Nihonkai Math. J.
Vol.24(2013), 93–102

A SPHERE THEOREM FOR RADIAL CURVATURE

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Abstract. We introduce a new constant for the surfaces of revolution homeomorphic to the 2-sphere. We prove a sphere theorem for radial curvature, assuming an inequality in the constant and the ratio of the difference of the maximal distance to the base point from the diameter of the reference surface and the injectivity radius of the base point. Namely, if a compact pointed Riemannian n-manifold which is referred to a surface of revolution satisfies the inequality, then it is topologically an n-sphere.

1. Introduction

We study the radial curvature and topology of pointed and compact Riemannian n-manifolds. Many attempts have been made to extend the classical (diameter) sphere theorems. The notion of radial curvature was first introduced by Klingenberg [7]. The topological sphere theorems for pointed manifolds with positive radial curvature have been investigated in [13] and [12]. When the reference surface of a compact pointed manifold is a compact surface of revolution, Lee [10], [11] proved topological sphere theorems under certain restrictions on the diameter of M and on the supremum of the distance from the base point. Indeed, the restriction on the diameter was needed, since the Alexandrov-Toponogov comparison theorem for spherical model surfaces was not established for every geodesic triangle but only for narrow triangles. Kondo-Ohta [8] proved it under the assumptions that a compact surface is a von Mangoldt surface of revolution and that there exists a point $x \in M$ such that the base point is a critical point of the distance function to x. Recently

2010 *Mathematics Subject Classification.* Primary 53C20 ; Secondary 53C22.
Key words and phrases. Toponogov comparison theorem, sphere theorem, radial curvature, surface of revolution.
Research of the first author was partially supported by Grant-in-Aid for Scientific Research (C), 22540072.
Research of the second author was partially supported by Grant-in-Aid for Scientific Research (C) 22540106.

the Toponogov comparison theorem has been established in [4] under a certain restrictions on the cut locus and the critical points of distance function to the base point.

The purpose of the present paper is to establish a new topological sphere theorem for pointed compact manifolds whose reference surfaces are compact surfaces of revolution. Here the Toponogov comparison theoprem established in [4] plays an important role.

A compact model surface $(\widetilde{M}, \tilde{o})$ is by definition a compact Riemannian 2-manifold whose metric $ds^2_{\widetilde{M}}$ is expressed in terms of the polar coordinates around the base point $\tilde{o}$ as:

$$ds^2_{\widetilde{M}} = dr^2 + f(r)^2 d\theta^2, \quad (r, \theta) \in (0, \ell) \times \mathbf{S}^1, \ell < \infty. \tag{1.1}$$

Here, $r : \widetilde{M} \to \mathbf{R}$ is the distance function to $\tilde{o}$, and $f : (0, \ell) \to \mathbf{R}$ the warping function which is positive smooth and satisfies the Jacobi equation:

$$f'' + Kf = 0, \quad f(0) = f(\ell) = 0, \quad f'(0) = -f'(\ell) = 1.$$

Here, $K : [0, \ell] \to \mathbf{R}$ is the Gaussian curvature of $\widetilde{M}$.

Let $\tilde{o}_1 := (\ell, 0)$ be the farthest point from $\tilde{o}$ in $\widetilde{M}$. For an arbitrary fixed point $\tilde{p} \neq \tilde{o}, \tilde{o}_1$ in $\widetilde{M}$, we set $\theta(\tilde{p}) := 0$. We divide $(\widetilde{M}, \tilde{o})$ by a simple closed geodesic consisting of two meridians $\theta^{-1}(\{0\}) \cup \theta^{-1}(\{\pi\})$ into $\widetilde{M}^+_{\tilde{p}}$ and $\widetilde{M}^-_{\tilde{p}}$, where $\widetilde{M}^+_{\tilde{p}} := \theta^{-1}([0, \pi])$.

Let (M, o) be a pointed compact Riemannian n-manifold. A 2-plane $\Pi \subset M_x$ at a point $x \in M$ is called a *radial plane* iff it contains a vector tangent to a minimizing geodesic joining o to x. The sectional curvature $K_M(\Pi)$ of M with respect to a radial plane is called a *radial curvature of* (M, o). A compact pointed manifold (M, o) is said to be *referred to* a compact model surface $(\widetilde{M}, \tilde{o})$ if and only if all the radial sectional curvatures of M satisfy

$$K_M(\Pi) \geq K(d(o, x)), \quad x \in M. \tag{1.2}$$

Let $i : M, \ \widetilde{M} \to \mathbf{R}$ be the injectivity radius function of the exponential map on $\widetilde{M}$ and M respectively. Let $B(\tilde{x}, a) \subset \widetilde{M}$ be the open metric a-ball with center at $\tilde{x}$. Let $\mathrm{Cut}(\tilde{p})$ denote the cut locus of $\tilde{p} \in \widetilde{M}$.

For the statement of our theorem, we define some constants given on $\widetilde{M}$. Let c_1 be the supremum of those $c > 0$ which satisfy

(1) $f^{-1}(\{t\})$ consists of two points for all $t \in [0, c)$,
(2) $[f^{-1}]'(t) \neq 0$ for all $t \in [0, c)$.

Let r_1 and ${r_1}^*$ be such that $0 < r_1 \leq \ell - {r_1}^* < \ell$ and $f(r_1) = f(\ell - {r_1}^*) = c_1$. Then, $f'(r) > 0$ for $r \in [0, r_1)$ and $f'(r) < 0$ for $r \in (\ell - {r_1}^*, \ell]$.

We define a constant $c_2(r)$, $r > 0$, by

$$\begin{aligned} &c_2(r) \\ := \quad &\sup\{r^* \mid \mathrm{Cut}(\tilde{p}) \cap \mathrm{Int}(\widetilde{M}^+_{\tilde{p}}) \subset B(\tilde{o}, r) \text{ for } \tilde{p} \text{ with } r(\tilde{p}) > \ell - r^*\}. \end{aligned}$$

Here $\mathrm{Int}(\widetilde{M}^+_{\tilde{p}})$ is by definition the interior of the set $\widetilde{M}^+_{\tilde{p}}$.

We say that a minimizing geodesic segment T is *maximal* if any extension of T is not minimizing. Let $\Gamma(\tilde{p})$ be the maximal minimizing geodesic segment emanating from $\tilde{p}$ and tangent to the parallel $r^{-1}(\{r(\tilde{p})\})$ at $\tilde{p}$. The terminal point of $\Gamma(\tilde{p})$ is denoted by $\Gamma(\tilde{p})_e$. For every $r \in (0, r_1)$, we define $c_3(r)$ by

$$c_3(r) := \sup\{r^* \mid \Gamma(\tilde{p})_e \in B(\tilde{o}, r) \text{ for any } \tilde{p} \text{ with } r(\tilde{p}) > \ell - r^*\}.$$

Both $c_2(r)$ and $c_3(r)$ are monotone and non-decreasing in $r \in (0, r_1)$. Finally, we define the constant $\mu = \mu(\widetilde{M})$ by

$$\mu = \inf_{r \in (0, r_1)} \frac{\min\{c_2(r), c_3(r)\}}{r}.$$

Notice that the constants c_1, r_1, $c_2(r)$, $c_3(r)$ and μ are determined by the metric on $\widetilde{M}$. With this notation we state our theorem:

Theorem 1.1. *Assume that a compact pointed n-dimensional Riemannian manifold (M, o) is referred to $(\widetilde{M}, \tilde{o})$ with $\mu > 0$. Then, M is homeomorphic to an n-sphere, if there exists a point $p \in M$ such that*

$$\mu > \frac{\ell - d(o, p)}{\min\{r_1, i(o)\}}. \tag{1.3}$$

It has been proved in [5] that if M is referred to $\widetilde{M}$, then $\ell \geq \max\{d(o, x) \mid x \in M\}$, equality holding if and only if M is isometric to the warped product $[0, \ell] \times_f S^{n-1}(1)$ with warping function f. Here $S^{n-1}(1)$ denotes the unit sphere with dimension $n-1$. Hence, we note $\ell - d(o, p) \geq 0$.

Our theorem is thought of as a new version of the classical diameter sphere theorem [3], in which two assumptions are settled. No further assumption is needed. We will mention the difference of their assumptions in Remark 5.2.

The importance of injectivity radii in studying sphere theorems are seen in [1], [2].

We would like to thank Professor M. Tanaka and Dr. K. Kondo for notifying us of the cut loci in a surface of revolution which is symmetric with respect to the equator (see Example 5.3).

2. Preliminaries

2.1. Basic lemma

If $\tilde{p}$ and $r_2 \in (0, r_1)$ satisfy $d(\tilde{o}, \tilde{p}) > \ell - r_1^*$ and $f(r_2) = f(r(\tilde{p}))$, then the geodesic segment $\Gamma(\tilde{p})$ intersects $r^{-1}(\{r_1\})$ at most once before it meets $r^{-1}(\{r_2\})$. The terminal point of $\Gamma(\tilde{p})$ is denoted by $\Gamma(\tilde{p})_e$. Notice that $\Gamma(\tilde{p})$ converges to the meridian $\theta^{-1}(\{\pi/2\})$ as $\tilde{p} \to \tilde{o}_1$. We then have

$$\theta(\tilde{p}) < \theta(\Gamma(\tilde{p})_e) \leq \theta(\tilde{p}) + \pi.$$

Lemma 2.1 (Basic Lemma). *Let $(\widetilde{M}, \tilde{o})$ be a compact surface of revolution with metric* (1.1). *Then, there exists for an arbitrary given $r \in (0, \ell)$ a point $\tilde{p} \in \widetilde{M}$, $\tilde{p} \neq \tilde{o}_1$, such that the cut locus* $\mathrm{Cut}(\tilde{p})$ *is contained in r-ball $B(\tilde{o}, r)$ centered at $\tilde{o}$. In particular, $c_2(r) > 0$ for all $r > 0$.*

Proof. It follows from the continuity of i and $i(\tilde{o}_1) = \ell$ that there exists for an arbitrary given $r \in (0, \ell)$ a point $\tilde{p} \in \widetilde{M}$ such that

$$r + i(\tilde{p}) - d(\tilde{p}, \tilde{o}_1) > \ell.$$

In fact, if $\tilde{p}$ is sufficiently close to $\tilde{o}_1$ then $\ell - i(\tilde{p}) + d(\tilde{p}, \tilde{o}_1)$ is arbitrary small. We then have $d(\tilde{o}, \tilde{q}) < r$ for every cut point $\tilde{q} \in \mathrm{Cut}(\tilde{p})$. In fact, suppose contrary that $d(\tilde{o}, \tilde{q}) \geq r$, namely $d(\tilde{o}_1, \tilde{q}) \leq \ell - r$, then

$$\begin{aligned} d(\tilde{p}, \tilde{q}) &\leq d(\tilde{q}, \tilde{o}_1) + d(\tilde{p}, \tilde{o}_1) \\ &\leq \ell - r + d(\tilde{p}, \tilde{o}_1) < i(\tilde{p}). \end{aligned}$$

This implies that $\tilde{q} \notin \mathrm{Cut}(\tilde{p})$, a contradiction.

□

2.2. The Toponogov comparison theorem

We have recently established in [4] the Toponogov comparison theorem for (M, o) being referred to a general surface of revolution $(\widetilde{M}, \tilde{o})$ with its metric (1.1) for $\ell \leq \infty$. Some notations are needed for the statement of it.

A *geodesic triangle* $\Delta(\alpha\,\beta\,\gamma)$ is by definition a triple of minimizing geodesics $\alpha, \beta, \gamma : [0, 1] \to M$ parameterized proportionally to arc-length such that $\alpha(0) = \beta(0) = o$, $\gamma(0) = \alpha(1)$ and $\gamma(1) = \beta(1)$. Also $\Delta(\alpha\,\beta\,\gamma)$ is expressed by its corners as $\Delta(\alpha\,\beta\,\gamma) = \Delta(o\,\alpha(1)\,\beta(1))$. A geodesic triangle $\Delta(\alpha\,\beta\,\gamma)$ is said to be *narrow* iff $\alpha[0, 1]$ is contained in the union of convex balls centered at points on $\beta[0, 1]$.

Let (M, o) be referred to $(\widetilde{M}, \tilde{o})$. For an arbitrary fixed point $p \in M$, $p \neq o$, we set $\tilde{p} \in \widetilde{M}$ by $\tilde{p} = (d(o, p), 0)$. For a point $q \in M$, its reference point $\tilde{q} \in \widetilde{M}$ is defined so as to satisfy

$$d(\tilde{o},\tilde{q}) = d(o,q), \quad d(\tilde{p},\tilde{q}) = d(p,q).$$

The reference point always exists if $\Delta(o\,p\,q)$ is narrow (see [6]).

The pair of points o and p together forms a Lipschitz map

$$F_p : M \to \mathbf{R}^2, \quad F_p(x) := (d(o,x), d(p,x)), \quad x \in M$$
$$\widetilde{F}_{\tilde{p}} : \widetilde{M} \to \mathbf{R}^2, \quad \widetilde{F}_{\tilde{p}}(\tilde{x}) := (d(\tilde{o},\tilde{x}), d(\tilde{p},\tilde{x})), \quad \tilde{x} \in \widetilde{M}.$$

Let $E(o,p;r) := \{x \in M \mid d(o,x) + d(p,x) = r\}$ be the ellipsoid with foci at o and p and with radius $r > d(o,p)$. Let $d_r : E(o,p;r) \to \mathbf{R}$ be the distance function to o restricted to $E(o,p;r)$, and $E_p(r) \subset M$ the set of all points where d_r attains local maximum. We then have from Lemma 14 in [4],

$$E(p) := \bigcup_{r > d(o,p)} E_p(r) \subset \mathrm{Cut}(o) \tag{2.1}$$

With this notation we state (see Theorem 7 in [4]):

Theorem 2.2 (The Topopnogov comparison theorem for radial curvature)**.** *Let (M,o) be referred to $(\widetilde{M},\tilde{o})$. Assume that a point $p \in M$, $p \neq o$, satisfies*

$$F_p(E(p)) \cap \widetilde{F}_{\tilde{p}}(\mathrm{Cut}(\tilde{p}) \cap \mathrm{Int}(\widetilde{M}^+_{\tilde{p}})) = \emptyset. \tag{2.2}$$

Then every geodesic triangle $\Delta(o\,p\,q) \subset M$ admits its corresponding geodesic triangle $\Delta(\tilde{o}\,\tilde{p}\,\tilde{q}) \subset \widetilde{M}$ such that

$$\angle\, o\,p\,q \geq \angle\, \tilde{o}\,\tilde{p}\,\tilde{q}, \quad \angle\, o\,q\,p \geq \angle \tilde{o}\,\tilde{q}\,\tilde{p}, \quad \angle\, p\,o\,q \geq \angle\, \tilde{p}\,\tilde{o}\,\tilde{q}. \tag{2.3}$$

Moreover, for every $q \in M$ and every geodesic triangle $\Delta(\tilde{o}\,\tilde{p}\,\tilde{q}) \subset \widetilde{M}$, there exists a geodesic triangle $\Delta(o\,p\,q) \subset M$ satisfying (2.3)*.*

3. Lemmas

From now on let (M,o) be a compact pointed Riemannian manifold which is referred to $(\widetilde{M},\tilde{o})$ with its metric (1.1). Let $T(x,y)$ for $x,y \in M$, ($T(\tilde{x},\tilde{y})$ for $\tilde{x},\tilde{y} \in \widetilde{M}$, respectively) be a minimizing geodesic joining x to y, ($\tilde{x}$ to $\tilde{y}$, respectively). The following Lemmas are useful for the proof of our theorem.

Lemma 3.1 (ATCT)**.** *Assume that there exists a point $p \in M$ such that $d(o,p) > \ell - c_2(i(o))$. Then, every $T(p,q) \subset M$ joining p to every point $q \in M$ admits a corresponding $T(\tilde{p},\tilde{q}) \subset \widetilde{M}$ satisfying* (2.3)*. Moreover, there exists for every point $q \in M$ and for every $T(\tilde{p},\tilde{q})$ a $T(p,q)$ in M satisfying* (2.3)*.*

Proof. It follows from the definition of $c_2(r)$ that $\mathrm{Cut}(\tilde{p}) \cap \mathrm{Int}(\widetilde{M}_{\tilde{p}}^+) \subset B(\tilde{o}, i(o))$, namely $d(\tilde{o}, \mathrm{Cut}(\tilde{p}) \cap \mathrm{Int}(\widetilde{M}_{\tilde{p}}^+)) < i(o)$. Since $E(p) \subset \mathrm{Cut}(o)$ from (2.1) and since $\mathrm{Cut}(o) \subset M \setminus B(o, i(o))$, we have $d(o, E(p)) \geq i(o)$. Then (2.2) is satisfied for p. We conclude the proof by Theorem 2.2. $\square$

We say that $\widetilde{M}$ is *without conjugate points in half* if all points $\tilde{p} \in \widetilde{M}$ have no point conjugate to $\tilde{p}$ along geodesics from $\tilde{p}$ in $\mathrm{Int}(\widetilde{M}_{\tilde{p}}^+)$. If $\widetilde{M}$ is without conjugate conjugate points in half, then all geodesics contained in $\mathrm{Int}(\widetilde{M}_{\tilde{p}}^+)$ are minimizing, and, in particular, $c_2(r) = \ell$ for all $r > 0$. Any von Mangoldt surface is without conjugate points in half (see [18]).

We say that $\Gamma(\tilde{p})_e$ is *cross-cutting* if its turn angle is π around $\tilde{o}$, namely $\angle(\tilde{p}\,\tilde{o}\,\Gamma(\tilde{p})_e) = \theta(\Gamma(\tilde{p})_e) - \theta(\tilde{p}) = \pi$. Since any simply connected biangle domain bounded by two minimizing geodesic segments has a point conjugate to its vertexes in its interior, $\Gamma(\tilde{p})_e$ is cross-cutting for any point $\tilde{p}$ other than $\tilde{o}$ and $\tilde{o}_1$ if $\widetilde{M}$ is without conjugate points in half.

Remark 3.2. If $\widetilde{M}$ is without conjugate points in half, then Theorem 2.2 is true for all points $\tilde{p} \in \widetilde{M}$. Therefore, we do not need Lemma 3.1.

Since (M, o) is referred to $(\widetilde{M}, \tilde{o})$, there exists a unique point $o^* \in M$ such that $d(o, o^*) = \max\{d(o, x) \,|\, x \in M\} \leq \ell$, equality holding if and only if M is isometric to the warped product manifold $[0, \ell] \times S^{n-1}(1)$ with warping function f. Here, the uniqueness of o^* will be proved later.

Lemma 3.3. *Let $c = \min\{r_1, i(o)\}$. Assume that a farthest point o^* to o in M satisfies $d(o, o^*) > \ell - \min\{c_2(c), c_3(c)\}$. Then, M is topologically an n-sphere.*

Proof. Let $\tilde{o}^* \in \widetilde{M}$ be a reference point of o^*, namely $\tilde{o}^* = (d(o, o^*), 0)$. Since $d(\tilde{o}, \tilde{o}^*) > \ell - c_3(c)$, the endpoint $\Gamma(\tilde{o}^*)_e$ of $\Gamma(\tilde{o}^*)$ is contained in $B(\tilde{o}, c)$. Set $c' = (c + r(\Gamma(\tilde{o}^*)_e))/2$. We then have $c > c' > r(\Gamma(\tilde{o}^*)_e)$. Let $N = \{p \in M \,|\, d(p, o) > d(\tilde{o}, \Gamma(\tilde{o}^*)_e) = r(\Gamma(\tilde{o}^*)_e)\}$ and $N' := \{p \in M \,|\, d(o, p) > c'\}$. Then, we have $B(o, c) \cup N' = M$ and $N' \subset N$, since $c > c' > r(\Gamma(\tilde{o}^*)_e)$.

Obviously, there exists no critical point of the distance function to o in $B(o, c) \setminus \{o\}$ because of $i(o) \geq c$. We will prove that there exists no critical point of the distance function to o^* in N'. Then the proof of this lemma will complete.

Let $\widetilde{D}$ denote the domain bounded by $\Gamma(\tilde{o}^*)$, $T(\tilde{o}, \tilde{o}^*)$ and $T(\tilde{o}, \Gamma(\tilde{o}^*)_e)$. It follows from the Clairault relation that $r(T(\tilde{o}^*, \tilde{x})(t))$, $0 \leq t \leq d(\tilde{o}^*, \tilde{x})$, is monotone decreasing for t if $\tilde{x} \in \widetilde{D} \cap r^{-1}([r(\Gamma(\tilde{o}^*)_e), \ell])$. In particular, we have $\angle(\tilde{o}^*\,\tilde{x}\,\tilde{o}) > \pi/2$ for all $\tilde{x} \in \widetilde{D} \cap r^{-1}([r(\Gamma(\tilde{o}^*)_e), \ell])$.

Let $S(o, c') := \{p \in M \,|\, d(o, p) = c'\}$. We first claim that all the reference points $\tilde{q}$ of $q \in S(o, c')$ are contained in the domain $\widetilde{D}$.

If $\theta(\Gamma(\tilde{o}^*)_e) = \pi$, nothing is left to prove because of Lemma 3.1.

Suppose that $\theta(\Gamma(\tilde{o}^*)_e) < \pi$. Suppose for indirect proof that there exists a point $q \in S(o, c')$ such that $\tilde{q} \notin \widetilde{D}$. Let q_1 be a point in $T(o, o^*)$ with $d(o, q_1) = c'$. Since $c' < i(o)$, we see $S(o, c')$ is diffeomorphic to an $(n-1)$-sphere. Hence, there exists a curve $g(t)$, $0 \le t \le 1$, in $S(o, c')$ connecting $g(0) = q_1$ and $g(1) = q$. The reference curve $\tilde{c}(t) = \widetilde{F}_{\tilde{o}^*}{}^{-1}(F_{o^*}(g(t)))$, $0 \le t \le 1$, moves on the parallel $r^{-1}(\{c'\}) = S(\tilde{o}, c')$ in $\widetilde{M}_{\tilde{p}}^+$ from $\tilde{q}_1$ to $\tilde{q}$. Hence, there exists a $t_0 \in (0, 1)$ such that $\theta(\Gamma(\tilde{o}^*) \cap S(\tilde{o}, c')) < \theta(\tilde{g}(t_0)) < \theta(\Gamma(\tilde{o}^*)_e)$ and $T(\tilde{o}^*, \tilde{g}(t_0))$ contains a point $\tilde{x}$ with $r(\tilde{x}) > r(\tilde{o}^*)$. In fact, since $r(\Gamma(\tilde{o}^*)_e) < c'$, we have the point $\tilde{q}_2$ where the parallel $S(\tilde{o}, c')$ intersects the meridian through $\Gamma(\tilde{o}^*)_e$. Then, all points $\tilde{g}(t_0)$ lying in the subarc of $S(\tilde{o}, c')$ between $\Gamma(\tilde{o}^*) \cap S(\tilde{o}, c')$ and $\tilde{q}_2$ satisfy this property. Actually, since $\Gamma(\tilde{o}^*)$ is tangent to the parallel $r^{-1}(\{r(\tilde{o}^*)\})$ at $\tilde{o}^*$, we have $T(\tilde{o}^*, \tilde{g}(t_0)) \not\subset r^{-1}([0, r(\tilde{o}^*)])$. Then, it follows from Lemma 3.1 that there exists a point $x \in M$ such that $d(o, x) > d(o, o^*)$, contradicting the choice of o^*. Thus all the reference points for $S(o, c')$ is contained in $\widetilde{D}$.

We secondly claim that all the reference points $\tilde{q}$ of $q \in N'$ are contained in $\widetilde{D} \cap r^{-1}((c', \ell])$.

Suppose for indirect proof that there exists a reference point $\tilde{q} \notin \widetilde{D} \cap r^{-1}((c', \ell])$ of $q \in N'$, namely $r(\tilde{q}) > c'$ but $\tilde{q} \notin \widetilde{D}$. Let $\widetilde{T}(o^*, q)(t)$, $0 \le t \le d(o^*, q)$, be the reference curve of a minimizing geodesic $T(o^*, q)(t)$. From the definition of o^*, there exists a $t_0 > 0$ such that $\widetilde{T}(o^*, q)(t) \in \widetilde{D} \cap r^{-1}((c', \ell])$ for $t \in [0, t_0]$.

There exists no $t \in [0, d(o^*, q)]$ such that $r(\widetilde{T}(o^*, q)(t)) = c'$. In fact, suppose contrary, then $r(T(\tilde{o}^*, \tilde{q})(s)) = c'$ for some $s \in [0, d(o^*, q)]$ because of Lemma 3.1. Recall that the strip bounded by two parallels $r^{-1}(\{r(\tilde{o}^*)\})$ and $r^{-1}(\{r(\Gamma(\tilde{o}^*)_e)\})$ is foliated by minimizing geodesic segments $R_\theta(\Gamma(\tilde{o}^*))$, $0 \le \theta \le 2\pi$, where R_θ is the rotation with angle θ of $\widetilde{M}$ around $\tilde{o}$. From the fact that the reference points for $S(o, c')$ is contained in $\widetilde{D}$, it is impossible that $r(T(\tilde{o}^*, \tilde{q})(0)) > c'$, $r(T(\tilde{o}^*, \tilde{q})(s)) = c'$ and $r(T(\tilde{o}^*, \tilde{q})(d(o^*, q)) > c'$, since, otherwise, $T(\tilde{o}^*, \tilde{q})$ intersects $R_\theta(\Gamma(\tilde{o}^*))$ twice for some θ.

Thus, it follows that there exists $t_1 \ge t_0$ such that $\widetilde{T}(o^*, q)(t_1) \in \Gamma(\tilde{o}^*)$ and $r(\widetilde{T}(o^*, q)(t_1)) > c'$.

Then, as was seen before, the existence of a point $\widetilde{T}(o^*, q)(t_1 + \varepsilon) \notin \widetilde{D} \cap r^{-1}((c', \ell])$ for a sufficiently small ε and Lemma 3.1 implies that there exists an $x \in N'$ such that its reference point $\tilde{x}$ satisfies $r(\tilde{x}) > r(\tilde{o}^*)$, a contradiction. This completes the proof of the second claim.

Let $q \in N'$. Since, as was mentioned above, $r(T(\tilde{o}^*, \tilde{q})(t))$ is monotone decreasing in $t \in [0, d(o^*, q))]$, we have $\angle(\tilde{o}^* \, \tilde{q} \, \tilde{o}) > \pi/2$. From Lemma 3.1, we have $\angle(o^* \, q \, o) > \pi/2$. Thus there exists no critical point of the distance function to o^* in $N' \setminus \{o^*\}$. $\square$

Notice that, as was just seen, the reference points of N' are contained in the domain $\widetilde{D} \setminus r^{-1}((0, r(\Gamma(\tilde{o}^*)_e)))$. In particular, the farthest point o^* to o is unique, since $r(\Gamma(\tilde{o}^*)(t))$ is monotone decreasing in $t \in [0, d(\tilde{o}^*, \Gamma(\tilde{p})_e)]$.

Remark 3.4. If $\widetilde{M}$ is without conjugate points in half, then the distance function to o^* has no critical point in $N' := \{p \in M \mid d(o,p) > r(\Gamma(\tilde{o}^*)_e)\}$.

In fact, the Toponogov comparison theorem is true from Remark 3.2. Under the assumption, the domain $\widetilde{D}$ is bounded by the meridians $\theta^{-1}(0)$, $\theta^{-1}(\pi)$ and the minimizing geodesic $\Gamma(\tilde{o}^*)$, and, therefore, the reference points for $S(o, r(\Gamma(\tilde{o}^*)_e))$ is always contained in $\widetilde{D}$. This proves the argument in Remark 3.4.

4. Proof of Theorem 1.1

If $i(o) \geq r_1$, we then have

$$\frac{\min\{c_2(r_1), c_3(r_2)\}}{r_1} \geq \mu > \frac{\ell - d(o, o^*)}{r_1}.$$

Thus, we have $\min\{c_2(r_1), c_3(r_1)\} > \ell - d(o, o^*)$, meaning $d(o, o^*) > \ell - \min\{c_2(r_1), c_3(r_1)\}$. If $i(o) < r_1$, we then have

$$\frac{\min\{c_2(i(o)), c_3(i(o))\}}{i(o)} \geq \mu > \frac{\ell - d(o, o^*)}{i(o)}.$$

From the similar argument, we have $d(o, o^*) > \ell - \min\{c_2(i(o)), c_3(i(o))\}$. Therefore, Lemma 3.3 completes the proof of Theorem 1.1.

5. Examples

We first study the sphere with constant radius as a reference surface.

Example 5.1. We have $r_1 = \pi/(2\sqrt{\lambda})$ and $\mu = 1$ for the sphere with constant radius $1/\sqrt{\lambda}$.

Remark 5.2. The assumptions and the proof ideas of our theorem are compared with those of the classical diameter theorem [3], which is stated: Let M be a connected, complete Riemannian manifold with sectional curvature $K_M \geq \lambda > 0$ and diameter $\operatorname{diam}(M) > \pi/(2\sqrt{\lambda})$. Then M is homeomorphic to the n-sphere. The most important point is that one endpoint of the diameter is a critical point of the distance function to the other endpoint. From this point of view, Kondo and Ohta [8] extended the diameter sphere theorem, assuming that the base point is a critical point of the distance function to a certain point. In our theorem the base point o is not a critical point of the distance function to o^*, in general.

More generally, the following example is remarkable.

Example 5.3. Sinclair and Tanaka [17] determined the cut locus of a 2-sphere $\widetilde{M}$ of revolution satisfying that $K(t)$ is monotone and $K(t) = K(\ell - t)$ for $t \in [0, \ell/2]$. When $K(t)$, $0 \le t \le \ell/2$, is monotone non-decreasing, the cut locus of a point $\tilde{p}$ is a sub-arc of the parallel $r^{-1}(\{\ell - r(\tilde{p})\})$. Therefore, $c_2(r) = c_3(r) = r$ for all $r \in [0, \ell/2]$. Thus we have $\mu = 1$.

We have an estimate of μ for a von Mangoldt surface of revolution.

Example 5.4. Let $\widetilde{M}$ be a von Mangoldt surface, namely the curvature function $K(r)$ is monotone non-increasing in $r \in [0, \ell]$. Then there exists a unique $r_1 = \ell - {r_1}^*$ such that $f'(r_1) = 0$. Let $\tilde{p}$ be a point with $r(\tilde{p}) > \ell - \min\{r_1, \ell - r_1\}$. Then, $f(r(\Gamma(\tilde{o}^*)_e)) \ge f(r(\tilde{p}))$ because of the Clairaut relation. Since $K(r) \ge K(r')$ for all $0 \le r \le r_1 \le r' \le \ell$, we have $f(r) \le f(\ell - r)$ in $r \in [0, \min\{r_1, \ell - r_1\}]$. This implies that $\ell - r(\tilde{p}) \le r(\Gamma(\tilde{o}^*)_e)$. Therefore, we have $\mu \le 1$.

The following example suggests us that the constant μ defined from the curvature function K and the assumption (1.3) are important to study a sphere theorem.

Example 5.5. Let $0 < a < b$. Let S_1 and S_2 be circles whose lengths are $2a$ and $2b$, respectively. Let $T = S_1 \times S_2$ and $\tilde{o} \in T$. Let o^* be the furthest point to o in T. Then, we have $i(o) = a$ and $d(o, o^*) = \sqrt{a^2 + b^2} > b$.

This example shows that for any small $\varepsilon > 0$ and large $\ell > 0$ there exists a torus T satisfying $i(o) < \varepsilon$ and $d(o, o^*) = \ell$.

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Received October 28, 2013
Revised December 4, 2013