[X^H, Y^V]_u = (\nabla_X Y)^V_u, \tag{2.2}$$

$$d\pi([X^H, Y^H]_u) = [X, Y]_{\pi(u)}, \tag{2.3}$$

$$K([X^H, Y^H]_u) = -R(X_{\pi(u)}, Y_{\pi(u)})u, \tag{2.4}$$

where K is the connection map $K : TTM \to TM$ which maps $A \in TTM$ to its vertical component A^V and R is the Riemannian curvature tensor of M defined by

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z.$$

Let ρ^* be the Ricci $*$-tensor defined by

$$\rho^*(x, y) = \frac{1}{2}\mathrm{trace}(z \mapsto R(x, Jy)Jz).$$

Ricci $*$-tensor is not symmetric but satisfies $\rho^*(JX, JY) = \rho^*(Y, X)$. If M is a Kähler manifold, ρ^* coincides with the Riemannian Ricci tensor.

The sectional curvature $H(u)$ of the plane in $T_{\pi(u)}M$ determined by u and Ju is called the holomorphic sectional curvature. If $H(u)$ is a constant for all $u \in T_pM$ and all $p \in M$, M is called a space of constant holomorphic sectional curvature. It is well known that the curvature tensor R of a Kähler manifold of constant holomorphic sectional curvature $H(u) = 4c$ is of the form

$$\begin{aligned} R(X, Y, Z, W) &= g(R(X, Y)Z, W) \\ &= c\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W) + g(X, JW)g(Y, JZ) \\ &\qquad - g(X, JZ)g(Y, JW) - 2g(X, JY)g(Z, JW)\}. \end{aligned}$$

Next, we recall the sixteen classes of almost Hermitian manifolds established in [2]. We denote by $\mathscr{W}$ the set of all almost Hermitian manifolds of dimension $2n$ (≥ 6).

Let Ω be the Kähler form of $M = (M, J, g) \in \mathscr{W}$, where Ω is defined by

$$\Omega(X, Y) = g(X, JY).$$

Making use of the invariant subspaces $\mathscr{W}_1, \ldots, \mathscr{W}_4$ of the unitary representation, we can classify $\mathscr{W}$ into following sixteen classes.

(1) $\mathscr{K}$ = Kähler manifolds: $\nabla\Omega = 0$.

(2) $\mathscr{W}_1 = \mathscr{NK}$ = nearly Kähler manifolds: $(\nabla_X\Omega)(X, Y) = 0$.

(3) $\mathscr{W}_2 = \mathscr{AK}$ = almost Kähler manifolds: $d\Omega = 0$.

(4) $\mathscr{W}_3 = \mathscr{H} \cap \mathscr{SK}$ = Hermitian semi-Kähler manifolds:

$$(\nabla_X\Omega)(Y, Z) - (\nabla_{JX}\Omega)(JY, Z) = \delta\Omega = 0.$$

(5) $\mathscr{W}_4$:

$$\begin{aligned}(\nabla_X\Omega)(Y, Z) = -\frac{1}{2(n-1)}\{&g(X, Y)\delta\Omega(Z) - g(X, Z)\delta\Omega(Y)\\ &- g(X, JY)\delta\Omega(JZ) + g(X, JZ)\delta\Omega(JY)\}.\end{aligned}$$

(6) $\mathscr{W}_1 \cup \mathscr{W}_2 = \mathscr{QK}$ = quasi-Kähler manifolds:

$$(\nabla_X\Omega)(Y, Z) + (\nabla_{JX}\Omega)(JY, Z) = 0.$$

(7) $\mathscr{W}_1 \cup \mathscr{W}_3$:

$$(\nabla_X\Omega)(X, Y) - (\nabla_{JX}\Omega)(JX, Y) = \delta\Omega = 0.$$

(8) $\mathscr{W}_1 \cup \mathscr{W}_4$:

$$(\nabla_X\Omega)(X, Y) = -\frac{1}{2(n-1)}\{\|X\|^2\delta\Omega(Y) - g(X, Y)\delta\Omega(X) - g(JX, Y)\delta\Omega(JX)\}.$$

(9) $\mathscr{W}_2 \cup \mathscr{W}_3$:

$$\underset{X,Y,Z}{\mathfrak{S}}\{(\nabla_X\Omega)(Y, Z) - (\nabla_{JX}\Omega)(JY, Z)\} = \delta\Omega = 0,$$

where $\mathfrak{S}$ denotes the cyclic sum.

(10) $\mathscr{W}_2 \cup \mathscr{W}_4$:

$$\underset{X,Y,Z}{\mathfrak{S}}\{(\nabla_X\Omega)(Y, Z) - g(X, JY)\delta\Omega(JZ)/(n-1)\} = 0.$$

(11) $\mathscr{W}_3 \cup \mathscr{W}_4 = \mathscr{H}$ = Hermitian manifolds:

$$(\nabla_X\Omega)(Y, Z) - (\nabla_{JX}\Omega)(JY, Z) = 0.$$

(12) $\mathscr{W}_1 \cup \mathscr{W}_2 \cup \mathscr{W}_3 = \mathscr{SK}$ = semi-Kähler manifolds: $\delta\Omega = 0$.

(13) $\mathscr{W}_1 \cup \mathscr{W}_2 \cup \mathscr{W}_4$:

$$\begin{aligned}(\nabla_X\Omega)(Y, Z) + (\nabla_{JX}\Omega)(JY, Z) = -\frac{1}{n-1}\{&g(X, Y)\delta\Omega(Z)\\ - g(X, Z)\delta\Omega(Y) - g(X, JY)\delta\Omega(JZ) &+ g(X, JZ)\delta\Omega(JY)\}.\end{aligned}$$

(14) $\mathscr{W}_1 \cup \mathscr{W}_3 \cup \mathscr{W}_4$:

$$(\nabla_X \Omega)(X, Y) - (\nabla_{JX} \Omega)(JX, Y) = 0.$$

(15) $\mathscr{W}_2 \cup \mathscr{W}_3 \cup \mathscr{W}_4$:

$$\underset{X,Y,Z}{\mathfrak{S}} \{(\nabla_X \Omega)(Y, Z) - (\nabla_{JX} \Omega)(JY, Z)\} = 0.$$

(16) $\mathscr{W}$ = almost Hermitian manifolds: No condition.

3. Almost Hermitian structures on TM

First, we introduce the almost Hermitian structure (J_1, G_1) and almost complex structures J_2, J_3 on TM over an almost Hermitian manifold $M = (M, J, g)$ constructed in [4].

Let $f_1, h_1, k_1 : [0, \infty) \to \mathbb{R}$ be positive C^∞-functions such that $(f_1 - h_1)/t$ and $(f_1 - k_1)/t$ are C^∞ at $t = 0$. Moreover, let $\alpha_1, \beta_1, \gamma_1 : [0, \infty) \to \mathbb{R}$ be C^∞-functions satisfying $\alpha_1 > 0$, $\alpha_1 + t\beta_1 > 0$, $\alpha_1 + t\gamma_1 > 0$. We define an almost complex structure J_1 and a Riemannian metric G_1 on TM respectively by

$$\begin{cases} J_1 X_u^H = f_1\, X_u^V + \dfrac{h_1 - f_1}{t} g(X, u) u_u^V + \dfrac{k_1 - f_1}{t} g(X, Ju)(Ju)_u^V, \\ J_1 X_u^V = -\dfrac{1}{f_1}\, X_u^H + \dfrac{h_1 - f_1}{t\, f_1 h_1} g(X, u) u_u^H + \dfrac{k_1 - f_1}{t\, f_1 k_1} g(X, Ju)(Ju)_u^H, \end{cases}$$

$$\begin{cases} G_1(X_u^H, Y_u^H) = \alpha_1 g(X, Y) + \beta_1 g(X, u) g(Y, u) + \gamma_1 g(X, Ju) g(Y, Ju), \\ G_1(X_u^V, Y_u^V) = \varphi_1 g(X, Y) + \psi_1 g(X, u) g(Y, u) + \xi_1 g(X, Ju) g(Y, Ju), \\ G_1(X_u^H, Y_u^V) = 0, \end{cases}$$

for $u \in TM$ and $X, Y \in T_{\pi(u)}M$, where $t = \|u\|^2$ and

$$\varphi_1 = \frac{\alpha_1}{f_1^2}, \quad \psi_1 = \frac{\alpha_1(f_1^2 - h_1^2) + t f_1^2 \beta_1}{t f_1^2 h_1^2}, \quad \xi_1 = \frac{\alpha_1(f_1^2 - k_1^2) + t f_1^2 \gamma_1}{t f_1^2 k_1^2}.$$

We may assume that $k_1 - f_1 \neq 0$. Note that (J_1, G_1) coincide with (J, G) in [3] and [5] if $k_1 - f_1 = \gamma_1 = 0$.

The almost complex structure J_2 is defined by

$$\begin{cases} J_2 X_u^H = f_2 (JX)_u^V - \dfrac{k_2 - f_2}{t} g(X, Ju) u_u^V + \dfrac{h_2 - f_2}{t} g(X, u)(Ju)_u^V, \\ J_2 X_u^V = \dfrac{1}{f_2} (JX)_u^H + \dfrac{h_2 - f_2}{t f_2 h_2} g(X, Ju) u_u^H - \dfrac{k_2 - f_2}{t f_2 k_2} g(X, u)(Ju)_u^H, \end{cases}$$

where $f_2, h_2, k_2 : [0, \infty) \to \mathbb{R}$ are positive C^∞-functions such that $(h_2 - f_2)/t$ and $(k_2 - f_2)/t$ are C^∞ at $t = 0$.

The almost complex structure J_3 is defined by

$$\begin{cases} J_3 X_u^H = \lambda (JX)_u^H + \dfrac{h_3 - \lambda}{\lambda t h_3} g(X, Ju) u_u^H + \dfrac{h_3 - \lambda}{t} g(X, u)(Ju)_u^H, \\ J_3 X_u^V = \mu (JX)_u^V + \dfrac{k_3 - \mu}{\mu t k_3} g(X, Ju) u_u^V + \dfrac{k_3 - \mu}{t} g(X, u)(Ju)_u^V, \end{cases}$$

where $\lambda = \pm 1$, $\mu = \pm 1$ and $h_3, k_3 : [0, \infty) \to \mathbb{R}$ are nowhere-zero C^∞-functions such that $(h_3 - \lambda)/t$ and $(k_3 - \mu)/t$ are C^∞ at $t = 0$. The almost complex structure J_3 preserves the horizontal and the vertical subspaces respectively. We may assume that $h_3 - \lambda \neq 0$.

The triple $\{J_1, J_2, J_3\}$ defines an almost hyper-complex structure on TM if

$$f_1 = f_2, \quad h_1 k_1 = h_2 k_2, \quad \lambda = -1, \quad \mu = 1, \quad h_3 = -\frac{h_1}{k_2}, \quad k_3 = \frac{k_1}{k_2}.$$

Namely, the equalities

$$J_1 J_2 = J_3, \quad J_2 J_3 = J_1, \quad J_3 J_1 = J_2$$

hold. See [4] for more details.

Now, we give Riemannian metrics G_i $(i = 2, 3)$ which makes (J_i, G_i) an almost Hermitian structure on TM. We define G_i $(i = 2, 3)$ by

$$\begin{cases} G_i(X_u^H, Y_u^H) = \alpha_i g(X, Y) + \beta_i g(X, u) g(Y, u) + \gamma_i g(X, Ju) g(Y, Ju), \\ G_i(X_u^V, Y_u^V) = \varphi_i g(X, Y) + \psi_i g(X, u) g(Y, u) + \xi_i g(X, Ju) g(Y, Ju), \\ G_i(X_u^H, Y_u^V) = 0. \end{cases}$$

For G_2, we assume that $\alpha_2, \beta_2, \gamma_2 : [0, \infty) \to \mathbb{R}$ are C^∞-functions satisfying $\alpha_2 > 0$, $\alpha_2 + t\beta_2 > 0$, $\alpha_2 + t\gamma_2 > 0$ and put

$$\varphi_2 = \frac{\alpha_2}{f_2^2}, \quad \psi_2 = \frac{\alpha_2(f_2^2 - k_2^2) + t f_2^2 \gamma_2}{t f_2^2 k_2^2}, \quad \xi_2 = \frac{\alpha_2(f_2^2 - h_2^2) + t f_2^2 \beta_2}{t f_2^2 h_2^2}.$$

For G_3, we assume that $\alpha_3, \beta_3, \gamma_3, \varphi_3, \psi_3, \xi_3 : [0, \infty) \to \mathbb{R}$ are C^∞-functions satisfying $\alpha_3 > 0$, $\alpha_3 + t\beta_3 > 0$, $\alpha_3 + t\gamma_3 > 0$, $\varphi_3 > 0$, $\varphi_3 + t\psi_3 > 0$, $\varphi_3 + t\xi_3 > 0$ and

$$\alpha_3 + t\beta_3 = h_3^2(\alpha_3 + t\gamma_3), \quad \varphi_3 + t\psi_3 = k_3^2(\varphi_3 + t\xi_3).$$

It is easy to verify that (J_2, G_2) and (J_3, G_3) are almost Hermitian structures on TM.

We denote by $\nabla^{(i)}$ $(i = 1, 2, 3)$ the Riemannian connection with respect to G_i. Then, by direct computations, we have

$$\begin{aligned} &G_i(\nabla^{(i)}_{X_u^H} Y^H, Z_u^H) \qquad (3.1) \\ &= \alpha_i g(\nabla_X Y, Z) + \beta_i g(\nabla_X Y, u) g(Z, u) + \gamma_i g(\nabla_X Y, Ju) g(Z, Ju) \\ &\quad - \frac{\gamma_i}{2} \{ g((\nabla_X J)Y, u) g(Z, Ju) + g((\nabla_X J)Z, u) g(Y, Ju) \end{aligned}$$

$$
\begin{aligned}
&\quad + g((\nabla_Y J)X, u)g(Z, Ju) + g((\nabla_Y J)Z, u)g(X, Ju) \\
&\quad - g((\nabla_Z J)X, u)g(Y, Ju) - g((\nabla_Z J)Y, u)g(X, Ju)\},
\end{aligned}
$$

$$
\begin{aligned}
&G_i(\nabla^{(i)}_{X^H_u} Y^H, Z^V_u) \qquad (3.2)\\
&= -\{\alpha_i' g(X, Y) + \beta_i' g(X, u)g(Y, u) + \gamma_i' g(X, Ju)g(Y, Ju)\}g(Z, u) \\
&\quad - \frac{\beta_i}{2}\{g(X, Z)g(Y, u) + g(Y, Z)g(X, u)\} \\
&\quad - \frac{\gamma_i}{2}\{g(X, JZ)g(Y, Ju) + g(Y, JZ)g(X, Ju)\} \\
&\quad - \frac{\varphi_i}{2} g(R(X, Y)u, Z) - \frac{\xi_i}{2} g(R(X, Y)u, Ju)g(Z, Ju),
\end{aligned}
$$

$$
\begin{aligned}
&G_i(\nabla^{(i)}_{X^H_u} Y^V, Z^V_u) \qquad (3.3)\\
&= \varphi_i g(\nabla_X Y, Z) + \psi_i g(\nabla_X Y, u)g(Z, u) + \xi_i g(\nabla_X Y, Ju)g(Z, Ju) \\
&\quad - \frac{\xi_i}{2}\{g((\nabla_X J)Y, u)g(Z, Ju) + g((\nabla_X J)Z, u)g(Y, Ju)\},
\end{aligned}
$$

$$
G_i(\nabla^{(i)}_{X^V_u} Y^H, Z^H_u) = -G_i(\nabla^{(i)}_{Y^H_u} Z^H, X^V_u), \qquad (3.4)
$$

$$
\begin{aligned}
&G_i(\nabla^{(i)}_{X^V_u} Y^H, Z^V_u) \qquad (3.5)\\
&= -\frac{\xi_i}{2}\{g((\nabla_Y J)X, u)g(Z, Ju) + g((\nabla_Y J)Z, u)g(X, Ju)\},
\end{aligned}
$$

$$
\begin{aligned}
&G_i(\nabla^{(i)}_{X^V_u} Y^V, Z^V_u) \qquad (3.6)\\
&= \varphi_i'\{g(X, u)g(Y, Z) + g(Y, u)g(X, Z) - g(Z, u)g(X, Y)\} \\
&\quad + \psi_i' g(X, u)g(Y, u)g(Z, u) \\
&\quad + \xi_i'\{g(X, u)g(Y, Ju)g(Z, Ju) + g(X, Ju)g(Y, u)g(Z, Ju) \\
&\qquad - g(X, Ju)g(Y, Ju)g(Z, u)\} \\
&\quad + \psi_i g(X, Y)g(Z, u) \\
&\quad - \xi_i\{g(X, JZ)g(Y, Ju) + g(Y, JZ)g(X, Ju)\}.
\end{aligned}
$$

4. Almost Hermitian structure (J_1, G_1)

For simplicity, we put

$$
\begin{aligned}
A_1 &= \frac{\alpha_1(f_1 - h_1) + tf_1\beta_1}{tf_1h_1}, \\
B_1 &= \frac{\alpha_1(f_1 - k_1) + tf_1\gamma_1}{tf_1k_1}, \\
C_1 &= \gamma_1' h_1 + \frac{\gamma_1(h_1 - f_1)}{t} - \frac{(\beta_1 - \gamma_1)(k_1 - f_1)}{2t},
\end{aligned}
$$

$$
\begin{aligned}
D_1 &= \frac{\gamma_1'}{k_1} - \frac{\alpha_1'(k_1 - f_1)}{tf_1k_1} - \frac{\gamma_1(h_1 - f_1)}{2tf_1h_1} - \frac{\gamma_1(k_1 - f_1)}{2tf_1k_1}, \\
E_1 &= 2A_1' + \frac{\alpha_1'(h_1 - f_1)}{tf_1h_1} - \frac{\beta_1'}{h_1} + \frac{\beta_1(h_1 - f_1)}{2tf_1h_1} - \frac{2\varphi_1'(h_1 - f_1)}{t} - h_1\psi_1', \\
F_1 &= 2B_1' + \frac{\alpha_1'(k_1 - f_1)}{tf_1k_1} - \frac{\gamma_1'}{k_1} + \frac{\gamma_1(k_1 - f_1)}{2tf_1k_1} - \frac{(\varphi_1' + \xi_1)(k_1 - f_1)}{t} - k_1\xi_1', \\
H_1 &= \frac{\alpha_1(h_1 - f_1)}{tf_1^2} - \frac{\beta_1}{2f_1} + h_1\varphi_1', \\
I_1 &= \alpha_1'h_1 - \frac{\beta_1 f_1}{2}, \\
K_1 &= \frac{\beta_1(k_1 - f_1)}{2tf_1k_1} + \frac{2\xi_1(h_1 - f_1)}{t} + h_1\xi_1', \\
L_1 &= \frac{\gamma_1(h_1 - f_1)}{2tf_1h_1} + \frac{(\varphi_1' + \xi_1)(k_1 - f_1)}{t} + k_1\xi_1', \\
N_1 &= A_1 - \frac{2\alpha_1}{f_1}\left(\log\frac{\alpha_1}{f_1}\right)', \\
R_1 &= 2h_1\left(\log\frac{\alpha_1 + t\gamma_1}{k_1}\right)' - \frac{k_1(\alpha_1 + t\beta_1)}{t(\alpha_1 + t\gamma_1)} + \frac{h_1}{t}, \\
S_1 &= 2h_1\left(\log\frac{\alpha_1}{f_1}\right)' - \frac{\beta_1 f_1}{\alpha_1} + \frac{h_1 - f_1}{t}.
\end{aligned}
$$

We denote by Ω_1 the Kähler form of (J_1, G_1). By direct computations, which we omit here, we have

$$X_u^H\Omega_1(Y^H, Z^H) = X_u^V\Omega_1(Y^H, Z^H) = 0, \tag{4.1}$$

$$
\begin{aligned}
&X_u^H\Omega_1(Y^H, Z^V) \\
&= -f_1\varphi_1\{g(\nabla_X Y, Z) + g(\nabla_X Z, Y)\} \\
&\quad - A_1\{g(\nabla_X Y, u)g(Z, u) + g(\nabla_X Z, u)g(Y, u)\} \\
&\quad - B_1\{g(\nabla_X Y, Ju)g(Z, Ju) + g(\nabla_X Z, Ju)g(Y, Ju)\} \\
&\quad + B_1\{g((\nabla_X J)Y, u)g(Z, Ju) + g((\nabla_X J)Z, u)g(Y, Ju)\},
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
&X_u^V\Omega_1(Y^H, Z^V) \\
&= -2(f_1\varphi_1)'g(X, u)g(Y, Z) \\
&\quad - 2A_1'g(X, u)g(Y, u)g(Z, u) - 2B_1'g(X, u)g(Y, Ju)g(Z, Ju) \\
&\quad - A_1\{g(X, Y)g(Z, u) + g(X, Z)g(Y, u)\} \\
&\quad + B_1\{g(X, JY)g(Z, Ju) + g(X, JZ)g(Y, Ju)\},
\end{aligned}
\tag{4.3}
$$

$$X_u^H\Omega_1(Y^V, Z^V) = X_u^V\Omega_1(Y^V, Z^V) = 0. \tag{4.4}$$

Making use of (3.1)–(3.6) and (4.1)–(4.4), we can obtain the covariant derivative $\nabla^{(1)}\Omega_1$. For instance,

$$
\begin{aligned}
&(\nabla^{(1)}_{X^H_u}\Omega_1)(Y^H_u, Z^H_u) &(4.5)\\
&= X^H_u\Omega_1(Y^H, Z^H) - G_1(\nabla^{(1)}_{X^H_u}Y^H, J_1Z^H_u) + G_1(\nabla^{(1)}_{X^H_u}Z^H, J_1Y^H_u)\\
&= I_1\{g(X,Y)g(Z,u) - g(X,Z)g(Y,u)\}\\
&\quad - \frac{\gamma_1 f_1}{2}\{g(X,JY)g(Z,Ju) - g(X,JZ)g(Y,Ju) - 2g(Y,JZ)g(X,Ju)\}\\
&\quad - C_1 g(X,Ju)\{g(Y,u)g(Z,Ju) - g(Y,Ju)g(Z,u)\} + \frac{f_1\varphi_1}{2}g(R(X,u)Y,Z)\\
&\quad + \frac{B_1}{2}\{R(X,Y,u,Ju)g(Z,Ju) - R(X,Z,u,Ju)g(Y,Ju)\}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&(\nabla^{(1)}_{X^H_u}\Omega_1)(Y^H_u, Z^V_u) &(4.6)\\
&= \frac{B_1(k_1-f_1)}{2k_1}g((\nabla_X J)Y,u)g(Z,Ju) - \frac{B_1(k_1-f_1)}{2f_1}g((\nabla_X J)Z,u)g(Y,Ju)\\
&\quad - \frac{\gamma_1}{2k_1}g((\nabla_Y J)X,u)g(Z,Ju) - \frac{\gamma_1}{2f_1}g((\nabla_Y J)Z,u)g(X,Ju)\\
&\quad + \frac{\gamma_1}{2f_1}g((\nabla_Z J)X,u)g(Y,Ju) + \frac{\gamma_1}{2f_1}g((\nabla_Z J)Y,u)g(X,Ju)\\
&\quad - \frac{\gamma_1(h_1-f_1)}{2tf_1h_1}g((\nabla_u J)X,u)g(Y,Ju)g(Z,u)\\
&\quad - \frac{\gamma_1(h_1-f_1)}{2tf_1h_1}g((\nabla_u J)Y,u)g(X,Ju)g(Z,u)\\
&\quad - \frac{\gamma_1(k_1-f_1)}{2tf_1k_1}g((\nabla_{Ju} J)X,u)g(Y,Ju)g(Z,Ju)\\
&\quad - \frac{\gamma_1(k_1-f_1)}{2tf_1k_1}g((\nabla_{Ju} J)Y,u)g(X,Ju)g(Z,Ju),\\
&(\nabla^{(1)}_{X^H_u}\Omega_1)(Y^V_u, Z^V_u) &(4.7)\\
&= -I_1\{g(X,Y)g(Z,u) - g(X,Z)g(Y,u)\}\\
&\quad + \frac{\gamma_1}{2k_1}\{g(X,JY)g(Z,Ju) - g(X,JZ)g(Y,Ju)\}\\
&\quad + D_1 g(X,Ju)\{g(Y,u)g(Z,Ju) - g(Y,Ju)g(Z,u)\}\\
&\quad - \frac{\gamma_1}{f_1}g(X,Ju)g(Y,JZ) - \frac{\varphi_1}{2f_1}R(X,u,Y,Z)\\
&\quad - \frac{\xi_1}{2f_1}\{R(X,Y,u,Ju)g(Z,Ju) - R(X,Z,u,Ju)g(Y,Ju)\}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\varphi_1(h_1 - f_1)}{2tf_1h_1}\{R(X,u,u,Y)g(Z,u) - R(X,u,u,Z)g(Y,u)\} \\
&+ \frac{\xi_1(h_1 - f_1)}{2tf_1h_1}R(X,u,u,Ju)\{g(Y,u)g(Z,Ju) - g(Y,Ju)g(Z,u)\} \\
&- \frac{\varphi_1(k_1 - f_1)}{2tf_1k_1}\{R(X,Ju,u,Y)g(Z,Ju) - R(X,Ju,u,Z)g(Y,Ju)\},
\end{aligned}
$$

$$
\begin{aligned}
&(\nabla^{(1)}_{X^V_u}\Omega_1)(Y^H_u, Z^H_u) \qquad (4.8)\\
&= \frac{f_1\xi_1}{2}g(X,Ju)\{g((\nabla_Y J)Z,u) - g((\nabla_Z J)Y,u)\} \\
&+ \frac{k_1\xi_1}{2}\{g((\nabla_Y J)X,u)g(Z,Ju) - g((\nabla_Z J)X,u)g(Y,Ju)\},
\end{aligned}
$$

$$
\begin{aligned}
&(\nabla^{(1)}_{X^V_u}\Omega_1)(Y^H_u, Z^V_u) \qquad (4.9)\\
&= \frac{f_1H_1}{h_1}g(X,Y)g(Z,u) - H_1g(X,Z)g(Y,u) \\
&- \frac{2f_1B_1 - \gamma_1}{2k_1}g(X,JY)g(Z,Ju) + \frac{2f_1B_1 - \gamma_1}{2f_1}g(X,JZ)g(Y,Ju) \\
&+ f_1\xi_1 g(X,Ju)g(Y,JZ) \\
&- E_1g(X,u)g(Y,u)g(Z,u) - F_1g(X,u)g(Y,Ju)g(Z,Ju) \\
&- K_1g(X,Ju)g(Y,u)g(Z,Ju) + L_1g(X,Ju)g(Y,Ju)g(Z,u) \\
&+ \frac{\varphi_1}{2f_1}R(Y,Z,u,X) + \frac{\xi_1}{2f_1}R(Y,Z,u,Ju)g(X,Ju) \\
&- \frac{\varphi_1(h_1 - f_1)}{2tf_1h_1}R(X,u,u,Y)g(Z,u) \\
&+ \frac{\varphi_1(k_1 - f_1)}{2tf_1k_1}R(X,u,Y,Ju)g(Z,Ju) \\
&- \frac{\xi_1(h_1 - f_1)}{2tf_1h_1}R(Y,u,u,Ju)g(X,Ju)g(Z,u) \\
&- \frac{\xi_1(k_1 - f_1)}{2tf_1k_1}R(Y,Ju,u,Ju)g(X,Ju)g(Z,Ju),
\end{aligned}
$$

$$
\begin{aligned}
&(\nabla^{(1)}_{X^V_u}\Omega_1)(Y^V_u, Z^V_u) \qquad (4.10)\\
&= \frac{\xi_1}{2f_1}\{g((\nabla_Z J)Y,u)g(X,Ju) - g((\nabla_Y J)Z,u)g(X,Ju)\} \\
&+ \frac{\xi_1}{2f_1}\{g((\nabla_Z J)X,u)g(Y,Ju) - g((\nabla_Y J)X,u)g(Z,Ju)\} \\
&+ \frac{\xi_1(h_1 - f_1)}{2tf_1h_1}g((\nabla_u J)X,u)\{g(Y,u)g(Z,Ju) - g(Y,Ju)g(Z,u)\}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\xi_1(h_1 - f_1)}{2tf_1h_1} g((\nabla_u J)Y, u) g(X, Ju) g(Z, u) \\
&+ \frac{\xi_1(h_1 - f_1)}{2tf_1h_1} g((\nabla_u J)Z, u) g(X, Ju) g(Y, u) \\
&- \frac{\xi_1(k_1 - f_1)}{2tf_1k_1} g((\nabla_{Ju} J)Y, u) g(X, Ju) g(Z, Ju) \\
&+ \frac{\xi_1(k_1 - f_1)}{2tf_1k_1} g((\nabla_{Ju} J)Z, u) g(X, Ju) g(Y, Ju).
\end{aligned}
$$

Making use of (4.5)–(4.10), we can obtain the exterior derivative $d\Omega_1$. For instance,

$$
\begin{aligned}
d\Omega_1(X_u^H, Y_u^H, Z_u^H) &= \underset{X,Y,Z}{\mathfrak{S}} (\nabla^{(1)}_{X_u^H} \Omega_1)(Y_u^H, Z_u^H) \\
&= B_1 \underset{X,Y,Z}{\mathfrak{S}} R(X, Y, u, Ju) g(Z, Ju).
\end{aligned} \tag{4.11}
$$

Similarly, we have

$$
\begin{aligned}
&d\Omega_1(X_u^H, Y_u^H, Z_u^V) \\
&= B_1\{g((\nabla_X J)Y, u) - g((\nabla_Y J)X, u)\} g(Z, Ju) \\
&\quad + B_1\{g((\nabla_X J)Z, u) g(Y, Ju) - g((\nabla_Y J)Z, u) g(X, Ju)\},
\end{aligned} \tag{4.12}
$$

$$
\begin{aligned}
&d\Omega_1(X_u^H, Y_u^V, Z_u^V) \\
&= N_1\{g(X, Y) g(Z, u) - g(X, Z) g(Y, u)\} \\
&\quad + B_1\{g(X, JY) g(Z, Ju) - g(X, JZ) g(Y, Ju)\} \\
&\quad + 2B_1' g(X, Ju)\{g(Y, u) g(Z, Ju) - g(Y, Ju) g(Z, u)\} \\
&\quad - 2B_1 g(X, Ju) g(Y, JZ),
\end{aligned} \tag{4.13}
$$

$$
d\Omega_1(X_u^V, Y_u^V, Z_u^V) = 0. \tag{4.14}
$$

To compute the co-derivative $\delta\Omega_1$, we first choose the orthonormal basis of $T_{\pi(u)}M$ of the form $\{e_1 = u/\sqrt{t}, e_2 = Je_1, e_3, e_4 = Je_3, \ldots, e_{2n-1}, e_{2n} = Je_{2n-1}\}$ and put

$$
E_1 = \frac{1}{\sqrt{\alpha_1 + t\beta_1}} (e_1)_u^H, \quad E_2 = \frac{1}{\sqrt{\alpha_1 + t\gamma_1}} (e_2)_u^H, \quad E_i = \frac{1}{\sqrt{\alpha_1}} (e_i)_u^H,
$$

$$
E_{2n+1} = \frac{h_1}{\sqrt{\alpha_1 + t\beta_1}} (e_1)_u^V, \quad E_{2n+2} = \frac{k_1}{\sqrt{\alpha_1 + t\gamma_1}} (e_2)_u^V, \quad E_{2n+i} = \frac{f_1}{\sqrt{\alpha_1}} (e_i)_u^V,
$$

for $i = 3, 4, \ldots, 2n$. Then $\{E_i\}_{i=1,\ldots,4n}$ forms an orthonormal basis of T_uTM. In case $t = 0$ (or $u = 0$), we choose any orthonormal basis $\{e_i\}_{i=1,\ldots,2n}$ of $T_{\pi(u)}M$ and put

$$
E_i = \frac{1}{\sqrt{\alpha_1(0)}} (e_i)_0^H, \quad E_{2n+i} = \frac{f_1(0)}{\sqrt{\alpha_1(0)}} (e_i)_0^V
$$

for $i = 1, 2, \ldots, 2n$. Then $\{E_i\}_{i=1,\ldots,4n}$ forms an orthonormal basis of T_0TM. Taking account of the assumptions for f_1, h_1 and k_1, we do not need to distinguish the case between $t \neq 0$ and $t = 0$. Compute

$$\delta\Omega_1(X) = -\sum_{i=1}^{4n} (\nabla^{(1)}_{E_i}\Omega_1)(E_i, X)$$

directly, we obtain

$$\delta\Omega_1(X^H_u) = -\{R_1 + 2(n-1)S_1\}g(X,u), \tag{4.15}$$

$$\delta\Omega_1(X^V_u) = \frac{f_1 B_1}{\alpha_1 k_1} g(X, Ju)\delta\Omega(u) - \frac{k_1 B_1}{f_1(\alpha_1 + t\gamma_1)} g((\nabla_{Ju}J)u, X). \tag{4.16}$$

Now, for a constant c, we consider next conditions.

$P^{(1)}_1$: $M = (M, J, g)$ is a Kähler manifold.

$P^{(1)}_2$: M is a space of constant holomorphic sectional curvature $4c$.

$P^{(1)}_3$: $c\alpha_1 - \gamma_1 f_1^2 = 0$.

$P^{(1)}_4$: $B_1 = 0$.

$P^{(1)}_5$: $2h_1(\log\alpha_1)' - \dfrac{\beta_1 f_1}{\alpha_1} - \dfrac{c}{f_1} = 0$.

$P^{(1)}_6$: $2h_1(\log f_1)' - \dfrac{h_1 - f_1}{t} - \dfrac{c}{f_1} = 0$.

$P^{(1)}_7$: $S_1 = 0$.

$P^{(1)}_8$: $f_1(f_1 - k_1) + ct = 0$.

$P^{(1)}_9$: M is a nearly Kähler manifold.

$P^{(1)}_{10}$: $R_1 + 2(n-1)S_1 = 0$.

$P^{(1)}_{11}$: $T^{(1)}_1(u, X, Y, Z) = T^{(1)}_2(u, X, Y, Z) = 0$ for any u, X, Y, Z,

where, $T^{(1)}_1$ and $T^{(1)}_2$ are defined respectively by

$$\begin{aligned}
&T^{(1)}_1(u, X, Y, Z) \\
&= (k_1 - f_1)\{g(Y, JZ)g(X, Ju) - g(X, JZ)g(Y, Ju)\} \\
&\quad + \frac{(k_1 - f_1)^2}{tk_1} g(Z, Ju)\{g(X, Ju)g(Y, u) - g(Y, Ju)g(X, u)\} \\
&\quad + \frac{(k_1 - f_1)f_1 B_1}{k_1} g(X, JY)g(Z, Ju) - \frac{1}{2k_1} R(X, Y, u, Ju)g(Z, Ju) \\
&\quad - \frac{1}{2f_1}\{R(Z, X, u, Ju)g(Y, Ju) - R(Z, Y, u, Ju)g(X, Ju)\} \\
&\quad - \frac{h_1 - f_1}{2tf_1h_1} g(Z, u)\{R(X, u, u, Ju)g(Y, Ju) - R(Y, u, u, Ju)g(X, Ju)\}
\end{aligned}$$

$$- \frac{k_1 - f_1}{2tf_1k_1} g(Z, Ju)\{R(X, Ju, u, Ju)g(Y, Ju) - R(Y, Ju, u, Ju)g(X, Ju)\},$$

$$T_2^{(1)}(u, X, Y, Z)$$

$$= \frac{k_1 - f_1}{f_1k_1} \underset{X,Y,Z}{\mathfrak{S}} g(X, JY)g(Z, Ju) - \frac{1}{2f_1^2k_1} \underset{X,Y,Z}{\mathfrak{S}} R(X, Y, u, Ju)g(Z, Ju)$$

$$+ \frac{h_1 - f_1}{2tf_1^2h_1k_1} \underset{X,Y,Z}{\mathfrak{S}} R(X, u, u, Ju)\{g(Y, u)g(Z, Ju) - g(Z, u)g(Y, Ju)\}.$$

Note that if $P_3^{(1)}$ and $P_4^{(1)}$ then $P_8^{(1)}$, if $P_5^{(1)}$ and $P_6^{(1)}$ then $P_7^{(1)}$, if $P_1^{(1)}$, $P_2^{(1)}$ and $P_8^{(1)}$ then $P_{11}^{(1)}$. Further, if $P_4^{(1)}$ and $P_7^{(1)}$ then $R_1 = 0$ and hence $P_{10}^{(1)}$.

Making use of $\nabla^{(1)}\Omega_1$, $d\Omega_1$ and $\delta\Omega_1$, we can write down the conditions for the sixteen classes and obtain the following.

Theorem 1. *Let $M = (M, J, g)$ be an almost Hermitian manifold of dimension $2n \geq 4$. For the almost Hermitian manifold $TM = (TM, J_1, G_1)$,*

(1) *$TM \in \mathscr{K}$ if and only if $P_1^{(1)}$–$P_6^{(1)}$.*
(2) *$TM \in \mathscr{W}_1 = \mathscr{NK}$ if and only if $P_1^{(1)}$–$P_6^{(1)}$.*
(3) *$TM \in \mathscr{W}_2 = \mathscr{AK}$ if and only if $P_4^{(1)}$ and $P_7^{(1)}$.*
(4) *$TM \in \mathscr{W}_3 = \mathscr{H} \cap \mathscr{SK}$ if and only if $P_1^{(1)}$, $P_2^{(1)}$, $P_6^{(1)}$, $P_8^{(1)}$ and $P_{10}^{(1)}$.*
(5) *$TM \in \mathscr{W}_4$ if and only if $P_1^{(1)}$–$P_6^{(1)}$.*
(6) *$TM \in \mathscr{W}_1 \cup \mathscr{W}_2 = \mathscr{QK}$ if and only if $P_4^{(1)}$ and $P_7^{(1)}$.*
(7) *$TM \in \mathscr{W}_1 \cup \mathscr{W}_3$ if and only if $P_1^{(1)}$, $P_2^{(1)}$, $P_6^{(1)}$, $P_8^{(1)}$ and $P_{10}^{(1)}$.*
(8) *$TM \in \mathscr{W}_1 \cup \mathscr{W}_4$ if and only if $P_1^{(1)}$–$P_6^{(1)}$.*
(9) *$TM \in \mathscr{W}_2 \cup \mathscr{W}_3$ if and only if "$P_1^{(1)}$, $P_{10}^{(1)}$ and $P_{11}^{(1)}$" or "$P_4^{(1)}$ and $P_7^{(1)}$".*
(10) *$TM \in \mathscr{W}_2 \cup \mathscr{W}_4$ if and only if $P_4^{(1)}$ and $P_7^{(1)}$.*
(11) *$TM \in \mathscr{W}_3 \cup \mathscr{W}_4 = \mathscr{H}$ if and only if $P_1^{(1)}$, $P_2^{(1)}$, $P_6^{(1)}$ and $P_8^{(1)}$.*
(12) *$TM \in \mathscr{W}_1 \cup \mathscr{W}_2 \cup \mathscr{W}_3 = \mathscr{SK}$ if and only if "$P_9^{(1)}$ and $P_{10}^{(1)}$" or "$P_4^{(1)}$ and $P_7^{(1)}$".*
(13) *$TM \in \mathscr{W}_1 \cup \mathscr{W}_2 \cup \mathscr{W}_4$ if and only if $P_4^{(1)}$ and $P_7^{(1)}$.*
(14) *$TM \in \mathscr{W}_1 \cup \mathscr{W}_3 \cup \mathscr{W}_4$ if and only if $P_1^{(1)}$, $P_2^{(1)}$, $P_6^{(1)}$ and $P_8^{(1)}$*
(15) *$TM \in \mathscr{W}_2 \cup \mathscr{W}_3 \cup \mathscr{W}_4$ if and only if "$P_1^{(1)}$ and $P_{11}^{(1)}$" or $P_4^{(1)}$.*

Thus, if we restrict to the space $\{(TM, J_1, G_1)\}$ of almost Hermitian manifolds, $\mathscr{K} = \mathscr{NK} = \mathscr{W}_4 = \mathscr{W}_1 \cup \mathscr{W}_4$, $\mathscr{AK} = \mathscr{QK} = \mathscr{W}_2 \cup \mathscr{W}_4 = \mathscr{W}_1 \cup \mathscr{W}_2 \cup \mathscr{W}_4$, $\mathscr{W}_3 = \mathscr{W}_1 \cup \mathscr{W}_3$ and $\mathscr{H} = \mathscr{W}_1 \cup \mathscr{W}_3 \cup \mathscr{W}_4$.

Example 1. Unless otherwise specified, we assume that the parameter functions f_1, h_1, k_1, α_1, β_1, γ_1 satisfy the conditions for defining (J_1, G_1).

(A) Let M be an almost Hermitian manifold. If

$$\alpha_1 = f_1, \quad \beta_1 = \frac{h_1 - f_1}{t}, \quad \gamma_1 = \frac{k_1 - f_1}{t},$$

then $TM \in \mathscr{AK}$ ([4]).

(B) Let M be a Kähler manifold of constant holomorphic sectional curvature $4c$. If

$$h_1 = \frac{f_1^2 - ct}{f_1 - 2tf_1'}, \quad k_1 = \frac{f_1^2 + ct}{f_1},$$

then $TM \in \mathscr{H}$. An example of f_1 is $f_1 = \sqrt{e^{-t} + |c|t}$.

(C) Let M be a Kähler manifold of constant holomorphic sectional curvature $4c$. If α_1, β_1, γ_1 are same as (A) and h_1, k_1 as (B), then $TM \in \mathscr{K}$ ([4]). Moreover, if α_1, β_1, γ_1 are same as (A) and f_1, h_1, k_1 do not satisfy $P_3^{(1)}$ or $P_5^{(1)} = P_6^{(1)}$, then $TM \in \mathscr{AK} - \mathscr{K}$. For example,

$$h_1 = \frac{f_1^2 + ct}{f_1}, \quad k_1 = \frac{f_1^2 - ct}{f_1 - 2tf_1'},$$

and choose a function f_1 satisfying $h_1 \neq k_1$, such as $f_1 = \sqrt{e^{-t} + |c|t}$.

(D) If

$$\alpha_1 = f_1, \quad \beta_1 = \frac{\{(2n+1)t + 2n - 1\}h_1}{\{2(n-1)t + 2n - 1\}t} - \frac{f_1}{t}, \quad \gamma_1 = 0, \quad k_1 = \frac{f_1}{t+1},$$

then $TM \in \mathscr{SK} - \mathscr{AK}$.

5. Almost Hermitian structure (J_2, G_2)

For simplicity, we put

$$\begin{aligned}
A_2 &= \frac{\alpha_2(f_2 - h_2) + tf_2\beta_2}{tf_2h_2}, \\
B_2 &= \frac{\alpha_2(f_2 - k_2) + tf_2\gamma_2}{tf_2k_2}, \\
C_2 &= \beta_2'k_2 + \frac{\beta_2(k_2 - f_2)}{t} + \frac{(\beta_2 - \gamma_2)(h_2 - f_2)}{2t}, \\
D_2 &= \frac{\beta_2'}{h_2} - \frac{\alpha_2'(h_2 - f_2)}{tf_2h_2} - \frac{\beta_2(h_2 - f_2)}{2tf_2h_2} - \frac{\beta_2(k_2 - f_2)}{2tf_2k_2}, \\
E_2 &= 2A_2' + \frac{\alpha_2'(h_2 - f_2)}{tf_2h_2} - \frac{\beta_2'}{h_2} + \frac{\beta_2(h_2 - f_2)}{2tf_2h_2} - \frac{(\varphi_2' + \xi_2)(h_2 - f_2)}{t} - \xi_2'h_2, \\
F_2 &= 2B_2' + \frac{\alpha_2'(k_2 - f_2)}{tf_2k_2} - \frac{\gamma_2'}{k_2} + \frac{\gamma_2(k_2 - f_2)}{2tf_2k_2} - \frac{2\varphi_2'(k_2 - f_2)}{t} - \psi_2'k_2, \\
H_2 &= B_2 - \frac{\gamma_2}{2f_2} + k_2(\varphi_2' - \psi_2), \\
I_2 &= \alpha_2'k_2 - \frac{\gamma_2f_2}{2}, \\
K_2 &= \frac{\beta_2(k_2 - f_2)}{2tf_2k_2} + \frac{(\varphi_2' + \xi_2)(h_2 - f_2)}{t} + h_2\xi_2',
\end{aligned}$$

$$L_2 = \frac{\gamma_2(h_2 - f_2)}{2tf_2h_2} + \frac{2\xi_2(k_2 - f_2)}{t} + k_2\xi_2',$$

$$N_2 = B_2 - \frac{2\alpha_2}{f_2}\left(\log\frac{\alpha_2}{f_2}\right)',$$

$$O_2 = B_2 - \frac{\gamma_2}{2k_2} - f_2\varphi_2',$$

$$R_2 = 2k_2\left(\log\frac{\alpha_2 + t\beta_2}{h_2}\right)' - \frac{h_2(\alpha_2 + t\gamma_2)}{t(\alpha_2 + t\beta_2)} + \frac{k_2}{t},$$

$$S_2 = 2k_2\left(\log\frac{\alpha_2}{f_2}\right)' - \frac{f_2\gamma_2}{\alpha_2} + \frac{k_2 - f_2}{t}.$$

We denote by Ω_2 the Kähler form of (J_2, G_2). The covariant derivative $\nabla^{(2)}\Omega_2$ is given by

$$\begin{aligned}
&(\nabla^{(2)}_{X^H_u}\Omega_2)(Y^H_u, Z^H_u) \qquad (5.1)\\
&= -I_2\{g(X,Y)g(Z,Ju) - g(X,Z)g(Y,Ju)\}\\
&\quad - \frac{\beta_2 f_2}{2}\{g(X,JY)g(Z,u) - g(X,JZ)g(Y,u) - 2g(Y,JZ)g(X,u)\}\\
&\quad - C_2 g(X,u)\{g(Y,u)g(Z,Ju) - g(Y,Ju)g(Z,u)\}\\
&\quad + \frac{f_2\varphi_2}{2}\{R(X,Y,u,JZ) - R(X,Z,u,JY)\}\\
&\quad + \frac{A_2}{2}\{R(X,Y,u,Ju)g(Z,u) - R(X,Z,u,Ju)g(Y,u)\},
\end{aligned}$$

$$\begin{aligned}
&(\nabla^{(2)}_{X^H_u}\Omega_2)(Y^H_u, Z^V_u) \qquad (5.2)\\
&= -f_2\varphi_2 g((\nabla_X J)Y, Z)\\
&\quad - \frac{2k_2B_2 - \gamma_2}{2k_2} g((\nabla_X J)Y, u)g(Z,u) + \frac{2A_2 - \xi_2 h_2}{2} g((\nabla_X J)Z, u)g(Y,u)\\
&\quad + \frac{\gamma_2}{2f_2} g((\nabla_X J)Z, Ju)g(Y,Ju) - \frac{f_2\xi_2}{2} g((\nabla_X J)Y, Ju)g(Z,Ju)\\
&\quad + \frac{\gamma_2}{2k_2} g((\nabla_Y J)X, u)g(Z,u) + \frac{\gamma_2}{2f_2} g((\nabla_Y J)Z, Ju)g(X,Ju)\\
&\quad - \frac{\gamma_2}{2f_2} g((\nabla_{JZ} J)X, u)g(Y,Ju) - \frac{\gamma_2}{2f_2} g((\nabla_{JZ} J)Y, u)g(X,Ju)\\
&\quad - \frac{\gamma_2(h_2 - f_2)}{2tf_2h_2} g((\nabla_u J)X, u)g(Y,Ju)g(Z,Ju)\\
&\quad - \frac{\gamma_2(h_2 - f_2)}{2tf_2h_2} g((\nabla_u J)Y, u)g(X,Ju)g(Z,Ju)\\
&\quad + \frac{\gamma_2(k_2 - f_2)}{2tf_2k_2} g((\nabla_{Ju} J)X, u)g(Y,Ju)g(Z,u)
\end{aligned}$$

$$+\frac{\gamma_2(k_2-f_2)}{2tf_2k_2}g((\nabla_{Ju}J)Y,u)g(X,Ju)g(Z,u),$$

$$(\nabla^{(2)}_{X^H_u}\Omega_2)(Y^V_u,Z^V_u) \tag{5.3}$$

$$\begin{aligned}
&= I_2\{g(X,JY)g(Z,u)-g(X,JZ)g(Y,u)\}\\
&\quad+\frac{\beta_2}{2h_2}\{g(X,Y)g(Z,Ju)-g(X,Z)g(Y,Ju)\}-\frac{\beta_2}{f_2}g(X,u)g(Y,JZ)\\
&\quad+D_2g(X,u)\{g(Y,u)g(Z,Ju)-g(Y,Ju)g(Z,u)\}\\
&\quad+\frac{\varphi_2}{2f_2}\{R(X,JY,u,Z)-R(X,JZ,u,Y)\}\\
&\quad+\frac{\xi_2}{2f_2}\{R(X,JY,u,Ju)g(Z,Ju)-R(X,JZ,u,Ju)g(Y,Ju)\}\\
&\quad-\frac{\varphi_2(h_2-f_2)}{2tf_2h_2}\{R(X,u,u,Y)g(Z,Ju)-R(X,u,u,Z)g(Y,Ju)\}\\
&\quad+\frac{\varphi_2(k_2-f_2)}{2tf_2k_2}\{R(X,Ju,u,Y)g(Z,u)-R(X,Ju,u,Z)g(Y,u)\}\\
&\quad-\frac{\xi_2(k_2-f_2)}{2tf_2k_2}R(X,Ju,u,Ju)\{g(Y,u)g(Z,Ju)-g(Y,Ju)g(Z,u)\},
\end{aligned}$$

$$(\nabla^{(2)}_{X^V_u}\Omega_2)(Y^H_u,Z^H_u) \tag{5.4}$$

$$\begin{aligned}
&= \frac{f_2\xi_2}{2}g(X,Ju)\{g((\nabla_YJ)Z,Ju)-g((\nabla_ZJ)Y,Ju)\}\\
&\quad+\frac{h_2\xi_2}{2}\{g((\nabla_YJ)X,u)g(Z,u)-g((\nabla_ZJ)X,u)g(Y,u)\},
\end{aligned}$$

$$(\nabla^{(2)}_{X^V_u}\Omega_2)(Y^H_u,Z^V_u) \tag{5.5}$$

$$\begin{aligned}
&= -\frac{2h_2(A_2-f_2\xi_2)-\beta_2}{2h_2}g(X,Y)g(Z,Ju)+\frac{2f_2A_2-\beta_2}{2f_2}g(X,JZ)g(Y,u)\\
&\quad+H_2g(X,Z)g(Y,Ju)-O_2g(X,JY)g(Z,u)\\
&\quad+f_2\xi_2g(X,Ju)g(Y,Z)\\
&\quad-E_2g(X,u)g(Y,u)g(Z,Ju)+F_2g(X,u)g(Y,Ju)g(Z,u)\\
&\quad+K_2g(X,Ju)g(Y,u)g(Z,u)+L_2g(X,Ju)g(Y,Ju)g(Z,Ju)\\
&\quad+\frac{\varphi_2}{2f_2}R(Y,JZ,X,u)-\frac{\xi_2}{2f_2}R(Y,JZ,u,Ju)g(X,Ju)\\
&\quad-\frac{\varphi_2(h_2-f_2)}{2tf_2h_2}R(X,u,u,Y)g(Z,Ju)\\
&\quad-\frac{\varphi_2(k_2-f_2)}{2tf_2k_2}R(X,u,Y,Ju)g(Z,u)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\xi_2(h_2 - f_2)}{2tf_2h_2} R(Y, u, u, Ju) g(X, Ju) g(Z, Ju) \\
&+ \frac{\xi_2(k_2 - f_2)}{2tf_2k_2} R(Y, Ju, u, Ju) g(X, Ju) g(Z, u),
\end{aligned}
$$

$$
\begin{aligned}
&(\nabla^{(2)}_{X^V_u} \Omega_2)(Y^V_u, Z^V_u) \qquad (5.6)\\
&= \frac{\xi_2}{2f_2} \{ g((\nabla_{JY} J)X, u) g(Z, Ju) - g((\nabla_{JZ} J)X, u) g(Y, Ju) \} \\
&+ \frac{\xi_2}{2f_2} \{ g((\nabla_{JY} J)Z, u) g(X, Ju) - g((\nabla_{JZ} J)Y, u) g(X, Ju) \} \\
&- \frac{\xi_2(k_2 - f_2)}{2tf_2k_2} g((\nabla_{Ju} J)X, u) \{ g(Y, u) g(Z, Ju) - g(Y, Ju) g(Z, u) \} \\
&- \frac{\xi_2(h_2 - f_2)}{2tf_2h_2} g((\nabla_u J)Y, u) g(X, Ju) g(Z, Ju) \\
&+ \frac{\xi_2(h_2 - f_2)}{2tf_2h_2} g((\nabla_u J)Z, u) g(X, Ju) g(Y, Ju) \\
&+ \frac{\xi_2(k_2 - f_2)}{2tf_2k_2} g((\nabla_{Ju} J)Y, u) g(X, Ju) g(Z, u) \\
&- \frac{\xi_2(k_2 - f_2)}{2tf_2k_2} g((\nabla_{Ju} J)Z, u) g(X, Ju) g(Y, u).
\end{aligned}
$$

The exterior derivative $d\Omega_2$ is given by

$$
\begin{aligned}
&d\Omega_2(X^H_u, Y^H_u, Z^H_u) \qquad (5.7)\\
&= f_2\varphi_2 \underset{X,Y,Z}{\mathfrak{S}} R(X, Y, u, JZ) + A_2 \underset{X,Y,Z}{\mathfrak{S}} R(X, Y, u, Ju) g(Z, u),
\end{aligned}
$$

$$
\begin{aligned}
&d\Omega_2(X^H_u, Y^H_u, Z^V_u) \qquad (5.8)\\
&= -f_2\varphi_2 \{ g((\nabla_X J)Y, Z) - g((\nabla_Y J)X, Z) \} \\
&\quad - B_2 \{ g((\nabla_X J)Y, u) - g((\nabla_Y J)X, u) \} g(Z, u) \\
&\quad + A_2 \{ g((\nabla_X J)Z, u) g(Y, u) - g((\nabla_Y)Z, u) g(X, u) \},
\end{aligned}
$$

$$
\begin{aligned}
&d\Omega_2(X^H_u, Y^V_u, Z^V_u) \qquad (5.9)\\
&= -N_2 \{ g(X, JY) g(Z, u) - g(X, JZ) g(Y, u) \} \\
&\quad + A_2 \{ g(X, Y) g(Z, Ju) - g(X, Z) g(Y, Ju) \} \\
&\quad + 2A_2' g(X, u) \{ g(Y, u) g(Z, Ju) - g(Y, Ju) g(Z, u) \} \\
&\quad - 2A_2 g(X, u) g(Y, JZ),
\end{aligned}
$$

$$
d\Omega_2(X^V_u, Y^V_u, Z^V_u) = 0. \qquad (5.10)
$$

The co-derivative $\delta\Omega_2$ is given by

$$
\delta\Omega_2(X^H_u) = \{ R_2 + 2(n-1) S_2 \} g(X, u), \qquad (5.11)
$$

$$\delta\Omega_2(X_u^V) = \frac{1}{f_2}\delta\Omega(X) + \frac{f_2 - k_2}{tf_2k_2}g(X,u)\delta\Omega(u) + \frac{f_2\xi_2}{\alpha_2}g(X,Ju)\delta\Omega(Ju) - \frac{h_2A_2}{f_2(\alpha_2 + t\beta_2)}g((\nabla_u J)u, X). \tag{5.12}$$

Now, for a constant c, we consider next conditions.

$P_1^{(2)}$: $M = (M, J, g)$ is a Kähler manifold.

$P_2^{(2)}$: M is a space of constant holomorphic sectional curvature $4c$.

$P_3^{(2)}$: $c\alpha_2 - \beta_2 f_2^2 = 0$.

$P_4^{(2)}$: $A_2 = 0$.

$P_5^{(2)}$: $2k_2(\log\alpha_2)' - \frac{\gamma_2 f_2}{\alpha_2} - \frac{c}{f_2} = 0$.

$P_6^{(2)}$: $2k_2(\log f_2)' - \frac{k_2 - f_2}{t} - \frac{c}{f_2} = 0$.

$P_7^{(2)}$: $S_2 = 0$.

$P_8^{(2)}$: $f_2(f_2 - h_2) + ct = 0$.

$P_9^{(2)}$: M is a nearly Kähler manifold.

$P_{10}^{(2)}$: M is a semi-Kähler manifold.

$P_{11}^{(2)}$: $R_2 + 2(n-1)S_2 = 0$.

$P_{12}^{(2)}$: $T_1^{(2)}(u, X, Y, Z) = T_2^{(2)}(u, X, Y, Z) = 0$ for any u, X, Y, Z,

where, $T_1^{(2)}$ and $T_2^{(2)}$ are defined respectively by

$$\begin{aligned}
&T_1^{(2)}(u, X, Y, Z)\\
&= -(h_2 - f_2)\{g(X,Z)g(Y,u) - g(Y,Z)g(X,u)\}\\
&\quad + \frac{(h_2 - f_2)^2}{th_2}g(Z,Ju)\{g(X,Ju)g(Y,Ju) - g(X,u)g(Y,Ju)\}\\
&\quad + \frac{(h_2 - f_2)f_2}{h_2}g(X,JY)g(Z,Ju) - \frac{1}{2h_2}R(X,Y,u,Ju)g(Z,Ju)\\
&\quad + \frac{1}{2f_2}\{R(JZ,X,u,Ju) - R(JZ,Y,u,Ju)g(X,u)\}\\
&\quad - \frac{h_2 - f_2}{2tf_2h_2}g(Z,Ju)\{R(X,u,u,Ju)g(Y,u) - R(Y,u,u,Ju)g(X,u)\}\\
&\quad + \frac{k_2 - f_2}{2tf_2k_2}g(Z,u)\{R(X,Ju,u,Ju)g(Y,u) - R(Y,Ju,u,Ju)g(X,u)\},
\end{aligned}$$

$T_2^{(2)}(u, X, Y, Z)$

$$= \frac{h_2 - f_2}{f_2 h_2} \underset{X,Y,Z}{\mathfrak{S}} g(X, JY)g(Z, u) - \frac{1}{2f_2^2 h_2} \underset{X,Y,Z}{\mathfrak{S}} R(Y, Z, u, Ju)g(X, Ju)$$
$$+ \frac{k_2 - f_2}{2t f_2^2 h_2 k_2} \underset{X,Y,Z}{\mathfrak{S}} R(X, u, u, Ju)\{g(Y, u)g(Z, Ju) - g(Z, u)g(Y, Ju)\}.$$

Note that if $P_3^{(2)}$ and $P_4^{(2)}$ then $P_8^{(2)}$, if $P_5^{(2)}$ and $P_6^{(2)}$ then $P_7^{(2)}$, if $P_1^{(2)}$, $P_2^{(2)}$ and $P_8^{(2)}$ then $P_{12}^{(2)}$. Further, if $P_4^{(2)}$ and $P_7^{(2)}$ then $R_2 = 0$ and hence $P_{11}^{(2)}$.

Making use of $\nabla^{(2)}\Omega_2$, $d\Omega_2$ and $\delta\Omega_2$, we have the following.

Theorem 2. *Let $M = (M, J, g)$ be an almost Hermitian manifold of dimension $2n \geq 4$. For the almost Hermitian manifold $TM = (TM, J_2, G_2)$,*

(1) *$TM \in \mathscr{K}$ if and only if $P_1^{(2)}$–$P_6^{(2)}$.*
(2) *$TM \in \mathscr{W}_1 = \mathscr{NK}$ if and only if $P_1^{(2)}$–$P_6^{(2)}$.*
(3) *$TM \in \mathscr{W}_2 = \mathscr{AK}$ if and only if $P_1^{(2)}$, $P_4^{(2)}$ and $P_7^{(2)}$.*
(4) *$TM \in \mathscr{W}_3 = \mathscr{H} \cap \mathscr{SK}$ if and only if $P_1^{(2)}$, $P_2^{(2)}$, $P_6^{(2)}$, $P_8^{(2)}$ and $P_{11}^{(2)}$.*
(5) *$TM \in \mathscr{W}_4$ if and only if $P_1^{(2)}$–$P_6^{(2)}$.*
(6) *$TM \in \mathscr{W}_1 \cup \mathscr{W}_2 = \mathscr{QK}$ if and only if $P_1^{(2)}$, $P_4^{(2)}$ and $P_7^{(2)}$.*
(7) *$TM \in \mathscr{W}_1 \cup \mathscr{W}_3$ if and only if $P_1^{(2)}$, $P_2^{(2)}$, $P_6^{(2)}$, $P_8^{(2)}$ and $P_{11}^{(2)}$.*
(8) *$TM \in \mathscr{W}_1 \cup \mathscr{W}_4$ if and only if $P_1^{(2)}$–$P_6^{(2)}$.*
(9) *$TM \in \mathscr{W}_2 \cup \mathscr{W}_3$ if and only if "$P_1^{(2)}$, $P_{11}^{(2)}$ and $P_{12}^{(2)}$" or "$P_4^{(2)}$ and $P_7^{(2)}$".*
(10) *$TM \in \mathscr{W}_2 \cup \mathscr{W}_4$ if and only if $P_1^{(2)}$, $P_4^{(2)}$ and $P_7^{(2)}$.*
(11) *$TM \in \mathscr{W}_3 \cup \mathscr{W}_4 = \mathscr{H}$ if and only if $P_1^{(2)}$, $P_2^{(2)}$, $P_6^{(2)}$ and $P_8^{(2)}$.*
(12) *$TM \in \mathscr{W}_1 \cup \mathscr{W}_2 \cup \mathscr{W}_3 = \mathscr{SK}$ if and only if "$P_9^{(2)}$ and $P_{11}^{(2)}$" or "$P_{10}^{(2)}$, $P_4^{(2)}$ and $P_7^{(2)}$".*
(13) *$TM \in \mathscr{W}_1 \cup \mathscr{W}_2 \cup \mathscr{W}_4$ if and only if $P_1^{(2)}$, $P_4^{(2)}$ and $P_7^{(2)}$.*
(14) *$TM \in \mathscr{W}_1 \cup \mathscr{W}_3 \cup \mathscr{W}_4$ if and only if $P_1^{(2)}$, $P_2^{(2)}$, $P_6^{(2)}$ and $P_8^{(2)}$*
(15) *$TM \in \mathscr{W}_2 \cup \mathscr{W}_3 \cup \mathscr{W}_4$ if and only if "$P_1^{(2)}$ and $P_4^{(2)}$" or "$P_1^{(2)}$ and $P_{12}^{(2)}$".*

Thus, if we restrict to the space $\{(TM, J_2, G_2)\}$ of almost Hermitian manifolds, $\mathscr{K} = \mathscr{NK} = \mathscr{W}_4 = \mathscr{W}_1 \cup \mathscr{W}_4$, $\mathscr{AK} = \mathscr{QK} = \mathscr{W}_2 \cup \mathscr{W}_4 = \mathscr{W}_1 \cup \mathscr{W}_2 \cup \mathscr{W}_4$, $\mathscr{W}_3 = \mathscr{W}_1 \cup \mathscr{W}_3$ and $\mathscr{H} = \mathscr{W}_1 \cup \mathscr{W}_3 \cup \mathscr{W}_4$.

Example 2. Unless otherwise specified, we assume that the parameter functions f_2, h_2, k_2, α_2, β_2, γ_2 satisfy the conditions for defining (J_2, G_2).

(A) Let M be a Kähler manifold. If

$$\alpha_2 = f_2, \quad \beta_2 = \frac{h_2 - f_2}{t}, \quad \gamma_2 = \frac{k_2 - f_2}{t},$$

then $TM \in \mathscr{AK}$.

(B) Let M be a Kähler manifold of constant holomorphic sectional curvature $4c$. If

$$h_2 = \frac{f_2^2 + ct}{f_2}, \quad k_2 = \frac{f_2^2 - ct}{f_2 - 2t f_2'},$$

then $TM \in \mathscr{H}$. An example of f_2 is $f_2 = \sqrt{e^{-t} + |c|t}$.

(C) Let M be a Kähler manifold of constant holomorphic sectional curvature $4c$. If α_2, β_2, γ_2 are same as (A) and h_2, k_2 as (B), then $TM \in \mathscr{K}$. Moreover, if α_2, β_2, γ_2 are same as (A) and f_2, h_2, k_2 do not satisfy $P_3^{(2)}$ or $P_5^{(2)} = P_6^{(2)}$, then $TM \in \mathscr{AK} - \mathscr{K}$. For example,

$$h_2 = \frac{f_2^2 - ct}{f_2 - 2tf_2'}, \quad k_2 = \frac{f_2^2 + ct}{f_2},$$

and choose a function f_2 satisfying $h_2 \neq k_2$, such as $f_2 = \sqrt{e^{-t} + |c|t}$.

(D) Let M be a semi-Kähler manifold. If α_2, β_2, γ_2 are same as (A), then $TM \in \mathscr{SK}$. In particular, if M is an a Kähler manifold, then $TM \in \mathscr{AK}$.

(E) Let M be a nearly Kähler manifold. If

$$\alpha_2 = f_2, \quad \beta_2 = 0, \quad \gamma_2 = \frac{\{(2n+1)t + 2n - 1\}k_2}{\{2(n-1)t + 2n - 1\}t} - \frac{f_2}{t}, \quad h_2 = \frac{f_2}{t+1},$$

then $TM \in \mathscr{SK} - \mathscr{AK}$ even if M is a Kähler manifold.

6. Almost Hermitian structure (J_3, G_3)

For simplicity, we put

$$\begin{aligned}
A_3 &= \frac{\alpha_3(\lambda - h_3) + \lambda t\beta_3}{\lambda t h_3}, \\
B_3 &= \frac{\varphi_3(\mu - k_3) + \mu t\psi_3}{\mu t k_3}, \\
C_3 &= \frac{\beta_3(k_3 - \mu)}{\mu t k_3} + \frac{\beta_3(h_3 - \lambda)}{2t} - \frac{\beta_3'}{k_3}, \\
D_3 &= \frac{\gamma_3(k_3 - \mu)}{\mu t k_3} - \frac{\gamma_3(h_3 - \lambda)}{2\lambda t h_3} - \frac{\gamma_3'}{k_3}, \\
E_3 &= 2A_3' - \gamma_3' h_3 - \frac{\beta_3'}{h_3} - \frac{\alpha_3'(h_3 - \lambda)}{t} - \frac{\gamma_3(h_3 - \lambda)}{2t} + \frac{\alpha_3'(h_3 - \lambda)}{\lambda t h_3} + \frac{\beta_3(h_3 - \lambda)}{2\lambda t h_3}, \\
F_3 &= 2B_3' - \xi_3' k_3 - \frac{\psi_3'}{k_3} - \frac{(k_3 - \mu)(\varphi_3' + \xi_3)}{t} + \frac{2\varphi_3'(k_3 - \mu)}{\mu t k_3}, \\
H_3 &= A_3 - \frac{\beta_3}{2h_3} - \frac{\lambda\gamma_3}{2}, \\
I_3 &= B_3 - \mu\xi_3 + \frac{\varphi_3' - \psi_3}{k_3}, \\
K_3 &= \frac{(k_3 - \mu)(\beta_3 - \gamma_3)}{2t} - \frac{\beta_3'}{h_3} + \frac{(h_3 - \lambda)(2\alpha_3' + \beta_3)}{2\lambda t h_3}, \\
L_3 &= \frac{(k_3 - \mu)(\beta_3 - \gamma_3)}{2t} + \gamma_3' h_3 + \frac{(h_3 - \lambda)(2\alpha_3' + \gamma_3)}{2t},
\end{aligned}$$

$$
\begin{aligned}
N_3 &= \frac{\lambda}{t\alpha_3} - \frac{h_3}{t(\alpha_3 + t\beta_3)}, \\
R_3 &= \frac{1}{k_3}(\log(\alpha_3 + t\beta_3)(\alpha_3 + t\gamma_3))', \\
S_3 &= \frac{1}{k_3}(\log \alpha_3\varphi_3)' - \frac{\mu B_3}{k_3\varphi_3}.
\end{aligned}
$$

We denote by Ω_3 the Kähler form of (J_3, G_3). The covariant derivative $\nabla^{(3)}\Omega_3$ is given by

$$
\begin{aligned}
&(\nabla^{(3)}_{X^H_u}\Omega_3)(Y^H_u, Z^H_u) \qquad (6.1)\\
&= -\lambda\alpha_3 g((\nabla_X J)Y, Z) \\
&\quad - \frac{2A_3 - \gamma_3 h_3}{2}\{g((\nabla_X J)Y, u)g(Z, u) - g((\nabla_X J)Z, u)g(Y, u)\} \\
&\quad + \frac{\gamma_3 h_3}{2}\{g((\nabla_Y J)X, u)g(Z, u) - g((\nabla_Z J)X, u)g(Y, u)\} \\
&\quad - \frac{\lambda\gamma_3}{2}\{g((\nabla_X J)Y, Ju)g(Z, Ju) - g((\nabla_X J)Z, Ju)g(Y, Ju) \\
&\qquad - g((\nabla_Y J)Z, Ju)g(X, Ju) + g((\nabla_Z J)Y, Ju)g(X, Ju) \\
&\qquad - g((\nabla_{JY} J)X, u)g(Z, Ju) + g((\nabla_{JZ} J)X, u)g(Y, Ju) \\
&\qquad - g((\nabla_{JY} J)Z, u)g(X, Ju) + g((\nabla_{JZ} J)Y, u)g(X, Ju)\} \\
&\quad - \frac{\gamma_3(h_3 - \lambda)}{2\lambda t h_3} g((\nabla_u J)Y, u)g(X, Ju)g(Z, Ju) \\
&\quad + \frac{\gamma_3(h_3 - \lambda)}{2\lambda t h_3} g((\nabla_u J)Z, u)g(X, Ju)g(Y, Ju) \\
&\quad + \frac{\gamma_3(h_3 - \lambda)}{2t} g((\nabla_{Ju} J)X, u)g(Y, u)g(Z, Ju) \\
&\quad + \frac{\gamma_3(h_3 - \lambda)}{2t} g((\nabla_{Ju} J)Z, u)g(X, Ju)g(Y, u) \\
&\quad - \frac{\gamma_3(h_3 - \lambda)}{2t} g((\nabla_{Ju} J)X, u)g(Y, Ju)g(Z, u) \\
&\quad - \frac{\gamma_3(h_3 - \lambda)}{2t} g((\nabla_{Ju} J)Y, u)g(X, Ju)g(Z, u), \\
&(\nabla^{(3)}_{X^H_u}\Omega_3)(Y^H_u, Z^V_u) \qquad (6.2)\\
&= -\frac{\alpha_3'}{k_3} g(X, Y)g(Z, Ju) + \lambda\alpha_3' g(X, JY)g(Z, u) \\
&\quad - \frac{\mu\beta_3 + h_3\gamma_3}{2\mu h_3} g(X, Z)g(Y, Ju) + \frac{\mu\beta_3 + \gamma_3 h_3}{2} g(X, JZ)g(Y, u) \\
&\quad + \frac{\gamma_3(\lambda - \mu)}{2} g(Y, Z)g(X, Ju) - \frac{\beta_3(\lambda - \mu)}{2} g(Y, JZ)g(X, u)
\end{aligned}
$$

$$
\begin{aligned}
&+ C_3 g(X,u)g(Y,u)g(Z,Ju) + D_3 g(X,Ju)g(Y,Ju)g(Z,Ju)\\
&+ K_3 g(X,u)g(Y,Ju)g(Z,u) + L_3 g(X,Ju)g(Y,u)g(Z,u)\\
&+ \frac{\mu\varphi_3}{2}R(X,Y,u,JZ) + \frac{\lambda\varphi_3}{2}R(X,JY,u,Z)\\
&+ \frac{B_3}{2}R(X,Y,u,Ju)g(Z,u) + \frac{\lambda\xi_3}{2}R(X,JY,u,Ju)g(Z,Ju)\\
&+ \frac{\varphi_3(h_3-\lambda)}{2\lambda t h_3}R(X,u,u,Z)g(Y,Ju)\\
&+ \frac{\varphi_3(h_3-\lambda)}{2t}R(X,Ju,u,Z)g(Y,u)\\
&+ \frac{\xi_3(h_3-\lambda)}{2\lambda t h_3}R(X,u,u,Ju)g(Y,Ju)g(Z,Ju)\\
&+ \frac{\xi_3(h_3-\lambda)}{2t}R(X,Ju,u,Ju)g(Y,u)g(Z,Ju),
\end{aligned}
$$

$$
\begin{aligned}
&(\nabla^{(3)}_{X^H_u}\Omega_3)(Y^V_u, Z^V_u) \qquad (6.3)\\
&= -\mu\varphi_3 g((\nabla_X J)Y, Z)\\
&\quad - \frac{2B_3-\xi_3 k_3}{2}\{g((\nabla_X J)Y,u)g(Z,u) - g((\nabla_X J)Z,u)g(Y,u)\}\\
&\quad - \frac{\mu\xi_3}{2}\{g((\nabla_X J)Y,Ju)g(Z,Ju) - g((\nabla_X J)Z,Ju)g(Y,Ju)\},
\end{aligned}
$$

$$
\begin{aligned}
&(\nabla^{(3)}_{X^V_u}\Omega_3)(Y^H_u, Z^H_u) \qquad (6.4)\\
&= -H_3\{g(X,Y)g(Z,Ju) - g(X,Z)g(Y,Ju)\}\\
&\quad - H_3\{g(X,JY)g(Z,u) - g(X,JZ)g(Y,u)\}\\
&\quad - E_3 g(X,u)\{g(Y,u)g(Z,Ju) - g(Y,Ju)g(Z,u)\}\\
&\quad + \frac{\lambda\varphi_3}{2}\{R(Y,JZ,X,u) - R(Z,JY,X,u)\}\\
&\quad - \frac{\lambda\xi_3}{2}g(X,Ju)\{R(Y,JZ,u,Ju) - R(Z,JY,u,Ju)\}\\
&\quad - \frac{\varphi_3(h_3-\lambda)}{2\lambda t h_3}\{R(X,u,u,Y)g(Z,Ju) - R(X,u,u,Z)g(Y,Ju)\}\\
&\quad + \frac{\varphi_3(h_3-\lambda)}{2t}\{R(X,u,Y,Ju)g(Z,u) - R(X,u,Z,Ju)g(Y,u)\}\\
&\quad - \frac{\xi_3(h_3-\lambda)}{2\lambda t h_3}R(Y,u,u,Ju)g(X,Ju)g(Z,Ju)\\
&\quad + \frac{\xi_3(h_3-\lambda)}{2\lambda t h_3}R(Z,u,u,Ju)g(X,Ju)g(Y,Ju)\\
&\quad - \frac{\xi_3(h_3-\lambda)}{2t}R(Y,Ju,u,Ju)g(X,Ju)g(Z,u)
\end{aligned}
$$

$$+\frac{\xi_3(h_3-\lambda)}{2t}R(Z,Ju,u,Ju)g(X,Ju)g(Y,u),$$

$$(\nabla^{(3)}_{X^V_u}\Omega_3)(Y^H_u,Z^V_u) \tag{6.5}$$

$$=\frac{\xi_3k_3}{2}g((\nabla_YJ)X,u)g(Z,u)+\frac{\mu\xi_3}{2}g((\nabla_YJ)Z,Ju)g(X,Ju)$$

$$+\frac{\lambda\xi_3}{2}\{g((\nabla_{JY}J)X,u)g(Z,Ju)+g((\nabla_{JY}J)Z,u)g(X,Ju)\}$$

$$+\frac{\xi_3(h_3-\lambda)}{2\lambda th_3}g((\nabla_uJ)X,u)g(Y,Ju)g(Z,Ju)$$

$$+\frac{\xi_3(h_3-\lambda)}{2\lambda th_3}g((\nabla_uJ)Z,u)g(X,Ju)g(Y,Ju)$$

$$+\frac{\xi_3(h_3-\lambda)}{2t}g((\nabla_{Ju}J)X,u)g(Y,u)g(Z,Ju)$$

$$+\frac{\xi_3(h_3-\lambda)}{2t}g((\nabla_{Ju}J)Z,u)g(X,Ju)g(Y,u),$$

$$(\nabla^{(3)}_{X^V_u}\Omega_3)(Y^V_u,Z^V_u) \tag{6.6}$$

$$=-I_3\{g(X,Y)g(Z,Ju)-g(X,Z)g(Y,Ju)\}$$

$$-(B_3-\mu\varphi_3')\{g(X,JY)g(Z,u)-g(X,JZ)g(Y,u)\}$$

$$-F_3g(X,u)\{g(Y,u)g(Z,Ju)-g(Y,Ju)g(Z,u)\}.$$

The exterior derivative $d\Omega_3$ is given by

$$d\Omega_3(X^H_u,Y^H_u,Z^H_u) \tag{6.7}$$

$$=\lambda\alpha_3d\Omega(X,Y,Z)-A_3\underset{X,Y,Z}{\mathfrak{S}}\{g((\nabla_XJ)Y,u)-g((\nabla_YJ)X,u)\}g(Z,u),$$

$$d\Omega_3(X^H_u,Y^H_u,Z^V_u) \tag{6.8}$$

$$=2\lambda\alpha_3'g(X,JY)g(Z,u)$$

$$-2A_3'\{g(X,u)g(Y,Ju)-g(X,Ju)g(Y,u)\}g(Z,u)$$

$$-A_3\{g(X,Z)g(Y,Ju)-g(Y,Z)g(X,Ju)\}$$

$$+A_3\{g(X,JZ)g(Y,u)-g(Y,JZ)g(X,u)\}$$

$$+\mu\varphi_3R(X,Y,u,JZ)+B_3R(X,Y,u,Ju)g(Z,u).$$

$$d\Omega_3(X^H_u,Y^V_u,Z^V_u) \tag{6.9}$$

$$=-\mu\varphi_3g((\nabla_XJ)Y,Z)$$

$$-B_3\{g((\nabla_XJ)Y,u)g(Z,u)-g((\nabla_XJ)Z,u)g(Y,u)\},$$

$$d\Omega_3(X^V_u,Y^V_u,Z^V_u)=-2(B_3-\mu\varphi_3')\underset{X,Y,Z}{\mathfrak{S}}g(X,JY)g(Z,u). \tag{6.10}$$

The co-derivative $\delta\Omega_3$ is given by

$$\delta\Omega_3(X_u^H) = \lambda\delta\Omega(X) + \frac{h_3-\lambda}{t}g(X,u)\delta\Omega(u) + \frac{\lambda\gamma_3}{\alpha_3}g(X,Ju)\delta\Omega(Ju) \quad (6.11)$$
$$- \frac{\lambda h_3 A_3}{\alpha_3+t\beta_3}g((\nabla_u J)u,X),$$
$$\delta\Omega_3(X_u^V) = \{R_3 + 2(n-1)S_3\}g(X,Ju) \quad (6.12)$$
$$+ \varphi_3 N_3 g(R(u,Ju)u,X) - t^2\xi_3 N_3 H(u)g(X,Ju)$$
$$+ \frac{\lambda\varphi_3}{\alpha_3}\rho^*(X,Ju) + \frac{\lambda\xi_3}{\alpha_3}\rho^*(u,u)g(X,Ju).$$

Now, for a constant c, we consider next conditions.

$P_1^{(3)}$: $M = (M,J,g)$ is a Kähler manifold.

$P_2^{(3)}$: M is a space of constant holomorphic sectional curvature $4c$.

$P_3^{(3)}$: $\alpha_3' k_3 + c\lambda(\varphi_3 + t\psi_3) = 0$.

$P_4^{(3)}$: $\lambda = \mu$.

$P_5^{(3)}$: $c\varphi_3 = -\mu A_3$.

$P_6^{(3)}$: $\varphi_3' = \mu B_3$.

$P_7^{(3)}$: $\beta_3' = \dfrac{\beta_3(k_3-\mu)}{\mu t} + \dfrac{\beta_3 k_3(h_3-\lambda)}{2t} - ck_3\xi_3(2h_3-\lambda) - \dfrac{3c\varphi_3 k_3(h_3-\lambda)}{2t}$.

$P_8^{(3)}$: $2th_3' - (h_3^2-1)k_3 = 0$.

$P_9^{(3)}$: $T_1^{(3)}(u,X,Y,Z) = 0$ for any u,X,Y,Z.

$P_{10}^{(3)}$: $T_1^{(3)}(u,X,Y,Z) + T_2^{(3)}(u,X,Y,Z) = 0$ for any u,X,Y,Z.

$P_{11}^{(3)}$: M is a nearly Kähler manifold.

$P_{12}^{(3)}$: M is a semi-Kähler manifold and $A_3 = 0$.

$P_{13}^{(3)}$: $(6.12) = 0$,

where, $T_1^{(3)}$ and $T_2^{(3)}$ are defined respectively by

$$T_1^{(3)}(u,X,Y,Z)$$
$$= \frac{\varphi_3}{\lambda\mu t h_3}\{R(X,u,u,JZ)g(Y,Ju) - R(Y,u,u,JZ)g(X,Ju)\}$$
$$+ \frac{\varphi_3}{\mu t}\{R(X,Ju,u,JZ)g(Y,u) - R(Y,Ju,u,JZ)g(X,u)\}$$
$$+ \frac{\varphi_3}{th_3}\{R(JX,u,u,Z)g(Y,Ju) - R(JY,u,u,Z)g(X,Ju)\}$$

$$+\frac{\varphi_3}{\lambda t}\{R(X,u,u,Z)g(Y,u)-R(Y,u,u,Z)g(X,u)\}$$
$$+\frac{B_3}{\lambda t h_3}\{R(X,u,u,Ju)g(Y,Ju)-R(Y,u,u,Ju)g(X,Ju)\}g(Z,u)$$
$$+\frac{B_3}{t}\{R(X,Ju,u,Ju)g(Y,u)-R(Y,Ju,u,Ju)g(X,u)\}g(Z,u)$$
$$+\frac{\xi_3}{t h_3}\{R(JX,u,u,Ju)g(Y,Ju)-R(JY,u,u,Ju)g(X,Ju)\}g(Z,Ju)$$
$$+\frac{\xi_3}{\lambda t}\{R(X,u,u,Ju)g(Y,u)-R(Y,u,u,Ju)g(X,u)\}g(Z,Ju)$$
$$+\frac{\varphi_3(h_3-\lambda)}{\lambda t^2 h_3}R(Z,u,u,Ju)\{g(X,u)g(Y,Ju)-g(X,Ju)g(Y,u)\}$$
$$+\frac{\xi_3(h_3-\lambda)}{\lambda h_3}H(u)\{g(X,u)g(Y,Ju)-g(X,Ju)g(Y,u)\}g(Z,Ju),$$
$$T_2^{(3)}(u,X,Y,Z)$$
$$=-\frac{(\lambda-\mu)A_3}{\lambda}\{g(X,u)g(Y,Z)-g(Y,u)g(X,Z)\}$$
$$+\frac{(\lambda-\mu)A_3}{h_3}\{g(X,Ju)g(Y,JZ)-g(Y,Ju)g(X,JZ)\}$$
$$+\frac{(\lambda-\mu)(h_3-\lambda)A_3}{\lambda t h_3}g(Z,Ju)\{g(X,u)g(Y,Ju)-g(Y,u)g(X,Ju)\}.$$

Note that if $P_4^{(3)}$–$P_7^{(3)}$ then $P_8^{(3)}$, if $P_1^{(3)}$, $P_2^{(3)}$ and $P_4^{(3)}$ then $P_9^{(3)}$. Further, if $P_1^{(3)}$–$P_3^{(3)}$, $P_5^{(3)}$ and $P_6^{(3)}$ then $P_{13}^{(3)}$.

Making use of $\nabla^{(3)}\Omega_3$, $d\Omega_3$ and $\delta\Omega_3$, we have the following.

Theorem 3. *Let $M=(M,J,g)$ be an almost Hermitian manifold of dimension $2n\geq 4$. For the almost Hermitian manifold $TM=(TM,J_3,G_3)$,*

(1) *$TM\in\mathscr{K}$ if and only if $P_1^{(3)}$–$P_7^{(3)}$.*
(2) *$TM\in\mathscr{W}_1=\mathscr{NK}$ if and only if $P_1^{(3)}$–$P_7^{(3)}$.*
(3) *$TM\in\mathscr{W}_2=\mathscr{AK}$ if and only if $P_1^{(3)}$–$P_3^{(3)}$, $P_5^{(3)}$ and $P_6^{(3)}$.*
(4) *$TM\in\mathscr{W}_3=\mathscr{H}\cap\mathscr{SK}$ if and only if $P_1^{(3)}$, $P_4^{(3)}$, $P_8^{(3)}$, $P_9^{(3)}$ and $P_{13}^{(2)}$.*
(5) *$TM\in\mathscr{W}_4$ if and only if $P_1^{(3)}$–$P_7^{(3)}$.*
(6) *$TM\in\mathscr{W}_1\cup\mathscr{W}_2=\mathscr{QK}$ if and only if $P_1^{(3)}$–$P_3^{(3)}$, $P_5^{(3)}$ and $P_6^{(3)}$.*
(7) *$TM\in\mathscr{W}_1\cup\mathscr{W}_3$ if and only if $P_1^{(3)}$, $P_4^{(2)}$, $P_8^{(3)}$, $P_9^{(3)}$ and $P_{13}^{(3)}$.*
(8) *$TM\in\mathscr{W}_1\cup\mathscr{W}_4$ if and only if $P_1^{(3)}$–$P_7^{(3)}$.*
(9) *$TM\in\mathscr{W}_2\cup\mathscr{W}_3$ if and only if $P_1^{(3)}$, $P_{10}^{(3)}$ and $P_{13}^{(3)}$.*
(10) *$TM\in\mathscr{W}_2\cup\mathscr{W}_4$ if and only if $P_1^{(3)}$–$P_3^{(3)}$, $P_5^{(3)}$ and $P_6^{(3)}$.*
(11) *$TM\in\mathscr{W}_3\cup\mathscr{W}_4=\mathscr{H}$ if and only if $P_1^{(3)}$, $P_4^{(3)}$, $P_8^{(3)}$ and $P_9^{(3)}$.*
(12) *$TM\in\mathscr{W}_1\cup\mathscr{W}_2\cup\mathscr{W}_3=\mathscr{SK}$ if and only if "$P_{11}^{(3)}$ and $P_{13}^{(3)}$" or "$P_{12}^{(3)}$ and $P_{13}^{(3)}$".*
(13) *$TM\in\mathscr{W}_1\cup\mathscr{W}_2\cup\mathscr{W}_4$ if and only if $P_1^{(3)}$–$P_3^{(3)}$, $P_5^{(3)}$ and $P_6^{(3)}$.*

(14) $TM \in \mathscr{W}_1 \cup \mathscr{W}_3 \cup \mathscr{W}_4$ *if and only if* $P_1^{(3)}$, $P_4^{(3)}$, $P_8^{(3)}$ *and* $P_9^{(3)}$

(15) $TM \in \mathscr{W}_2 \cup \mathscr{W}_3 \cup \mathscr{W}_4$ *if and only if* $P_1^{(3)}$ *and* $P_{10}^{(3)}$.

Thus, if we restrict to the space $\{(TM, J_3, G_3)\}$ *of almost Hermitian manifolds,* $\mathscr{K} = \mathscr{NK} = \mathscr{W}_4 = \mathscr{W}_1 \cup \mathscr{W}_4$, $\mathscr{AK} = \mathscr{QK} = \mathscr{W}_2 \cup \mathscr{W}_4 = \mathscr{W}_1 \cup \mathscr{W}_2 \cup \mathscr{W}_4$, $\mathscr{W}_3 = \mathscr{W}_1 \cup \mathscr{W}_3$ *and* $\mathscr{H} = \mathscr{W}_1 \cup \mathscr{W}_3 \cup \mathscr{W}_4$.

Example 3. Unless otherwise specified, we assume that the parameter functions f_3, h_3, k_3, α_3, β_3, γ_3, φ_3, ψ_3, ξ_3 satisfy the conditions for defining (J_3, G_3).

(A) Let M be a Kähler manifold of constant negative holomorphic sectional curvature $4c$ ($c < 0$). We assume that h_3, k_3 are positive functions. If

$$\lambda = \mu = 1, \quad \alpha_3 = 1 - cte^t, \quad \beta_3 = \frac{h_3 - 1}{t} - ce^t(2h_3 - 1),$$
$$\varphi_3 = e^t, \quad \psi_3 = \frac{(tk_3 + k_3 - 1)e^t}{t},$$

then $TM \in \mathscr{AK}$.

(B) Under the same conditions as (A), we find that TM is a Kähler manifold if and only if $2th_3' = (h_3^2 - 1)k_3$. Thus, for example, if

$$h_3 = 2t + 1, \quad k_3 = \frac{1}{t+1},$$

then $TM \in \mathscr{K}$. Moreover if

$$h_3 = \frac{1}{t+1}, \quad k_3 = 2t + 1,$$

then $TM \in \mathscr{AK} - \mathscr{K}$.

(C) Let M be a Kähler manifold of constant holomorphic sectional curvature. If

$$\lambda = \mu = 1, \quad h_3 = 2t + 1, \quad k_3 = \frac{1}{t+1},$$

then $TM \in \mathscr{H}$.

(D) Let M be a Kähler manifold of constant holomorphic sectional curvature $4c$. Then, the condition $P_{13}^{(3)}$ becomes

$$R_3 + \frac{4ch_3(\varphi_3 + t\psi_3)}{k_3^2(\alpha_3 + t\beta_3)} + 2(n-1)\left\{S_3 + \frac{c\lambda(\varphi_3 + t\psi_3)}{\alpha_3 k_3^2}\right\} = 0.$$

If $c < 0$ and

$$\lambda = \mu = 1, \quad \alpha_3 = e^t, \quad \varphi_3 = e^{-t}, \quad \beta_3 = \psi_3 = 0,$$
$$h_3 = e^t, \quad k_3 = 1 - \frac{ct(2e^t + n - 1)}{(n-1)e^{2t}},$$

then $TM \in \mathscr{SK} - \mathscr{AK}$.

Acknowledgement. The authors would like to thank the referee for the careful review and the valuable comments.

References

[1] P. Dombrowski, *On the geometry of the tangent bundle*, J. Reine Angew. Math. **210** (1962), 73–88.

[2] A. Gray and L.M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl. **123** (1980), 35–58.

[3] T. Koike, T. Oguro and N. Watanabe, *Remarks on some almost Hermitian structure on the tangent bundle*, Nihonkai Math. J. **20** (2009), 25–32.

[4] M. Tahara, L. Vanhecke and Y. Watanabe, *New structures on tangent bundles*, Note di Mathematica **18** (1998), 131–141.

[5] M. Tahara and Y. Watanabe, *Natural almost Hermitian, Hermitian and Kähler metrics on the tangent bundles*, Math. J. Toyama Univ. **20** (1997), 149–160.

Division of Science, School of Science and Engineering, Tokyo Denki University, Saitama, 350-0394, JAPAN

E-mail address: `oguro@mail.dendai.ac.jp`

Received September 19, 2013