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CHARACTERIZATIONS OF REGULAR NORMS ON $\mathbb{R}^n$

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Abstract. In this paper, we study regular norms on $\mathbb{R}^n$. It is shown that the class of regular norms is a natural generalization of that of generalized Day-James type norms. Furthermore, using absolute norms, we give some characterizations of regular norms.

1. Introduction

Let AN_n denote the family of all absolute normalized norms on $\mathbb{R}^n$, where a norm $\|\cdot\|$ on $\mathbb{R}^n$ is said to be absolute if

$$\|(a_1, a_2, \dots, a_n)\| = \|(|a_1|, |a_2|, \dots, |a_n|)\|$$

for all $(a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, and normalized if

$$\|(1, 0, \dots, 0)\| = \|(0, 1, 0, \dots, 0)\| = \|(0, \dots, 0, 1)\| = 1.$$

As in Saito-Kato-Takahashi [8], AN_n and Ψ_n are in a one-to-one correspondence under the equation $\psi(s) = \|s\|_\psi$ for all $s \in \Delta_n$, where

$$\Delta_n = \left\{ (s_1, s_2, \dots, s_n) \in \mathbb{R}^n : \sum_{i=1}^{n} s_i = 1,\ s_i \geq 0 \text{ for } i = 1, 2, \dots, n \right\},$$

and Ψ_n is the set of all continuous convex functions on Δ_n which satisfy the following conditions:

$$\psi(1, 0, \dots, 0) = \psi(0, 1, 0, \dots, 0) = \psi(0, \dots, 0, 1) = 1,$$

and for all $i = 1, 2, \dots, n$ with $s_i < 1$,

$$\psi(s_1, s_2, \dots, s_n) \geq (1 - s_i)\psi\left(\frac{s_1}{1 - s_i}, \dots, \frac{s_{i-1}}{1 - s_i}, 0, \frac{s_{i+1}}{1 - s_i}, \dots, \frac{s_n}{1 - s_i}\right).$$

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We remark that the norm $\|\cdot\|_\psi$ associated with ψ is given by

$$\|(a_1, a_2, \ldots, a_n)\|_\psi = \begin{cases} \left(\sum_{i=1}^n |a_i|\right) \psi\left(\frac{|a_1|}{\sum_{i=1}^n |a_i|}, \ldots, \frac{|a_n|}{\sum_{i=1}^n |a_i|}\right) & \text{if } (a_1, a_2, \ldots, a_n) \neq 0, \\ 0 & \text{if } (a_1, a_2, \ldots, a_n) = 0. \end{cases}$$

From this result, we can consider many non-ℓ_p-type norms easily. Now let

$$\psi_p(s_1, s_2, \ldots, s_n) = \begin{cases} \left(\sum_{i=1}^n s_i^p\right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{s_1, s_2, \ldots, s_n\} & \text{if } p = \infty. \end{cases}$$

Then $\psi_p \in \Psi_n$ and, as is easily seen, the ℓ_p-norm $\|\cdot\|_p$ is associated with ψ_p. For some other results concerning absolute normalized norms, we refer the reader to [3, 4, 5, 7, 8, 9].

In [10], we showed that every n-dimensional real normed linear space is isometrically isomorphic to the space $\mathbb{R}^n$ endowed with a normal norm, where a norm $\|\cdot\|$ on $\mathbb{R}^n$ is said to be normal if it satisfies $\|\cdot\|_1 \leq \|\cdot\| \leq \|\cdot\|_\infty$. This generalizes the result of Alonso [1] which states that any two-dimensional real normed linear space is a generalized Day-James space (cf. Nilsrakoo and Saejung [6]). A generalized Day-James space is defined for each $\psi, \phi \in \Psi_2$ as the space $\mathbb{R}^2$ endowed with the norm

$$\|(a, b)\|_{\varphi,\psi} = \begin{cases} \|(a, b)\|_\varphi & \text{if } ab \geq 0, \\ \|(a, b)\|_\psi & \text{if } ab \leq 0. \end{cases}$$

Let NN_n denote the set of all normal norms on $\mathbb{R}^n$.

In this paper, we focus on the following type of norms on $\mathbb{R}^n$.

Definition 1.1 ([11])**.** *A norm $\|\cdot\|$ on $\mathbb{R}^n$ is said to be regular if it is normalized and*

$$\|(a_1, a_2, \ldots, a_n)\| \geq \max_{1\leq k\leq n} \|(a_1, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n)\|$$

for all $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. Let RN_n denote the family of all regular norms on $\mathbb{R}^n$.

In the previous paper [11], we studied orthogonal bases and a structure of finite dimensional normed linear spaces, and regular norms appeared on that occasion inevitably. So it is natural to consider the structure of regular norms on $\mathbb{R}^n$.

The aim of this paper is to present some characterizations of regular norms on $\mathbb{R}^n$ analogous to the work of Saito-Kato-Takahashi [8].

2. Preliminaries

We recall some results on AN_n, RN_n and NN_n. The following is an important characterization of absolute norms on $\mathbb{R}^n$. The proof can be found in [2, Proposition IV.1.1] (see, also [8, Lemma 4.1]).

Proposition 2.1. *A norm $\|\cdot\|$ on $\mathbb{R}^n$ is absolute if and only if it is monotone, that is, if $|a_i| \le |b_i|$ for all $i = 1, 2, \ldots, n$ then $\|(a_1, a_2, \ldots, a_n)\| \le \|(b_1, b_2, \ldots, b_n)\|$.*

In [11, Theorem 3.6], the RN_n version of this result was proved. To state this result, some preparations are needed. For each $n \ge 2$, define a $2^{n-1} \times n$ matrix R_n^+ by the formulas

$$R_2^+ = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad R_n^+ = \left(\begin{array}{c|c} 1 & \\ \vdots & R_{n-1}^+ \\ 1 & \\ \hline 1 & \\ \vdots & -R_{n-1}^+ \\ 1 & \end{array}\right) \quad (n \ge 3).$$

Fix a positive integer $n \ge 2$. Let

$$R_n^+ = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_{2^{n-1}} \end{pmatrix}.$$

Then, the Ω_i-quadrant of $\mathbb{R}^n$ is given by

$$\Omega_i = \theta_i \cdot \mathbb{R}_+^n = \{\theta_i \cdot x : x \in \mathbb{R}_+^n\},$$

where $\mathbb{R}_+$ is the set of all nonnegative real numbers, and $\theta_i \cdot x$ denotes the pointwise product of θ_i and x.

Using Ω_i-quadrants, we can consider the following weakened monotonicity.

Definition 2.2 ([11]). *A norm $\|\cdot\|$ is said to be semi-monotone if*

$$\|(a_1, a_2, \ldots, a_n)\| \le \|(b_1, b_2, \ldots, b_n)\|$$

whenever $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in \Omega_i$ for some i and $|a_k| \le |b_k|$ for all $k = 1, 2, \ldots, n$, or equivalently, whenever $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n)$ satisfies $a_k b_k \ge 0$ and $|a_k| \le |b_k|$ for all $k = 1, 2, \ldots, n$.

Regular norms on $\mathbb{R}^n$ are characterized by semi-monotonicity. We prove the following result only for the sake of completeness.

Proposition 2.3 ([11]). *A normalized norm $\|\cdot\|$ on $\mathbb{R}^n$ is regular if and only if it is semi-monotone.*

Proof. Every semi-monotone norm is clearly regular. So we only prove that regularity implies semi-monotonicity. Suppose that $\|\cdot\|$ is regular. Then

$$\|(a_1, a_2, \ldots, a_n)\| \geq \max_{1\leq k\leq n} \|(a_1, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n)\|$$

for all $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. If $t > 1$, we have

$$\begin{aligned}
&\|(a_1, a_2, \ldots, a_n)\| \\
&\leq (1 - t^{-1})\|(a_1, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n)\| \\
&\qquad\qquad\qquad + t^{-1}\|(a_1, \ldots, a_{k-1}, ta_k, a_{k+1}, \ldots, a_n)\| \\
&\leq (1 - t^{-1})\|(a_1, \ldots, a_{k-1}, ta_k, a_{k+1}, \ldots, a_n)\| \\
&\qquad\qquad\qquad + t^{-1}\|(a_1, \ldots, a_{k-1}, ta_k, a_{k+1}, \ldots, a_n)\| \\
&= \|(a_1, \ldots, a_{k-1}, ta_k, a_{k+1}, \ldots, a_n)\|
\end{aligned}$$

for all $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ and all $k = 1, 2, \ldots, n$.

Now, let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be elements of $\mathbb{R}^n$ such that $|a_i| \leq |b_i|$ and $a_ib_i \geq 0$ for all $i = 1, 2, \ldots, n$. Then, by the preceding paragraph, we obtain

$$\begin{aligned}
\|(a_1, a_2, \ldots, a_n)\| &\leq \|(b_1, a_2, \ldots, a_n)\| \\
&\leq \|(b_1, b_2, a_3, \ldots, a_n)\| \\
&\ \vdots \\
&\leq \|(b_1, b_2, \ldots, b_n)\|
\end{aligned}$$

Thus the norm $\|\cdot\|$ is semi-monotone. □

We conclude this section with the following basic relationship among AN_n, RN_n and NN_n.

Proposition 2.4. *$AN_n \subset RN_n \subset NN_n$. In particular, $RN_2 = NN_2$.*

3. Regular norms on $\mathbb{R}^3$

In this section, we consider the results in the case $\mathbb{R}^3$. The reason for this is that the results for $\mathbb{R}^3$ illustrate all the mechanism involved in the induction to follow.

A norm $\|\cdot\|$ on $\mathbb{R}^3$ is said to be regular if it is normalized and

$$\|(x, y, z)\| \geq \max\{\|(x, y, 0)\|, \|(x, 0, z)\|, \|(0, y, z)\|\}$$

for all $(x, y, z) \in \mathbb{R}^3$.

For each vector $p = (a, b, c) \in \mathbb{R}^3$, we define $\Omega(p)$ and $\Delta_3(p)$ by

$$\Omega(p) = \{(ax, by, cz) \in \mathbb{R}^3 : x, y, z \geq 0\},$$
$$\Delta_3(p) = \{(as, bt, cu) \in \mathbb{R}^3 : (s, t, u) \in \Delta_3\},$$

where

$$\Delta_3 = \{(s, t, u) \in \mathbb{R}^3 : s, t, u \geq 0,\ s + t + u = 1\}.$$

Recall that

$$R_3^+ = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{pmatrix},$$

and

$$\Omega_1 = \Omega(\theta_1) = \{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0\},$$
$$\Omega_2 = \Omega(\theta_2) = \{(x, y, -z) \in \mathbb{R}^3 : x, y, z \geq 0\},$$
$$\Omega_3 = \Omega(\theta_3) = \{(x, -y, -z) \in \mathbb{R}^3 : x, y, z \geq 0\},$$
$$\Omega_4 = \Omega(\theta_4) = \{(x, -y, z) \in \mathbb{R}^3 : x, y, z \geq 0\}.$$

Putting $\Theta_3 = \{\theta_1, \theta_2, \theta_3, \theta_4\}$, we have

$$\mathbb{R}^3 = \bigcup_{i=1}^{4} (\Omega(\theta_i) \cup \Omega(-\theta_i)).$$

The following lemmas are needed in the sequel.

Lemma 3.1. *Let $\theta_i, \theta_j \in \Theta_3$. Then*

$$\Omega(\theta_i) \cap \Omega(\theta_j) = \Omega\left(\frac{\theta_i + \theta_j}{2}\right),$$
$$\Omega(\theta_i) \cap \Omega(-\theta_j) = \Omega\left(\frac{\theta_i - \theta_j}{2}\right).$$

Lemma 3.2. *Let $\theta_i, \theta_j \in \Theta_3$. Then*

$$\Delta_3(\theta_i) \cap \Delta_3(\theta_j) = \Delta_3\left(\frac{\theta_i + \theta_j}{2}\right),$$
$$\Delta_3(\theta_i) \cap \Delta_3(-\theta_j) = \Delta_3\left(\frac{\theta_i - \theta_j}{2}\right).$$

Regular norms on $\mathbb{R}^3$ have the following property.

Proposition 3.3. *Suppose that* $\|\cdot\| \in RN_3$. *Let*

$$\begin{aligned}
\varphi_1(s,t,u) &= \|(s,t,u)\|, \\
\varphi_2(s,t,u) &= \|(s,t,-u)\|, \\
\varphi_3(s,t,u) &= \|(s,-t,-u)\|, \\
\varphi_4(s,t,u) &= \|(s,-t,u)\|,
\end{aligned}$$

for all $(s,t,u) \in \Delta_3$. *Then,* $(\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in (\Psi_3)^4$ *and*

$$\|(x,y,z)\| = \begin{cases} \|(x,y,z)\|_{\varphi_1} & \text{if } (x,y,z) \in \Omega(\theta_1) \cup \Omega(-\theta_1), \\ \|(x,y,z)\|_{\varphi_2} & \text{if } (x,y,z) \in \Omega(\theta_2) \cup \Omega(-\theta_2), \\ \|(x,y,z)\|_{\varphi_3} & \text{if } (x,y,z) \in \Omega(\theta_3) \cup \Omega(-\theta_3), \\ \|(x,y,z)\|_{\varphi_4} & \text{if } (x,y,z) \in \Omega(\theta_4) \cup \Omega(-\theta_4). \end{cases}$$

Proof. First, we show that $\varphi_i \in \Psi_3$ for $i = 1,2,3,4$. Take an arbitrary $(s,t,u) \in \Delta_3$. If $s < 1$ then $t + u > 0$ and

$$\begin{aligned}
\varphi_1(s) &= \|(s,t,u)\| \\
&\geq \|(0,t,u)\| \\
&= (t+u)\left\|\left(0, \frac{t}{t+u}, \frac{u}{t+u}\right)\right\| \\
&= (t+u)\varphi_1\left(0, \frac{t}{t+u}, \frac{u}{t+u}\right).
\end{aligned}$$

If $t < 1$ then $s + u > 0$ and

$$\begin{aligned}
\varphi_1(s) &= \|(s,t,u)\| \\
&\geq \|(s,0,u)\| \\
&= (s+u)\left\|\left(\frac{s}{s+u}, 0, \frac{u}{s+u}\right)\right\| \\
&= (s+u)\varphi_1\left(\frac{s}{s+u}, 0, \frac{u}{s+u}\right).
\end{aligned}$$

If $u < 1$ then $s + t > 0$ and

$$\begin{aligned}
\varphi_1(s) &= \|(s,t,u)\| \\
&\geq \|(s,t,0)\| \\
&= (s+t)\left\|\left(\frac{s}{s+t}, \frac{t}{s+t}, 0\right)\right\| \\
&= (s+t)\varphi_1\left(\frac{s}{s+t}, \frac{t}{s+t}, 0\right).
\end{aligned}$$

Thus we have $\varphi_1 \in \Psi_3$. Similarly, one can easily have $\varphi_2, \varphi_3, \varphi_4 \in \Psi_3$.

Next, we suppose that $x, y, z \geq 0$. It may be assumed that $(x, y, z) \neq 0$. Then we have

$$\begin{aligned}
\|(x, y, z)\| &= (x+y+z) \left\| \left(\frac{x}{x+y+z}, \frac{y}{x+y+z}, \frac{z}{x+y+z} \right) \right\| \\
&= (x+y+z)\varphi_1 \left(\frac{x}{x+y+z}, \frac{y}{x+y+z}, \frac{z}{x+y+z} \right) \\
&= \|(x, y, z)\|_{\varphi_1},
\end{aligned}$$

$$\begin{aligned}
\|(x, y, -z)\| &= (x+y+z) \left\| \left(\frac{x}{x+y+z}, \frac{y}{x+y+z}, \frac{-z}{x+y+z} \right) \right\| \\
&= (x+y+z)\varphi_2 \left(\frac{x}{x+y+z}, \frac{y}{x+y+z}, \frac{z}{x+y+z} \right) \\
&= \|(x, y, z)\|_{\varphi_2} \\
&= \|(x, y, -z)\|_{\varphi_2},
\end{aligned}$$

$$\begin{aligned}
\|(x, -y, -z)\| &= (x+y+z) \left\| \left(\frac{x}{x+y+z}, \frac{-y}{x+y+z}, \frac{-z}{x+y+z} \right) \right\| \\
&= (x+y+z)\varphi_3 \left(\frac{x}{x+y+z}, \frac{y}{x+y+z}, \frac{z}{x+y+z} \right) \\
&= \|(x, y, z)\|_{\varphi_3} \\
&= \|(x, -y, -z)\|_{\varphi_3},
\end{aligned}$$

$$\begin{aligned}
\|(x, -y, z)\| &= (x+y+z) \left\| \left(\frac{x}{x+y+z}, \frac{-y}{x+y+z}, \frac{z}{x+y+z} \right) \right\| \\
&= (x+y+z)\varphi_4 \left(\frac{x}{x+y+z}, \frac{y}{x+y+z}, \frac{z}{x+y+z} \right) \\
&= \|(x, y, z)\|_{\varphi_4} \\
&= \|(x, -y, z)\|_{\varphi_4}.
\end{aligned}$$

Thus we have

$$\|(x, y, z)\| = \begin{cases} \|(x, y, z)\|_{\varphi_1} & \text{if } (x, y, z) \in \Omega(\theta_1) \cup \Omega(-\theta_1), \\ \|(x, y, z)\|_{\varphi_2} & \text{if } (x, y, z) \in \Omega(\theta_2) \cup \Omega(-\theta_2), \\ \|(x, y, z)\|_{\varphi_3} & \text{if } (x, y, z) \in \Omega(\theta_3) \cup \Omega(-\theta_3), \\ \|(x, y, z)\|_{\varphi_4} & \text{if } (x, y, z) \in \Omega(\theta_4) \cup \Omega(-\theta_4). \end{cases}$$

This completes the proof. □

From this result, it turns out that RN_3 is a generalization of the class of generalized Day-James type norms.

Next, we present a characterization of regular norms on $\mathbb{R}^3$ similar to [8, Theorem 2.5].

Lemma 3.4. *Let $\|\cdot\| \in RN_3$ and let $F = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ be the element of $(\Psi_3)^4$ defined as in Proposition 3.3. If $i, j \in \{1, 2, 3, 4\}$, then*

$$\varphi_i(|s|, |t|, |u|) = \varphi_j(|s|, |t|, |u|)$$

for all $(s, t, u) \in \Delta_3((\theta_i + \theta_j)/2) \cup \Delta_3((\theta_i - \theta_j)/2)$.

Proof. By Proposition 3.3, we have $\|\cdot\|_{\varphi_i} = \|\cdot\|_{\varphi_j}$ on $\Omega((\theta_i+\theta_j)/2) \cup \Omega((\theta_i-\theta_j)/2)$. Take arbitrary $(s, t, u) \in \Delta_3((\theta_i + \theta_j)/2) \cup \Delta_3((\theta_i - \theta_j)/2)$. Since $\|\cdot\|_{\varphi_i}$ and $\|\cdot\|_{\varphi_j}$ are absolute, we obtain

$$\begin{aligned}
\varphi_i(|s|, |t|, |u|) &= \|(|s|, |t|, |u|)\|_{\varphi_i} \\
&= \|(s, t, u)\|_{\varphi_i} \\
&= \|(s, t, u)\|_{\varphi_j} \\
&= \|(|s|, |t|, |u|)\|_{\varphi_j} \\
&= \varphi_j(|s|, |t|, |u|).
\end{aligned}$$

Thus we have $\varphi_i(|s|, |t|, |u|) = \varphi_j(|s|, |t|, |u|)$ whenever $(s, t, u) \in \Delta_3((\theta_i + \theta_j)/2) \cup \Delta_3((\theta_i - \theta_j)/2)$. □

Let Φ_3 denote the family of all elements of $(\Psi_3)^4$ satisfying the following condition: If $i, j \in \{1, 2, 3, 4\}$, then

$$\varphi_i(|s|, |t|, |u|) = \varphi_j(|s|, |t|, |u|)$$

for all $(s, t, u) \in \Delta_3((\theta_i + \theta_j)/2) \cup \Delta_3((\theta_i - \theta_j)/2)$.

Our aim is to show that RN_3 and Φ_3 are in a one-to-one correspondence. To this end, some lemmas are needed.

Lemma 3.5. *Let $\psi \in \Psi_3$ and let $\|\cdot\|_{\widehat{\psi}}$ be a function on $\mathbb{R}^3$ defined by*

$$\|(x, y, z)\|_{\widehat{\psi}} = \max\{\|(x^+, y^+, z^+)\|_\psi, \|(x^-, y^-, z^-)\|_\psi\}$$

for all $(x, y, z) \in \mathbb{R}^3$, where $x \mapsto x^+$ and $x \mapsto x^-$ is defined by

$$x^+ = \max\{x, 0\} \quad \text{and} \quad x^- = \max\{-x, 0\},$$

respectively. Then $\|\cdot\|_{\widehat{\psi}}$ is a norm on $\mathbb{R}^3$. Moreover,

$$\|(x, y, z)\|_{\widehat{\psi}} = \|(x, y, z)\|_\psi$$

whenever $x, y, z \geq 0$ or $x, y, z \leq 0$.

The following lemma is the converse of Lemma 3.4.

Lemma 3.6. *Let $F = (\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in \Phi_3$. If $i, j \in \{1, 2, 3, 4\}$, then*

$$\|\cdot\|_{\varphi_i} = \|\cdot\|_{\varphi_j}$$

on $\Omega((\theta_i + \theta_j)/2) \cup \Omega((\theta_i - \theta_j)/2)$.

Proof. Let $i, j \in \{1, 2, 3, 4\}$. Suppose that $(x, y, z) \in \Omega((\theta_i+\theta_j)/2) \cup \Omega((\theta_i-\theta_j)/2)$. We may assume that $x \neq 0$. Then we have

$$\left(\frac{x}{|x|+|y|+|z|}, \frac{y}{|x|+|y|+|z|}, \frac{z}{|x|+|y|+|z|}\right) \in \Delta_3\left(\frac{\theta_i+\theta_j}{2}\right) \cup \Delta_3\left(\frac{\theta_i-\theta_j}{2}\right).$$

Since $F \in \Phi_3$, we have

$$\begin{aligned}
\|(x,y,z)\|_{\varphi_i} &= (|x|+|y|+|z|)\left\|\left(\frac{|x|}{|x|+|y|+|z|}, \frac{|y|}{|x|+|y|+|z|}, \frac{|z|}{|x|+|y|+|z|}\right)\right\|_{\varphi_i} \\
&= (|x|+|y|+|z|)\varphi_i\left(\frac{|x|}{|x|+|y|+|z|}, \frac{|y|}{|x|+|y|+|z|}, \frac{|z|}{|x|+|y|+|z|}\right) \\
&= (|x|+|y|+|z|)\varphi_j\left(\frac{|x|}{|x|+|y|+|z|}, \frac{|y|}{|x|+|y|+|z|}, \frac{|z|}{|x|+|y|+|z|}\right) \\
&= (|x|+|y|+|z|)\left\|\left(\frac{|x|}{|x|+|y|+|z|}, \frac{|y|}{|x|+|y|+|z|}, \frac{|z|}{|x|+|y|+|z|}\right)\right\|_{\varphi_j} \\
&= \|(x,y,z)\|_{\varphi_j}.
\end{aligned}$$

This completes the proof. □

We now present a characterization of regular norms on $\mathbb{R}^3$. Then some preparations are needed.

Theorem 3.7. *The following holds:*

(i) *Let $\|\cdot\| \in RN_3$ and let $F = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ be the element of $(\Psi_3)^4$ defined as in Proposition 3.3. Then $F \in \Phi_3$.*

(ii) *Let $F = (\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in \Phi_3$ and let $\|\cdot\|_F$ be the function on $\mathbb{R}^3$ defined by*

$$\|(x,y,z)\|_F = \max\{\|(x,y,z)\|_{\widehat{\varphi_1}}, \|(x,y,-z)\|_{\widehat{\varphi_2}}, \|(x,-y,-z)\|_{\widehat{\varphi_3}}, \|(x,-y,z)\|_{\widehat{\varphi_4}}\}$$

for all $(x, y, z) \in \mathbb{R}^3$. Then $\|\cdot\|_F \in RN_3$ and

$$\begin{aligned}
\varphi_1(s,t,u) &= \|(s,t,u)\|_F, \\
\varphi_2(s,t,u) &= \|(s,t,-u)\|_F, \\
\varphi_3(s,t,u) &= \|(s,-t,-u)\|_F, \\
\varphi_4(s,t,u) &= \|(s,-t,u)\|_F,
\end{aligned}$$

for all $(s, t, u) \in \Delta_3$.

Proof. (i) This is just a statement of Lemma 3.4.

(ii) Let $\|\cdot\|_{F,i}$ be the functions on $\mathbb{R}^n$ defined by

$$\begin{aligned}
\|(x,y,z)\|_{F,1} &= \|(x,y,z)\|_{\widehat{\varphi_1}}, \\
\|(x,y,z)\|_{F,2} &= \|(x,y,-z)\|_{\widehat{\varphi_2}}, \\
\|(x,y,z)\|_{F,3} &= \|(x,-y,-z)\|_{\widehat{\varphi_3}}, \\
\|(x,y,z)\|_{F,4} &= \|(x,-y,z)\|_{\widehat{\varphi_4}},
\end{aligned}$$

for all $(x,y,z) \in \mathbb{R}^3$. Then, it is easy to see that each $\|\cdot\|_{F,i}$ is a norm on $\mathbb{R}^3$. We note that

$$\|x\|_F = \max_{1 \le i \le 4} \|x\|_{\varphi_i,\infty}$$

for all $x \in \mathbb{R}^3$, which implies that $\|\cdot\|_F$ is a norm on $\mathbb{R}^n$.

Next, we prove that $\|x\|_F = \|x\|_{\varphi_i}$ whenever $x \in \Omega(\theta_i) \cup \Omega(-\theta_i)$. Let $x, y, z \ge 0$. Then we have

$$\|(x,y,z)\|_{F,1} = \|(x,y,z)\|_{\widehat{\varphi_1}} = \|(x,y,z)\|_{\varphi_1}$$

by Lemma 3.5. By Lemmas 2.1, 3.5 and 3.6, we also have

$$\begin{aligned}
\|(x,y,z)\|_{F,2} &= \|(x,y,-z)\|_{\widehat{\varphi_2}} \\
&= \max\{\|(x,y,0)\|_{\varphi_2}, \|(0,0,z)\|_{\varphi_2}\} \\
&= \max\{\|(x,y,0)\|_{\varphi_1}, \|(0,0,z)\|_{\varphi_1}\} \\
&\le \|(x,y,z)\|_{\varphi_1},
\end{aligned}$$

$$\begin{aligned}
\|(x,y,z)\|_{F,3} &= \|(x,-y,-z)\|_{\widehat{\varphi_3}} \\
&= \max\{\|(x,0,0)\|_{\varphi_3}, \|(0,y,z)\|_{\varphi_3}\} \\
&= \max\{\|(x,0,0)\|_{\varphi_1}, \|(0,y,z)\|_{\varphi_1}\} \\
&\le \|(x,y,z)\|_{\varphi_1},
\end{aligned}$$

and

$$\begin{aligned}
\|(x,y,z)\|_{F,4} &= \|(x,-y,z)\|_{\widehat{\varphi_4}} \\
&= \max\{\|(x,0,z)\|_{\varphi_4}, \|(0,y,0)\|_{\varphi_4}\} \\
&= \max\{\|(x,0,z)\|_{\varphi_1}, \|(0,y,0)\|_{\varphi_1}\} \\
&\le \|(x,y,z)\|_{\varphi_1}.
\end{aligned}$$

Thus we obtain

$$\|(x,y,z)\|_F = \max_{1 \le i \le 4} \|(x,y,z)\|_{\widehat{\varphi_i}} = \|(x,y,z)\|_{\varphi_1}$$

for all $(x, y, z) \in \Omega(\theta_1)$. This also implies $\|x\|_F = \|x\|_{\varphi_1}$ for all $x \in \Omega(-\theta_1)$. Similarly, one can prove that $\|x\|_F = \|x\|_{\varphi_i}$ whenever $x \in \Omega(\theta_i) \cup \Omega(-\theta_i)$. We remark that this shows

$$\begin{aligned}
\varphi_1(s,t,u) &= \|(s,t,u)\|_F, \\
\varphi_2(s,t,u) &= \|(s,t,-u)\|_F, \\
\varphi_3(s,t,u) &= \|(s,-t,-u)\|_F, \\
\varphi_4(s,t,u) &= \|(s,-t,u)\|_F
\end{aligned}$$

for all $(s, t, u) \in \Delta_3$.

Finally, take an arbitrary $(x, y, z) \in \mathbb{R}^3$. Then there exists $i \in \{1, 2, 3, 4\}$ such that $(x, y, z) \in \Omega(\theta_i) \cup \Omega(-\theta_i)$. We note that $(x, y, 0), (x, 0, z), (0, y, z) \in \Omega(\theta_i) \cup \Omega(-\theta_i)$, and so we have

$$\begin{aligned}
&\max\{\|(x,y,0)\|_F, \|(x,0,z)\|_F, \|(0,y,z)\|_F\} \\
&= \max\{\|(x,y,0)\|_{\varphi_i}, \|(x,0,z)\|_{\varphi_i}, \|(0,y,z)\|_{\varphi_i}\} \\
&\leq \|(x,y,z)\|_{\varphi_i} \\
&= \|(x,y,z)\|_F
\end{aligned}$$

by Lemma 2.1. Hence, we obtain $\|\cdot\|_F \in RN_3$. This completes the proof. $\square$

Thus, RN_3 and Φ_3 are in a one-to-one correspondence under the equation

$$\begin{aligned}
\varphi_1(s,t,u) &= \|(s,t,u)\|_F, \\
\varphi_2(s,t,u) &= \|(s,t,-u)\|_F, \\
\varphi_3(s,t,u) &= \|(s,-t,-u)\|_F, \\
\varphi_4(s,t,u) &= \|(s,-t,u)\|_F
\end{aligned}$$

for all $(s, t, u) \in \Delta_3$.

4. Regular norms on $\mathbb{R}^n$

In this section, we consider RN_n. For each vector $p = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, we define $\Omega(p)$ and $\Delta_n(p)$ by

$$\begin{aligned}
\Omega(p) &= \{(a_1t_1, a_2t_2, \ldots, a_nt_n) \in \mathbb{R}^n : t_1, t_2, \ldots, t_n \geq 0\}, \\
\Delta_n(p) &= \{(a_1s_1, a_2s_2, \ldots, a_ns_n) \in \mathbb{R}^n : (s_1, s_2, \ldots, s_n) \in \Delta_n\},
\end{aligned}$$

where

$$\Delta_n = \left\{(s_1, s_2, \ldots, s_n) \in \mathbb{R}^n : s_1, s_2, \ldots, s_n \geq 0,\ \sum_{i=1}^n s_i = 1\right\}.$$

Fix a positive integer $n \geq 2$. Recall that

$$R_n^+ = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_{2^{n-1}} \end{pmatrix},$$

$\Omega_i = \Omega(\theta_i)$ for each $i = 1, 2, \ldots, 2^{n-1}$, and

$$\mathbb{R}^n = \bigcup_{\theta \in \Theta_n} (\Omega(\theta) \cup \Omega(-\theta)),$$

where $\Theta_n = \{\theta_i : 1 \leq i \leq 2^{n-1}\}$.

The following lemmas are generalization of Lemmas 3.1 and 3.2.

Lemma 4.1. *Let* $\theta_i, \theta_j \in \Theta_n$. *Then*

$$\Omega(\theta_i) \cap \Omega(\theta_j) = \Omega\left(\frac{\theta_i + \theta_j}{2}\right),$$
$$\Omega(\theta_i) \cap \Omega(-\theta_j) = \Omega\left(\frac{\theta_i - \theta_j}{2}\right).$$

Lemma 4.2. *Let* $\theta_i, \theta_j \in \Theta_n$. *Then*

$$\Delta_n(\theta_i) \cap \Delta_n(\theta_j) = \Delta_n\left(\frac{\theta_i + \theta_j}{2}\right),$$
$$\Delta_n(\theta_i) \cap \Delta_n(-\theta_j) = \Delta_n\left(\frac{\theta_i - \theta_j}{2}\right).$$

For the convenience, henceforth, let

$$\theta_i = (r_{i1}, r_{i2}, \ldots, r_{in})$$

for all $i = 1, 2, \ldots, 2^{n-1}$.

Then, regular norms on $\mathbb{R}^n$ have the following property.

Proposition 4.3. *Let* $\|\cdot\| \in RN_n$. *For each* $i \in \{1, 2, \ldots, 2^{n-1}\}$, *let*

$$\varphi_i(s) = \|(r_{i1}s_1, r_{i2}s_2, \ldots, r_{in}s_n)\|.$$

for all $s = (s_1, s_2, \ldots, s_n) \in \Delta_n$. *Then,* $(\varphi_1, \varphi_2, \ldots, \varphi_{2^{n-1}}) \in (\Psi_n)^{2^{n-1}}$ *and* $\|x\| = \|x\|_{\varphi_i}$ *whenever* $x \in \Omega(\theta_i) \cup \Omega(-\theta_i)$.

Proof. Suppose that $i \in \{1, 2, \ldots, 2^{n-1}\}$. First, we show that $\varphi_i \in \Psi_n$. Take an arbitrary $(s_1, s_2, \ldots, s_n) \in \Delta_n$. For each $k \in \{1, 2, \ldots, n\}$ with $s_k < 1$, we obtain

$$\begin{aligned}
\varphi_i(s) &= \|(r_{i1}s_1, r_{i2}s_2, \ldots, r_{in}s_n)\| \\
&\geq \|(r_{i1}s_1, \ldots, , r_{i(k-1)}s_{k-1}, 0, , r_{i(k+1)}s_{k+1}, \ldots, r_{in}s_n)\| \\
&= (1 - s_k) \left\| \left(\frac{r_{i1}s_1}{1 - s_k}, \ldots, \frac{r_{i(k-1)}s_{k-1}}{1 - s_k}, 0, \frac{r_{i(k+1)}s_{k+1}}{1 - s_k}, \ldots, \frac{r_{in}s_n}{1 - s_k} \right) \right\| \\
&= (1 - s_k)\varphi_i \left(\frac{s_1}{1 - s_k}, \ldots, \frac{s_{k-1}}{1 - s_k}, 0, \frac{s_{k+1}}{1 - s_k}, \ldots, \frac{s_n}{1 - s_k} \right).
\end{aligned}$$

Thus we have $\varphi_i \in \Psi_n$.

Next, we suppose that $x = (r_{i1}t_1, r_{i2}t_2, \ldots, r_{in}t_n) \in \Omega(\theta_i)$. It may be assumed that $x \neq 0$. Then we have

$$\begin{aligned}
\|x\| &= \left(\sum_{\ell=1}^{n} t_\ell \right) \left\| \left(\frac{r_{i1}t_1}{\sum_{\ell=1}^{n} t_\ell}, \frac{r_{i2}t_2}{\sum_{\ell=1}^{n} t_\ell}, \ldots, \frac{r_{in}t_n}{\sum_{\ell=1}^{n} t_\ell} \right) \right\| \\
&= \left(\sum_{\ell=1}^{n} t_\ell \right) \varphi_i \left(\frac{t_1}{\sum_{\ell=1}^{n} t_\ell}, \frac{t_2}{\sum_{\ell=1}^{n} t_\ell}, \ldots, \frac{t_n}{\sum_{\ell=1}^{n} t_\ell} \right) \\
&= \|x\|_{\varphi_i}.
\end{aligned}$$

This also implies $\|x\| = \|x\|_{\varphi_i}$ if $x \in \Omega(-\theta_i)$. Hence we obtain $\|x\| = \|x\|_{\varphi_i}$ whenever $x \in \Omega(\theta_i) \cup \Omega(-\theta_i)$. This completes the proof. □

As in the case $n = 3$, Proposition 4.3 shows that the set RN_n is a generalization of that of generalized Day-James type norms.

We next present a characterization of regular norms on $\mathbb{R}^n$. A one-to-one correspondence between RN_n and a certain subset of $(\Psi_n)^{2^{n-1}}$ will be given.

Lemma 4.4. *Let $\|\cdot\| \in RN_n$ and let $F = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ be the element of $(\Psi_n)^{2^{n-1}}$ defined as in Proposition 4.3. If $i, j \in \{1, 2, \ldots, 2^{n-1}\}$, then*

$$\varphi_i(|s_1|, |s_2|, \ldots, |s_n|) = \varphi_j(|s_1|, |s_2|, \ldots, |s_n|)$$

for all $(s_1, s_2, \ldots, s_n) \in \Delta_n((\theta_i + \theta_j)/2) \cup \Delta_n((\theta_i - \theta_j)/2)$.

Proof. By Proposition 4.3, we have $\|\cdot\|_{\varphi_i} = \|\cdot\|_{\varphi_j}$ on $\Omega((\theta_i + \theta_j)/2) \cup \Omega((\theta_i - \theta_j)/2)$. Take an arbitrary $s = (s_1, s_2, \ldots, s_n) \in \Delta_n((\theta_i + \theta_j)/2) \cup \Delta_n((\theta_i - \theta_j)/2)$. Then, we note that

$$s \in \Omega \left(\frac{\theta_i + \theta_j}{2} \right) \cup \Omega \left(\frac{\theta_i - \theta_j}{2} \right)$$

and $|s| = (|s_1|, |s_2|, \ldots, |s_n|) \in \Delta_n$. Since $\|\cdot\|_{\varphi_i}$ and $\|\cdot\|_{\varphi_j}$ are absolute, we obtain

$$\varphi_i(|s|) = \|s\|_{\varphi_i} = \|s\|_{\varphi_j} = \varphi_j(|s|).$$

Thus we have

$$\varphi_i(|s_1|,|s_2|,\ldots,|s_n|)=\varphi_j(|s_1|,|s_2|,\ldots,|s_n|)$$

whenever $s\in\Delta_n((\theta_i+\theta_j)/2)\cup\Delta_n((\theta_i-\theta_j)/2)$. $\square$

Let Φ_n denote the set of all elements of $(\Psi_n)^{2^{n-1}}$ satisfying the following condition: If $i,j\in\{1,2,\ldots,2^{n-1}\}$, then

$$\varphi_i(|s_1|,|s_2|,\ldots,|s_n|)=\varphi_j(|s_1|,|s_2|,\ldots,|s_n|)$$

for all $(s_1,s_2,\ldots,s_n)\in\Delta_n((\theta_i+\theta_j)/2)\cup\Delta_n((\theta_i-\theta_j)/2)$.

Our aim is to show that RN_n and Φ_n are in a one-to-one correspondence.

Lemma 4.5. *Let $\psi\in\Psi_n$ and let $\|\cdot\|_{\widehat{\psi}}$ be the function on $\mathbb{R}^n$ defined by*

$$\|(x_1,x_2,\ldots,x_n)\|_{\widehat{\psi}}=\max\{\|(x_1^+,x_2^+,\ldots,x_n^+)\|_\psi,\|(x_1^-,x_2^-,\ldots,x_n^-)\|_\psi\}$$

for all $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$. Then $\|\cdot\|_{\widehat{\psi}}$ is a norm on $\mathbb{R}^n$. Moreover,

$$\|(x_1,x_2,\ldots,x_n)\|_{\widehat{\psi}}=\|(x_1,x_2,\ldots,x_n)\|_\psi$$

whenever $x_1,x_2,\ldots,x_n\geq 0$ or $x_1,x_2,\ldots,x_n\leq 0$.

The following lemma is the converse of Lemma 4.4 in a sense.

Lemma 4.6. *Let $F=(\varphi_1,\varphi_2,\ldots,\varphi_{2^{n-1}})\in\Phi_n$. If $i,j\in\{1,2,\ldots,2^{n-1}\}$, then*

$$\|\cdot\|_{\varphi_i}=\|\cdot\|_{\varphi_j}$$

on $\Omega((\theta_i+\theta_j)/2)\cup\Omega((\theta_i-\theta_j)/2)$.

Proof. Let $x=(x_1,x_2,\ldots,x_n)\in\Omega((\theta_i+\theta_j)/2)\cup\Omega((\theta_i-\theta_j)/2)$. We may assume that $x\neq 0$. Then by Lemmas 4.1 and 4.2, we have

$$\left(\frac{x_1}{\sum_{\ell=1}^n|x_\ell|},\frac{x_2}{\sum_{\ell=1}^n|x_\ell|},\ldots,\frac{x_n}{\sum_{\ell=1}^n|x_\ell|}\right)\in\Omega\left(\frac{\theta_i+\theta_j}{2}\right)\cup\Omega\left(\frac{\theta_i-\theta_j}{2}\right).$$

Since $\|\cdot\|_{\varphi_i}$ and $\|\cdot\|_{\varphi_i}$ are absolute, we have

$$\begin{aligned}\|x\|_{\varphi_i} &= \left(\sum_{\ell=1}^n |x_\ell|\right) \left\| \left(\frac{x_1}{\sum_{\ell=1}^n |x_\ell|}, \frac{x_2}{\sum_{\ell=1}^n |x_\ell|}, \dots, \frac{x_n}{\sum_{\ell=1}^n |x_\ell|} \right) \right\|_{\varphi_i} \\ &= \left(\sum_{\ell=1}^n |x_\ell|\right) \varphi_i \left(\frac{|x_1|}{\sum_{\ell=1}^n |x_\ell|}, \frac{|x_2|}{\sum_{\ell=1}^n |x_\ell|}, \dots, \frac{|x_n|}{\sum_{\ell=1}^n |x_\ell|} \right) \\ &= \left(\sum_{\ell=1}^n |x_\ell|\right) \varphi_j \left(\frac{|x_1|}{\sum_{\ell=1}^n |x_\ell|}, \frac{|x_2|}{\sum_{\ell=1}^n |x_\ell|}, \dots, \frac{|x_n|}{\sum_{\ell=1}^n |x_\ell|} \right) \\ &= \left(\sum_{\ell=1}^n |x_\ell|\right) \left\| \left(\frac{x_1}{\sum_{\ell=1}^n |x_\ell|}, \frac{x_2}{\sum_{\ell=1}^n |x_\ell|}, \dots, \frac{x_n}{\sum_{\ell=1}^n |x_\ell|} \right) \right\|_{\varphi_j} \\ &= \|x\|_{\varphi_j}.\end{aligned}$$

This completes the proof. □

Regular norms on $\mathbb{R}^n$ are characterized as follows:

Theorem 4.7. *The following holds:*

(i) *Let* $\|\cdot\| \in RN_n$ *and let* $F = (\varphi_1, \varphi_2, \dots, \varphi_n)$ *be an element of* $(\Psi_n)^{2^{n-1}}$ *defined as in Proposition 4.3. Then* $F \in \Phi_n$.

(ii) *Let* $F = (\varphi_1, \varphi_2, \dots, \varphi_{2^{n-1}}) \in \Phi_n$ *and let* $\|\cdot\|_F$ *be a function on* $\mathbb{R}^n$ *defined by*

$$\|(x_1, x_2, \dots, x_n)\|_F = \max_{1 \le i \le 2^{n-1}} \|(r_{i1}x_1, r_{i2}x_2, \dots, r_{in}x_n)\|_{\widehat{\varphi_i}}$$

for all $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. *Then* $\|\cdot\|_F \in RN_n$ *and*

$$\varphi_i(s) = \|(r_{i1}s_1, r_{i2}s_2, \dots, r_{in}s_n)\|_F$$

for all $s = (s_1, s_2, \dots, s_n) \in \Delta_n$ *and all* $i \in \{1, 2, \dots, n\}$.

Proof. (i) This is just a statement of Lemma 4.4.

(ii) For each $i \in \{1, 2, \dots, 2^{n-1}\}$, let $\|\cdot\|_{F,i}$ be the function on $\mathbb{R}^n$ defined by

$$\|x\|_{F,i} = \|(r_{i1}x_1, r_{i2}x_2, \dots, r_{in}x_n)\|_{\widehat{\varphi_i}}$$

for all $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Then, it is easy to see that $\|\cdot\|_{F,i}$ is a norm on $\mathbb{R}^n$. We note that

$$\|x\|_F = \max\{\|x\|_{\widehat{\varphi_i}} : 1 \le i \le 2^{n-1}\}$$

for all $x \in \mathbb{R}^n$, which implies that $\|\cdot\|_F$ is a norm on $\mathbb{R}^n$.

Next, we prove that $\|x\|_F = \|x\|_{\varphi_i}$ whenever $x \in \Omega(\theta_i) \cup \Omega(-\theta_i)$. Let

$$x = (r_{i1}t_1, r_{i2}t_2, \dots, r_{in}t_n) \in \Omega_i.$$

Then

$$(r_{i1}^2 t_1, r_{i2}^2 t_2, \dots, r_{in}^2 t_n) = (t_1, t_2, \dots, t_n),$$

and so, by Lemma 4.5,

$$\|(r_{i1}^2 t_1, r_{i2}^2 t_2, \dots, r_{in}^2 t_n)\|_{(\varphi_i)} = \|(t_1, t_2, \dots, t_n)\|_{\varphi_i}.$$

Take an arbitrary $j \in \{1, 2, \dots, 2^{n-1}\}$ with $i \neq j$. Put $y_k = r_{jk} r_{ik} t_k$ for all $k \in \{1, 2, \dots, n\}$. Then we have

$$y_k^+ = \begin{cases} t_k & \text{if } r_{ik} = r_{jk}, \\ 0 & \text{if } r_{ik} \neq r_{jk}. \end{cases}$$

and

$$y_k^- = \begin{cases} 0 & \text{if } r_{ik} = r_{jk}, \\ t_k & \text{if } r_{ik} \neq r_{jk}. \end{cases}$$

So, by Lemma 4.1, we have $z = (r_{i1} y_1^+, r_{i2} y_2^+, \dots, r_{in} y_n^+) \in \Omega((\theta_i + \theta_j)/2)$ and $w = (r_{i1} y_1^-, r_{i2} y_2^-, \dots, r_{in} y_n^-) \in \Omega((\theta_i - \theta_j)/2)$. Thus, from the absoluteness of $\|\cdot\|_{\varphi_i}$ and $\|\cdot\|_{\varphi_j}$, and Lemmas 2.1 and 4.6, we obtain

$$\begin{aligned} \|(y_1, y_2, \dots, y_n)\|_{\widehat{\varphi_j}} &= \max\{\|(y_1^+, y_2^+, \dots, y_n^+)\|_{\varphi_j}, \|(y_1^-, y_2^-, \dots, y_n^-)\|_{\varphi_j}\} \\ &= \max\{\|z\|_{\varphi_j}, \|w\|_{\varphi_j}\} \\ &= \max\{\|z\|_{\varphi_i}, \|w\|_{\varphi_i}\} \\ &= \max\{\|(y_1^+, y_2^+, \dots, y_n^+)\|_{\varphi_i}, \|(y_1^-, y_2^-, \dots, y_n^-)\|_{\varphi_i}\} \\ &\leq \|(t_1, t_2, \dots, t_n)\|_{\varphi_i}. \end{aligned}$$

Therefore we have $\|x\|_F = \|(t_1, t_2, \dots, t_n)\|_{\varphi_i} = \|x\|_{\varphi_i}$. Moreover, this implies $\|x\|_F = \|x\|_{\varphi_i}$ on $\Omega(-\theta_i)$. Hence we obtain $\|x\|_F = \|x\|_{\varphi_i}$ for any $x \in \Omega(\theta_i) \cup \Omega(-\theta_i)$. We remark that this implies

$$\varphi_i(s) = \|(r_{i1} s_1, r_{i2} s_2, \dots, r_{in} s_n)\|_F$$

for all $s = (s_1, s_2, \dots, s_n) \in \Delta_n$ and all $i \in \{1, 2, \dots, n\}$.

Finally, take an arbitrary $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Then there exists a positive integer $i \in \{1, 2, \dots, 2^{n-1}\}$ such that $x \in \Omega(\theta_i) \cup \Omega(-\theta_i)$. We note that

$$(x_1, \dots, x_{k-1}, 0, x_{k+1}, x_n) \in \Omega(\theta_i) \cup \Omega(-\theta_i)$$

for each $k = 1, 2, \dots, n$. Since $\|\cdot\|_{\varphi_i}$ is absolute, we have

$$\begin{aligned} \|(x_1, \dots, x_{k-1}, 0, x_{k+1}, x_n)\|_F &= \|(x_1, \dots, x_{k-1}, 0, x_{k+1}, x_n)\|_{\varphi_i} \\ &\leq \|(x_1, x_2, \dots, x_n)\|_{\varphi_i} \\ &= \|(x_1, x_2, \dots, x_n)\|_F \end{aligned}$$

by Lemma 2.1. Hence, the norm $\|\cdot\|_F$ is regular. This completes the proof. □

Thus, RN_n and Φ_n are in a one-to-one correspondence under the equation

$$\varphi_i(s) = \|(r_{i1}s_1, r_{i2}s_2, \dots, r_{in}s_n)\|_F$$

for all $s = (s_1, s_2, \dots, s_n) \in \Delta_n$ and all $i = 1, 2, \dots, 2^{n-1}$. Obviously, every absolute normalized norm $\|\cdot\|_\psi$ on $\mathbb{R}^n$ is a regular norm induced by $(\psi, \psi, \dots, \psi)$.

Finally, we show some simple facts about regular norms.

Proposition 4.8. $\Phi_2 = \Psi_2^2$.

Proof. Suppose that $(\varphi_1, \varphi_2) \in \Psi_2^2$. We recall that

$$R_2^+ = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

Thus we have

$$\Delta_2\left(\frac{(1,1)+(1,-1)}{2}\right) = \Delta_2(1,0) = \{(1,0)\},$$
$$\Delta_2\left(\frac{(1,1)-(1,-1)}{2}\right) = \Delta_2(0,1) = \{(0,1)\}.$$

Since $\varphi_i(1,0) = \varphi_i(0,1) = 1$ for $i = 1, 2$, we obtain $(\varphi_1, \varphi_2) \in \Phi_2$. □

This is the reason why generalized Day-James spaces can be defined for any choice of $\psi, \varphi \in \Psi_2$; however, if $n \geq 3$, then this is not the case.

Remark. Suppose that $n \geq 3$. Let $F = (\psi_2, \psi_\infty, \dots, \psi_\infty) \in (\Psi_n)^{2^{n-1}}$. We recall that $\theta_1 = (1, 1, \dots, 1)$ and $\theta_2 = (1, \dots, 1, -1)$, and so we have

$$\Delta_n\left(\frac{\theta_1+\theta_2}{2}\right) = \left\{(s_1, s_2, \dots, s_{n-1}, 0) \in (\mathbb{R}^+)^n : \sum_{k=1}^{n-1} s_k = 1\right\},$$
$$\Delta_n\left(\frac{\theta_1-\theta_2}{2}\right) = \{(0, \dots, 0, 1)\}.$$

In particular,

$$s = \left(\frac{1}{n-1}, \dots, \frac{1}{n-1}, 0\right) \in \Delta_n\left(\frac{\theta_1+\theta_2}{2}\right).$$

However, then we obtain

$$\psi_2(s) = \|s\|_2 = \frac{1}{\sqrt{n-1}} > \frac{1}{n-1} = \|s\|_\infty = \psi_\infty(s).$$

Thus $F \notin \Phi_n$, and so $\Phi_n \subsetneq (\Psi_n)^{2^{n-1}}$. Moreover, since $s \in \Omega(\theta_1) \cap \Omega(\theta_2)$, it is impossible that F induces a regular norm on $\mathbb{R}^n$ in the sense of Theorem 4.7.

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