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AN EXAMPLE OF 2×2 HYPERBOLIC CONSERVATION LAWS ADMITTING DELTA-SHOCK WAVES

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ABSTRACT. In this paper we show that for some specific initial data, the Riemann problem for a simple model of 2×2 hyperbolic conservation laws has solutions containing delta-shock waves.

1. Introduction

We consider the Riemann problem for a system of conservation laws,

$$u_t + f(v)_x = 0, \quad v_t + \left(\frac{v^2}{2}\right)_x = 0, \qquad t > 0, \ -\infty < x < \infty, \tag{1.1}$$

$$(u(x,0), v(x,0)) = \begin{cases} (u_-, v_-), & x < 0 \ , \\ (u_+, v_+), & x > 0 \ . \end{cases} \tag{1.2}$$

Here u and v are functions of t and x, and f is C^2 function of two real variables u and v. Our main assumption is that

$$\frac{d}{dv} f(v) > 0. \tag{1.3}$$

System (1.1) is a simple model of a hyperbolic system of conservation laws which do not contain the product u and v, admitting a delta-shock wave which is a Dirac delta function supported on a shock. Note that many systems of conservation laws which are discussed about the existence of delta-shock waves, contain the product u and v.

To investigate the validity of delta-shock waves, Tan, Zhang and Zheng [13] consider the Riemann problem for a hyperbolic system of conservation laws,

$$u_t + \left(u^2\right)_x = 0, \quad v_t + \left(uv\right)_x = 0, \tag{1.4}$$

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with initial data (1.2). They introduce a viscosity term in the first equation of (1.4),

$$u_t + \left(u^2\right)_x = \epsilon t u_{xx}, \quad v_t + \left(uv\right)_x = 0, \quad \epsilon > 0, \tag{1.5}$$

and by using the notion of delta-shock waves, discuss the existence of solutions to the Riemann problem (1.4), (1.2).

In [7], Hu extends the existence results of [13] by using a vanishing viscosity approach introduced by Dafermos[1] and by Tupciev [14]. (See [2], [4], [5], [6], [9], [10], [11], [12] and [15] for the applications of the vanishing viscosity approach.) He considers the regularized problem

$$u_t + \left(u^2\right)_x = \epsilon t u_{xx}, \quad v_t + \left(uv\right)_x = \epsilon t v_{xx}, \quad \epsilon > 0, \tag{1.6}$$

with initial data (1.2), and by using the notion of delta-shock waves, proves that self-similar solutions (u_ϵ, v_ϵ) of (1.6), (1.2) converge weakly to the solutions (u, v) of the Riemann problem (1.4), (1.2). See also [3] and [8].

By combining the vanishing viscosity approach with the notion of delta-shock waves, the existence of solutions to the Riemann problem for several examples of conservation laws can be discussed without difficulty. In fact, for many systems of conservation laws which contain the product u and v, the existence of solutions is discussed. One of the reasons for this is to avoid annoying arguments to determine a limit of the critical points of solutions to the regularized problem. However, the existence of solutions containing delta-shock waves generated by the vanishing viscosity approach, is guaranteed under a definite condition concerning a limit of the critical points of solutions to the regularized problem. Indeed, only when a limit of the critical points of solutions to the regularized problem is equal to the value of a discontinuity point of v, the existence results of solutions containing delta-shock waves in [7] and [8] are guaranteed (cf. [3] and [13]). For this reason, since there is a possibility that for all initial data $(u_\pm, v_\pm)$, the Riemann problem (1.4), (1.2) has no solutions containing delta-shock waves, we slightly doubt the validity of delta-shock waves generated by the vanishing viscosity approach.

The purpose of this paper is to further clarify the validity of delta-shock waves generated by the vanishing viscosity approach. Moreover, we show a negative association between the product u and v and the existence of delta-shock waves. Note that there has been very few study designed for these purposes. By using the vanishing viscosity approach, we shall prove that for some specific initial data, the Riemann problem (1.1), (1.2) has solutions containing delta-shock waves.

We now mention the construction in this paper. Section 2 is devoted to proving the existence of solutions to the regularized problem

$$u_t + f(v)_x = \epsilon t u_{xx}, \quad v_t + \left(\frac{v^2}{2}\right)_x = \epsilon t v_{xx}, \quad \epsilon > 0, \tag{1.7}$$

with initial data (1.2). The form of the viscosity operator on the right-hand side of (1.7) has been adopted so that the invariance property of (1.1) under the transformation $(t, x) \mapsto (at, ax)$, $a > 0$, is preserved by (1.7). As a consequence of this invariance property, the solution of (1.7), (1.2) is a function $\big(u_\epsilon(x/t), v_\epsilon(x/t)\big)$ of the single variable $\xi = x/t$, where $\big(u_\epsilon(x/t), v_\epsilon(x/t)\big)$ is the solution of the boundary–value problem

$$\epsilon u_\epsilon^{''}(\xi) = f(v_\epsilon)' - \xi u_\epsilon^{'}(\xi), \quad \epsilon v_\epsilon^{''}(\xi) = \Big(\frac{v_\epsilon^2}{2}\Big)' - \xi v_\epsilon^{'}(\xi), \tag{1.8}$$

$$\big(u_\epsilon(-\infty), v_\epsilon(-\infty)\big) = (u_-, v_-), \quad \big(u_\epsilon(\infty), v_\epsilon(\infty)\big) = (u_+, v_+), \tag{1.9}$$

where $'$ denotes differentiation with respect to ξ. Therefore, in Section 2 we show that for every fixed $\epsilon > 0$, the boundary–value problem (1.8), (1.9) has smooth solutions. In Section 3 we prove that for some specific initial data, limit solutions of (1.8), (1.9) generate the solutions which contain delta-shock waves, of the Riemann problem (1.1), (1.2).

2. Existence of solutions to (1.8), (1.9)

In this section, we prove the following result on the existence of solutions to (1.8), (1.9) under condition (1.3):

Theorem 2.1. *Assume that condition* (1.3) *is satisfied. Then, for every* $\epsilon > 0$, *there exists a smooth solution of* (1.8), (1.9).

This theorem follows from the following theorem, which is shown in Theorem 3.1 of Dafermos [1], under the assumption that we have a apriori bounds on the solution (cf. [10]):

Theorem 2.2. *Consider the two-parameter family of the boundary–value problems*

$$\epsilon u^{''}(\xi) = \mu f(v(\xi))' - \xi u^{'}(\xi), \quad \epsilon v^{''}(\xi) = \mu\Big(\frac{v^2(\xi)}{2}\Big)' - \xi v^{'}(\xi), \tag{2.1}$$

$$\big(u(-L), v(-L)\big) = (\mu u_-, \mu v_-), \quad \big(u(L), v(L)\big) = (\mu u_+, \mu v_+), \tag{2.2}$$

with parameters $L \geq 1$, $\mu \in [0, 1]$. *Assume that there exists a positive constant* M *depending at most on* $\epsilon, f, (u_-, v_-), (u_+, v_+)$ *(and thus independent of* L *and* μ*) such that any solution* $(u(\xi), v(\xi))$ *of* (2.1), (2.2) *satisfies*

$$\sup_{(-L,L)} |u(\xi)| \leq M, \quad \sup_{(-L,L)} |v(\xi)| \leq M. \tag{2.3}$$

Then, for every $\epsilon > 0$, *there exists a smooth solution of* (1.8), (1.9).

From Theorem 2.2 it is sufficient for the proof of Theorem 2.1 to prove a priori estimate (2.3). In proving estimate (2.3), one is helped by the following lemma:

Lemma 2.3. *Assume that condition* (1.3) *is satisfied and let* $(u(\xi), v(\xi))$ *be a solution of* (2.1), (2.2) *with* $1 \le L \le \infty$. *Then, one of the following holds:*
(i) *Both* $u(\xi)$ *and* $v(\xi)$ *are constant on* $(-L, L)$.
(ii) $u(\xi)$ *has, at most, one critical point, which* $u(\xi)$ *necessarily must attain a maximum (or minimum), in* $(-L, L)$, *while* $v(\xi)$ *is a strictly decreasing (or increasing) function with no critical point in* $(-L, L)$.

Proof. Note that when $\mu = 0$, a simple computation shows that both $u(\xi)$ and $v(\xi)$ are constant on $(-L, L)$. Suppose now that $\mu > 0$ and let $(u(\xi), v(\xi))$ be a nonconstant solution of (2.1), (2.2) on $(-L, L)$.

First, we investigate the property of $v(\xi)$. Suppose that τ is a critical point of $v(\xi)$. We then have

$$\epsilon v^{''}(\tau) = \mu v(\tau) v^{'}(\tau) - \tau v^{'}(\tau) = 0$$

so that $v^{''}(\tau) = 0$. However, by the uniqueness of solution to (2.1), (2.2) (cf. Lemma 4.1 in [1]), this means that $v(\xi)$ is a constant on $(-L, L)$ which implies a contradiction. Therefore, for the nonconstant function $v(\xi)$, $v(\xi)$ must be a strictly increasing function or a strictly decreasing function with no critical point in $(-L, L)$.

Next, we investigate the property of $u(\xi)$. Suppose that τ is a critical point of $u(\xi)$. We then have

$$\epsilon u^{''}(\tau) = \mu \frac{df(v)}{dv}\Big|_{\xi=\tau} v^{'}(\tau) - \tau u^{'}(\tau) = \mu \frac{df(v)}{dv}\Big|_{\xi=\tau} v^{'}(\tau)$$

so that there are the following three possibilities of behaviours at $\xi = \tau$: (I) $u^{''}(\tau) > 0$, $v^{'}(\tau) > 0$; (II) $u^{''}(\tau) < 0$, $v^{'}(\tau) < 0$; (III) $u^{''}(\tau) = 0$, $v^{'}(\tau) = 0$. However, by the uniqueness of solution to (2.1), (2.2), case (III) corresponds to $(u(\xi), v(\xi))$ being constant on $(-L, L)$, which implies a contradiction. Therefore, if τ is a critical point of $u(\xi)$, then we have either (I) or (II). In other words, for the nonconstant function $u(\xi)$, $u(\xi)$ must attain either a maximum or a minimum at the critical point τ. Note that it is clear that nonconstant function $u(\xi)$ has at most one critical point in $(-L, L)$. Thus the proof of Lemma 2.3 is complete. □

By Lemma 2.3, $v(\xi)$ is monotone. If $u(\xi)$ also is monotone, then a priori estimate (2.3) clearly holds. Therefore, it is sufficient for the proof of estimate (2.3) to deal with the following two cases:

Case 1: $u(\xi)$ is strictly increasing on $(-L, \tau)$, attains a maximum at τ, and is strictly decreasing on (τ, L), while $v(\xi)$ is strictly decreasing on $(-L, L)$.

Case 2: $u(\xi)$ is strictly decreasing on $(-L, \tau)$, attains a minimum at τ, and is strictly increasing on (τ, L), while $v(\xi)$ is strictly increasing on $(-L, L)$.

The following proposition plays an important role in proving various estimates on the solution:

Proposition 2.4. *Assume that condition* (1.3) *is satisfied and let* $(u(\xi), v(\xi))$ *be a solution of* (2.1), (2.2) *with* $1 \leq L \leq \infty$. *Moreover, denote by* $\bar{u} = \max\{|u_-|,\ |u_+|\}$ *the maximums of* $|u_\pm|$. *Then, we have the following:*
(i) *If* $u(\xi)$ *is strictly increasing on* $(-L, \tau)$, *attains a maximum at* τ, *and is strictly decreasing on* (τ, L), *while* $v(\xi)$ *is strictly decreasing on* $(-L, L)$, *then the following inequalities hold for some constant* $N \geq 0$ *which depends solely on* f, v_-, v_+*:*

$$\int_\alpha^\beta u(\xi)d\xi \leq (\beta - \alpha)\bar{u} + N \quad \textit{for every interval } (\alpha, \beta) \subset (-L, L), \tag{2.4}$$

$$-\bar{u} \leq u(\xi) \leq \bar{u} + \frac{N}{|\xi - \tau|} \qquad \textit{for all } \xi \in [-L, L]\backslash\tau. \tag{2.5}$$

(ii) *If* $u(\xi)$ *is strictly decreasing on* $(-L, \tau)$, *attains a minimum at* τ, *and is strictly increasing on* (τ, L), *while* $v(\xi)$ *is strictly increasing on* $(-L, L)$, *then the following inequalities hold for some constant* $N \geq 0$ *which depends solely on* f, v_-, v_+*:*

$$-\int_\alpha^\beta u(\xi)d\xi \leq (\beta - \alpha)\bar{u} + N \quad \textit{for every interval } (\alpha, \beta) \subset (-L, L), \tag{2.6}$$

$$-\bar{u} - \frac{N}{|\xi - \tau|} \leq u(\xi) \leq \bar{u} \qquad \textit{for all } \xi \in [-L, L]\backslash\tau. \tag{2.7}$$

Proof. We only prove (i), because (ii) is proved by arguments similar to the proof of (i).

We now prove inequality (2.4). If $u(\alpha) > \bar{u}$, then we set

$$\eta = \sup\big\{\xi \in [-L, \alpha) : u(\xi) \leq \bar{u}\big\}.$$

Note that this set is nonempty in view of the definition of $\bar{u}$. On the other hand, if $u(\alpha) \leq \bar{u}$, then we set

$$\eta = \inf\big\{\xi \in (\alpha, \beta) : u(\xi) \geq \bar{u}\big\}.$$

Note also that if this set is empty, then inequality (2.4) is satisfied for any $N \geq 0$. Similarly, if $u(\beta) > \bar{u}$, then we set

$$\theta = \inf\big\{\xi \in (\beta, L] : u(\xi) \leq \bar{u}\big\},$$

while if $u(\beta) \leq \bar{u}$, then we set

$$\theta = \sup\big\{\xi \in (\alpha, \beta) : u(\xi) \geq \bar{u}\big\}.$$

By the choices of η and θ, we have $u^{'}(\eta) \geq 0$, $u^{'}(\theta) \leq 0$ and

$$\int_\alpha^\beta \big[u(\xi) - \bar{u}\big]d\xi \leq \int_\eta^\theta \big[u(\xi) - \bar{u}\big]d\xi = -\int_\eta^\theta \xi u^{'}(\xi)d\xi. \tag{2.8}$$

Therefore, integrating the second equation in (2.1) over (η,θ), and using $u^{'}(\eta)\geq 0$, $u^{'}(\theta)\leq 0$ and (2.8), we obtain

$$\int_\alpha^\beta u(\xi)d\xi \leq (\beta-\alpha)\bar{u}+\mu f\big(v(\eta)\big)-\mu f\big(v(\theta)\big)\leq (\beta-\alpha)\bar{u}+N.$$

Therefore, inequality (2.4) is proved.

From inequality (2.4) we can easily check inequality (2.5). Indeed, if $\xi<\tau$, then we have

$$u(\xi)=\frac{1}{\tau-\xi}\int_\xi^\tau u(\xi)d\zeta \leq \frac{1}{\tau-\xi}\int_\xi^\tau u(\zeta)d\zeta \leq \bar{u}+\frac{N}{\tau-\xi}, \tag{2.9}$$

while if $\xi>\tau$, then we have

$$u(\xi)=\frac{1}{\xi-\tau}\int_\tau^\xi u(\xi)d\zeta \leq \frac{1}{\xi-\tau}\int_\tau^\xi u(\zeta)d\zeta \leq \bar{u}+\frac{N}{\xi-\tau}. \tag{2.10}$$

Thus inequality (2.5) is proved and the proof of Proposition 2.4 is complete. $\square$

We now prove Theorem 2.1 by proving estimates (2.3) in Cases 1 and 2. We only prove estimate (2.3) in Case 1, because (2.3) in Case 2 is proved by arguments similar to the proof of estimate (2.3) in Case 1.

In Case 1, it is sufficient to estimate $u(\tau)$ from above. In estimating $u(\tau)$ from above, we assume that $u(\tau)>\bar{u}\geq 0$, because it is clear that inequality (2.3) in the case of $u(\tau)\leq\bar{u}$ holds. We fix $\xi_0<\tau$ such that $u(\xi_0)=\bar{u}$, and let $\bar{\xi}$ denote the point in $(\tau,L]$ with the property $u(\bar{\xi})=u(\xi)$ for any $\xi\in[\xi_0,\tau)$. Integrating the first equation in (2.1) over $(\xi,\bar{\xi})$, we obtain

$$\epsilon u^{'}(\bar{\xi})-\epsilon u^{'}(\xi)=\mu\big(f(v(\bar{\xi}))-f(v(\xi))\big)-\int_\xi^{\bar{\xi}}\zeta u^{'}(\zeta)d\zeta. \tag{2.11}$$

Noting that $\xi\in[\xi_0,\tau)$, since $u^{'}(\bar{\xi})\leq 0$ and

$$-\int_\xi^{\bar{\xi}}\zeta u^{'}(\zeta)d\zeta=\int_\xi^{\bar{\xi}}\big(u(\zeta)-u(\xi)\big)d\zeta\geq 0, \tag{2.12}$$

inequality (2.11) gives

$$\epsilon u^{'}(\xi)\leq \mu\big(f(v(\bar{\xi}))-f(v(\xi))\big)\leq N, \tag{2.13}$$

where N is a nonnegative constant which depends solely on f,v_-,v_+.

Fixed $\xi_1\in[\xi_0,\tau)$, integrating (2.13) over (ξ_1,τ), we deduce

$$u(\tau)\leq u(\xi_1)+\frac{N}{\epsilon}(\tau-\xi_1). \tag{2.14}$$

If $\tau - \xi_0 < 1$, then, choosing $\xi_1 = \xi_0$, we have

$$u(\tau) \leq \bar{u} + \frac{N}{\epsilon}. \tag{2.15}$$

On the other hand, if $\tau - \xi_0 \geq 1$, then, choosing $\xi_1 = \tau - 1$ and using (2.5), we have

$$u(\tau) \leq \bar{u} + \frac{N}{|\xi_1 - \tau|} + \frac{N}{\epsilon}(\tau - \xi_1) = \bar{u} + \Big(1 + \frac{1}{\epsilon}\Big)N. \tag{2.16}$$

Thus estimate (2.3) in Case 1 is proved and the proof of Theorem 2.1 is complete.

3. Existence of delta-shock waves

In this section, we prove that for some specific initial data, the limit solutions of (1.8), (1.9) generate the solutions which contain delta-shock waves, of the Riemann problem (1.1), (1.2). More precisely, we prove the following theorem:

Theorem 3.1. *Assume that condition* (1.3) *is satisfied and let* $(u_\epsilon(\xi), v_\epsilon(\xi))$ *be a solution of the boundary-value problem* (1.8), (1.9). *If* $u_- = u_+$, $v_- + v_+ = 0$ *with* $v_- > v_+$, *then there exists a subsequence (still labeled by* $(u_\epsilon(\xi), v_\epsilon(\xi))$*) such that* $(u_\epsilon(\xi), v_\epsilon(\xi))$ *converges weakly to* $(H_u(\xi) + s\delta(\xi), H_v(\xi))$, *where*

$$H_u(\xi) = \begin{cases} u_-, & \xi < 0 \ , \\ u_+, & \xi > 0 \ , \end{cases} \qquad H_v(\xi) = \begin{cases} v_-, & \xi < 0 \ , \\ v_+, & \xi > 0 \ , \end{cases}$$

$\delta(\xi)$ *is the Dirac* δ*-function supported at* $\xi = 0$ *and* s *is the strength of* $\delta(\xi)$ *with* $s = f(v_-) - f(v_+)$. *Moreover, the limit function is a solution of the Riemann problem* (1.1), (1.2).

Remark 3.2. We remark that even if $u_- \neq u_+$, $v_- + v_+ = 0$ with $v_- > v_+$, then the assertion of Theorem 3.1 holds. However, as the purpose of this paper is concerned, we limit the discussion to the case of $u_- = u_+$, $v_- + v_+ = 0$ with $v_- > v_+$.

Before proving Theorem 3.1, we state some preliminary results. In the rest of this paper, we assume that condition (1.3) is satisfied.

The following lemma asserts that if $u_- \neq u_+$, $v_- > v_+$, then $u_\epsilon(\xi)$ is bell-shaped and $v_\epsilon(\xi)$ is monotone:

Lemma 3.3. *Let* $(u_\epsilon(\xi), v_\epsilon(\xi))$ *be a solution of* (1.8), (1.9) *and suppose that* $u_- = u_+$, $v_- > v_+$. *Then,* $u_\epsilon(\xi)$ *is strictly increasing on* $(-\infty, \tau_\epsilon)$, *attains a maximum at* τ_ϵ, *and is strictly decreasing on* (τ_ϵ, ∞), *while* $v_\epsilon(\xi)$ *is strictly decreasing on* $(-\infty, \infty)$.

Proof. By Lemma 2.3, it is sufficient to prove that $u_\epsilon(\xi)$ is nonconstant on $(-\infty, \infty)$.

If $u_\epsilon(\xi)$ is constant on $(-\infty, \infty)$, then we have $u_\epsilon^{'}(\xi) = u_\epsilon^{''}(\xi) = 0$ for $(-\infty, \infty)$. Therefore, it follows that $v_\epsilon^{'}(\xi) = 0$ for $(-\infty, \infty)$. But this contradicts to condition $v_- > v_+$. Thus, the proof of Lemma 3.3 is complete. □

The following result gives estimates for determining whether or not $u_\epsilon(\xi)$ is a function of uniformly bounded variation:

Proposition 3.4. *Let* $(u_\epsilon(\xi), v_\epsilon(\xi))$ *be a solution of* (1.8), (1.9). *Suppose that* $u_\epsilon(\xi)$ *is strictly increasing on* $(-\infty, \tau_\epsilon)$*, attains a maximum at* τ_ϵ*, and is strictly decreasing on* (τ_ϵ, ∞)*, while* $v_\epsilon(\xi)$ *is strictly decreasing on* $(-\infty, \infty)$*. If* $|\tau_\epsilon| > \delta$ *for any* $\delta > 0$*, then the following inequality holds for some constant* $N \geq 0$ *which depends solely on* f, v_-, v_+*:*

$$u_\epsilon(\xi) \leq \bar{u} + \frac{N}{\delta} \quad \text{for all } \xi \in (-\infty, \infty). \tag{3.1}$$

Proof. Suppose first that $\tau_\epsilon > \delta$. Then, for any $k > \tau_\epsilon$, integrating the first equation in (1.8) over (τ_ϵ, k), we have

$$\epsilon u_\epsilon^{'}(k) - \epsilon u_\epsilon^{'}(\tau_\epsilon) = f(v_\epsilon(k)) - f(v_\epsilon(\tau_\epsilon)) + \tau_\epsilon(u_\epsilon(\tau_\epsilon) - u_\epsilon(k)) + \int_{\tau_\epsilon}^{k} u_\epsilon(\xi) - u_\epsilon(k)\ d\xi$$

so that

$$0 > f(v_\epsilon(k)) - f(v_\epsilon(\tau_\epsilon)) + \tau_\epsilon(u_\epsilon(\tau_\epsilon) - u_\epsilon(k)). \tag{3.2}$$

Passing to $k \to \infty$ in (3.2), we obtain

$$0 \geq f(v_+) - f(v_\epsilon(\tau_\epsilon)) + \tau_\epsilon(u_\epsilon(\tau_\epsilon) - u_+). \tag{3.3}$$

Therefore, we have

$$u_\epsilon(\tau_\epsilon) \leq \frac{f(v_\epsilon(\tau_\epsilon)) - f(v_+)}{\tau_\epsilon} + u_+ \leq \frac{N}{\delta} + u_+. \tag{3.4}$$

Thus inequality (3.1) in the case of $\tau_\epsilon > \delta$ is proved.

Next, suppose that $\tau_\epsilon < -\delta$. Then, for any $k < \tau_\epsilon$, integrating the first equation in (1.8) over (k, τ_ϵ), we have

$$\epsilon u_\epsilon^{'}(\tau_\epsilon) - \epsilon u_\epsilon^{'}(k) = f(v_\epsilon(\tau_\epsilon)) - f(v_\epsilon(k) - \tau_\epsilon(u_\epsilon(\tau_\epsilon) - u_\epsilon(k)) + \int_{k}^{\tau_\epsilon} u_\epsilon(\xi) - u_\epsilon(k)\ d\xi$$

so that

$$0 > f(v_\epsilon(\tau_\epsilon)) - f(v_\epsilon(k)) - \tau_\epsilon(u_\epsilon(\tau_\epsilon) - u_\epsilon(k)). \tag{3.5}$$

Passing to $k \to -\infty$ in (3.5), we obtain

$$0 \geq f(v_\epsilon(\tau_\epsilon)) - f(v_-) - \tau_\epsilon(u_\epsilon(\tau_\epsilon) - u_-). \tag{3.6}$$

Therefore, we have

$$u_\epsilon(\tau_\epsilon) \leq \frac{f(v_\epsilon(\tau_\epsilon)) - f(v_-)}{\tau_\epsilon} + u_- \leq \frac{N}{\delta} + u_-. \tag{3.7}$$

Thus inequality (3.1) in the case of $\tau_\epsilon < -\delta$ is proved and the proof of Proposition 3.4 is complete. $\square$

From now on, let $\tau_\epsilon = \big\{\xi \in (-\infty, \infty) : u_\epsilon^{'}(\xi) = 0\big\}$ for every $\epsilon > 0$. When $u_- = u_+$, $v_- > v_+$, by possibly taking a subsequence, there exists τ_0 with $|\tau_0| \leq \infty$ such that $\tau_\epsilon \to \tau_0$ as $\epsilon \to 0+$. The following proposition provides the key ingredient in the existence proof of the solutions containing delta-shock waves:

Proposition 3.5. *Let* $(u_\epsilon(\xi), v_\epsilon(\xi))$ *be a solution of* (1.8), (1.9). *If* $u_- = u_+$, $v_- > v_+$, *then we have the following:*
(i) *If* $0 < |\tau_0| \leq \infty$, *then there exists a subsequence (still labeled by* $(u_\epsilon(\xi), v_\epsilon(\xi))$*) and a bounded function* $(u(\xi), v(\xi))$ *such that*

$$(u_\epsilon(\xi), v_\epsilon(\xi)) \to (u(\xi), v(\xi)), \quad \textit{a.e. as } \epsilon \to 0+. \tag{3.8}$$

Moreover, the limit function $(u(\xi), v(\xi))$ *satisfies*

$$f(v(\xi))^{'} - \xi u^{'}(\xi) = 0, \quad \Big(\frac{v^2(\xi)}{2}\Big)^{'} - \xi v^{'}(\xi) = 0 \tag{3.9}$$

in the sense of distributions at any $\xi \in (-\infty, \infty)$. *In particular, if* ξ_0 *is a point of discontinuity of* $(u(\xi), v(\xi))$, *then we have the Rankine-Hugoniot condition*

$$f(v(\xi_0+)) - f(v(\xi_0-)) = \xi_0\big(u(\xi_0+) - u(\xi_0-)\big), \quad \frac{v(\xi_0+) + v(\xi_0-)}{2} = \xi_0. \tag{3.10}$$

(ii) *If* $\tau_0 = 0$, *then there exists a subsequence (still labeled by* $(u_\epsilon(\xi), v_\epsilon(\xi))$*) and a locally integrable function* $(u(\xi), v(\xi))$ *such that*

$$(u_\epsilon(\xi), v_\epsilon(\xi)) \to (u(\xi), v(\xi)), \quad \textit{a.e. as } \epsilon \to 0+, \tag{3.11}$$

$$\begin{cases} |u(\xi)| \leq \dfrac{N}{|\xi|} + \bar{u}, & \xi \in (-\infty, \infty)\backslash 0\ , \\ |v(\xi)| \leq \bar{v}, & \xi \in (-\infty, \infty)\ , \end{cases} \tag{3.12}$$

where $\bar{v} = \max\{|v_-|,\ |v_+|\}$. *Moreover, the limit function* $(u(\xi), v(\xi))$ *satisfies*

$$f(v(\xi))^{'} - \xi u^{'}(\xi) = 0, \quad \Big(\frac{v^2(\xi)}{2}\Big)^{'} - \xi v^{'}(\xi) = 0 \tag{3.13}$$

in the sense of distributions at any $\xi \in (-\infty, \infty)\backslash 0$. *In particular, if* $\xi_0 \neq 0$ *is a point of discontinuity of* $(u(\xi), v(\xi))$, *then we have the Rankine-Hugoniot condition*

$$f(v(\xi_0+)) - f(v(\xi_0-)) = \xi_0\big(u(\xi_0+) - u(\xi_0-)\big), \quad \frac{v(\xi_0+) + v(\xi_0-)}{2} = \xi_0. \tag{3.14}$$

Proof. We prove the first assertion. By Proposition 3.4, $u_\epsilon(\xi)$ is uniformly bounded with respect to ϵ on $(-\infty, \infty)$. Therefore, by using arguments similar to Dafermos [1] and Dafermos and DiPerna [2], we see that the first assertion holds.

Next, we prove the second assertion. By (2.5), $u_\epsilon(\xi)$ is uniformly bounded with respect to ϵ on the interval $I_2 = [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$. Applying the Helly's theorem, there exists a convergent subsequence of $(u_\epsilon(\xi), v_\epsilon(\xi))$ (still labeled by $(u_\epsilon(\xi), v_\epsilon(\xi))$). Similarly, we can extract a convergent subsequence of $(u_\epsilon(\xi), v_\epsilon(\xi))$ on the interval $I_3 = [-3, -\frac{1}{3}] \cup [\frac{1}{3}, 3]$. Continue this process on each the interval the interval $I_n = [-n, -\frac{1}{n}] \cup [\frac{1}{n}, n]$, $n = 4, 5, \ldots$. Finally, by a standard diagonal process, we can construct a subsequence which converges at each $\xi \neq 0$ to a function $(u(\xi), v(\xi))$ defined on $(-\infty, \infty)\backslash 0$. Note that $v(\xi)$ is locally integrable on $(-\infty, \infty)$ by (2.4) and the Fatou's lemma. Since the limit function $(u(\xi), v(\xi))$ clearly satisfies (3.12), (3.13) and (3.14), the proof of Proposition 3.5 is complete. □

We now state some properties of the limit function $(u(\xi), v(\xi))$ constructed in Proposition 3.5.

The following lemma describes the behaviour of $(u(\xi), v(\xi))$ at the boundary:

Lemma 3.6. *The limit function $(u(\xi), v(\xi))$ satisfies the following boundary condition:*

$$(u(-\infty), v(-\infty)) = (u_-, v_-), \quad (u(\infty), v(\infty)) = (u_+, v_+). \tag{3.15}$$

Proof. We only prove equation (3.15) in the case of $\tau_0 = 0$, because equation (3.15) in the case of $0 < |\tau_0| \leq \infty$ is proved by arguments similar to Dafermos [1].

Let $\delta > 1$. From (2.5), we have

$$|u_\epsilon(\xi)| \leq \bar{u} + \frac{N}{\delta} \quad \text{for } \xi \in (-\infty, -\delta] \cup [\delta, \infty). \tag{3.16}$$

Setting

$$U_\epsilon(\xi) = \begin{pmatrix} u_\epsilon(\xi) \\ v_\epsilon(\xi) \end{pmatrix}, \quad F(U_\epsilon(\xi)) = \begin{pmatrix} f(v_\epsilon(\xi)) \\ \dfrac{v^2(\xi)}{2} \end{pmatrix},$$

from (1.8) we obtain

$$\frac{d}{d\xi}\Big(\exp\Big(\frac{\xi^2}{2\epsilon}\Big) U_\epsilon'(\xi)\Big) = \frac{1}{\epsilon} \nabla F(U_\epsilon(\xi)) U_\epsilon'(\xi) \exp\Big(\frac{\xi^2}{2\epsilon}\Big). \tag{3.17}$$

Integrating (3.17) over (δ, ξ), we have

$$\exp\Big(\frac{\xi^2}{2\epsilon}\Big) U_\epsilon'(\xi) - \exp\Big(\frac{\delta^2}{2\epsilon}\Big) U_\epsilon'(\delta) = \frac{1}{\epsilon} \int_\delta^\xi \nabla F(U_\epsilon(\zeta)) U_\epsilon'(\zeta) \exp\Big(\frac{\zeta^2}{2\epsilon}\Big) d\zeta. \tag{3.18}$$

Therefore, it follows from the Gronwall's inequality that

$$|U_\epsilon'(\xi)| \le |U_\epsilon'(\delta)| \exp\Big(\frac{\delta^2 + 2\alpha(\xi-\delta) - \xi^2}{2\epsilon}\Big), \quad \xi > \delta, \tag{3.19}$$

where $\alpha = \sup_{\xi\in[\delta,\infty)} |\nabla F(U_\epsilon(\xi))|$, which is a nonnegative constant which is independent of ϵ on account of (3.16).

Integrating (3.18) over $(\delta, \delta+1)$, we obtain

$$\begin{aligned} U_\epsilon'(\delta)\int_\delta^{\delta+1} \exp\Big(\frac{\delta^2-\xi^2}{2\epsilon}\Big)d\xi &= U_\epsilon(\delta+1) - U_\epsilon(\delta) - \frac{1}{\epsilon}\int_\delta^{\delta+1} F\big(U_\epsilon(\xi)\big)d\xi \\ &\quad + \frac{1}{\epsilon}F\big(U_\epsilon(\delta)\big)\int_\delta^{\delta+1} \exp\Big(\frac{\delta^2-\xi^2}{2\epsilon}\Big)d\xi \\ &\quad + \frac{1}{\epsilon^2}\int_\delta^{\delta+1}\int_\delta^{\xi} \zeta F\big(U_\epsilon(\zeta)\big)\exp\Big(\frac{\zeta^2-\xi^2}{2\epsilon}\Big)d\zeta d\xi. \end{aligned} \tag{3.20}$$

It is easy to check that for sufficiently small $\epsilon > 0$,

$$\int_\delta^{\delta+1} \exp\Big(\frac{\delta^2-\xi^2}{2\epsilon}\Big)d\xi \ge \int_\delta^{\delta+1} \exp\Big(\frac{(2\delta+1)(\delta-\xi)}{2\epsilon}\Big)d\xi \ge \frac{\epsilon}{2\delta+1}. \tag{3.21}$$

Therefore, by (3.16), (3.20) and (3.21), we have

$$|U_\epsilon'(\delta)| \le K\epsilon^{-3}, \tag{3.22}$$

where K is a positive constant which is independent of ϵ. Thus, inequalities (3.19) and (3.22) give

$$|U_\epsilon'(\xi)| \le K\epsilon^{-3} \exp\Big(\frac{\delta^2 + 2\alpha(\xi-\delta) - \xi^2}{2\epsilon}\Big), \quad \xi > \delta. \tag{3.23}$$

Recalling that $U_\epsilon(\xi) \to U(\xi) = (u(\xi), v(\xi))$, a.e. as $\epsilon \to 0+$, we obtain from (3.23) $U(\xi) = U_+ = (u_+, v_+)$ for $\xi > \max\{\delta, \alpha + |\alpha-\delta|\}$. By arguments similar to the case of $U(\infty) = U_+$, we can prove $U(-\infty) = U_- = (u_-, v_-)$ for $\xi < -\max\{\delta, \alpha+|\alpha-\delta|\}$. Thus the proof of Lemma 3.6 is complete. $\square$

The following lemma describes the form of $v(\xi)$:

Lemma 3.7. *Let $\sigma = \frac{v_- + v_+}{2}$ with $v_- > v_+$. Then, for any $\delta > 0$ we have the following:*

$$v(\xi) = \begin{cases} v_-, & \text{uniformly for } \xi < \sigma - \delta, \\ v_+, & \text{uniformly for } \xi > \sigma + \delta. \end{cases} \tag{3.24}$$

Proof. Since $v_\epsilon(\xi)$ is a strictly decreasing function, we can define $\xi_\epsilon = v_\epsilon(\xi_\epsilon)$ and $\xi_v = \lim_{\epsilon\to 0+} \xi_\epsilon$, $|\xi_v| < \infty$. Let ϵ be sufficiently small such that $\xi_\epsilon < \xi_v + \frac{\delta}{4} = \xi_0 - \frac{\delta}{4}$ for $\xi_0 = \xi_v + \frac{\delta}{2}$. From the second equation in (1.8), we have

$$v_\epsilon'(\xi) = v_\epsilon'(\xi_0)\exp\Big(\int_{\xi_0}^{\xi} \frac{v_\epsilon(\zeta) - \zeta}{\epsilon} d\zeta\Big). \tag{3.25}$$

Integrating (3.25) over $(\xi_0, \xi_0+\epsilon)$, we obtain

$$v_\epsilon(\xi_0+\epsilon) - v_\epsilon(\xi_0) = v_\epsilon'(\xi_0)\int_{\xi_0}^{\xi_0+\epsilon} \exp\Big(\frac{v_\epsilon(\zeta)-\zeta}{\epsilon}d\zeta\Big)d\xi. \tag{3.26}$$

Noting that

$$\begin{aligned}\int_{\xi_0}^{\xi_0+\epsilon} \exp\Big(\int_{\xi_0}^{\xi}\frac{v_\epsilon(\zeta)-\zeta}{\epsilon}d\zeta\Big)d\xi &\geq \int_{\xi_0}^{\xi_0+\epsilon} \exp\Big(\int_{\xi_0}^{\xi}\frac{v_+-\zeta}{\epsilon}d\zeta\Big)d\xi \\ &= \int_{\xi_0}^{\xi_0+\epsilon} \exp\Big(\frac{v_+}{\epsilon}(\xi-\xi_0) - \frac{1}{2\epsilon}(\xi^2-\xi_0^2)\Big)d\xi \\ &\geq \epsilon\int_0^1 \exp\Big(v_+\zeta - \xi_0\zeta - \frac{1}{2}\zeta^2\Big)d\zeta = A\epsilon,\end{aligned}$$

where $A = \int_0^1 \exp\big(v_+\zeta - \xi_0\zeta - \frac{1}{2}\zeta^2\big)d\zeta$, equation (3.26) gives

$$0 > v_\epsilon'(\xi_0) \geq \frac{v_+-v_-}{A\epsilon}. \tag{3.27}$$

Therefore, by (3.25) and (3.27), we obtain

$$|v_\epsilon'(\xi)| \leq \frac{v_--v_+}{A\epsilon}\exp\Big(\int_{\xi_0}^{\xi}\frac{v_\epsilon(\zeta)-\zeta}{\epsilon}d\zeta\Big). \tag{3.28}$$

From the definition of ξ_ϵ, it is clear that

$$v_\epsilon(\zeta)-\zeta = v_\epsilon(\zeta) - v_\epsilon(\xi_\epsilon) + \xi_\epsilon - \zeta = (\zeta-\xi_\epsilon)(v_\epsilon'(\theta_\epsilon)-1) \leq -\frac{\delta}{4}.$$

Thus we have for $\xi > \xi_0$,

$$|v_\epsilon'(\xi)| \leq \frac{v_--v_+}{A\epsilon}\exp\Big(-\frac{\delta}{4\epsilon}(\xi-\xi_0)\Big). \tag{3.29}$$

Here we take $\xi > \xi_v + \delta$. Noting that

$$\begin{aligned}|v_+ - v_\epsilon(\xi)| \leq \int_\xi^\infty |v_\epsilon'(\zeta)|d\zeta &\leq \frac{v_--v_+}{A\epsilon}\int_\xi^\infty \exp\Big(-\frac{\delta}{4\epsilon}(\zeta-\xi_0)\Big)d\zeta \\ &\leq \frac{4(v_--v_+)}{A\delta}\exp\Big(-\frac{\delta^2}{8\epsilon}\Big),\end{aligned}$$

we see that $\lim_{\epsilon\to0+} v_\epsilon(\xi) = v_+$ uniformly for $\xi > \xi_v + \delta$. Similarly, we can prove that $\lim_{\epsilon\to0+} v_\epsilon(\xi) = v_-$ uniformly for $\xi < \xi_v - \delta$.

Finally, we take $\phi(\xi) \in C_0^\infty(\xi_-, \xi_+)$ with $\xi_- < \xi_v < \xi_+$. Integrating the second equation in (1.8) over (ξ_-, ξ_+), we have

$$\epsilon\int_{\xi_-}^{\xi_+} v_\epsilon(\xi)\phi''(\xi)d\xi = \int_{\xi_-}^{\xi_+} v_\epsilon(\xi)(\phi(\xi)+\xi\phi'(\xi)) - \frac{v_\epsilon^2(\xi)}{2}\phi'(\xi)\,d\xi.$$

Letting $\epsilon \to 0+$, we have

$$\begin{aligned} 0 = &\int_{\xi_-}^{\xi_v} v_-(\phi(\xi) + \xi\phi^{'}(\xi)) - \frac{(v_-)^2}{2}\phi^{'}(\xi)\ d\xi \\ &+ \int_{\xi_v}^{\xi_+} v_+(\phi(\xi) + \xi\phi^{'}(\xi)) - \frac{(v_+)^2}{2}\phi^{'}(\xi)\ d\xi, \end{aligned}$$

so that

$$\Big\{(v_- - v_+)\xi_v - \frac{(v_-)^2 - (v_+)^2}{2}\Big\}\phi(\xi_v) = 0, \tag{3.30}$$

which means that $\xi_v = \sigma = \frac{v_- + v_+}{2}$ since $v_- > v_+$ and ϕ is arbitrary. Thus the proof of Lemma 3.7 is complete. □

The following lemma specifically describes the value of τ_0 in the case of $u_- = u_+$, $v_- + v_+ = 0$ with $v_- > v_+$:

Lemma 3.8. *Let $u_- = u_+$, $v_- + v_+ = 0$ with $v_- > v_+$. Then, we have the following:*

$$\lim_{\epsilon \to 0+} \tau_\epsilon = \tau_0 = 0. \tag{3.31}$$

Proof. By Lemma 3.7, we then have

$$v(\xi) = \begin{cases} v_-, & \xi < 0, \\ v_+, & \xi > 0. \end{cases}$$

If $\tau_0 \neq 0$, then by the Rankine-Hugoniot condition (3.10) for $\xi_0 = 0$, we have $f(v_-) = f(v_+)$. But this contradicts to condition (1.3). Thus the proof of Lemma 3.8 is complete. □

We now proceed to the proof of Theorem 3.1.

We first show the form of $u(\xi)$. Let $\delta > 0$. For each $\phi(\xi) \in C_0^\infty(\delta, \xi_+)$ with $\xi_+ > \delta$, integrating the first equation in (1.8) which is multiplied by $\phi(\xi)$, over (δ, ξ_+), we get

$$\epsilon \int_\delta^{\xi_+} u_\epsilon(\xi)\phi^{''}(\xi)d\xi = \int_\delta^{\xi_+} u_\epsilon(\xi)(\phi(\xi) + \xi\phi^{'}(\xi)) - f(v_\epsilon(\xi))\phi^{'}(\xi)\ d\xi. \tag{3.32}$$

Letting $\epsilon \to 0+$, we have

$$\int_\delta^{\xi_+} u(\xi)\Big(\frac{d}{d\xi}(\xi\phi(\xi))\Big)d\xi = \int_\delta^{\xi_+} f(v_+)\phi^{'}(\xi)d\xi = 0. \tag{3.33}$$

Thus we see that $u(\xi) = u_+$ for $\xi > 0$ since δ and ϕ are arbitrary. Similarly, we can prove that $u(\xi) = u_-$ for $\xi < 0$.

Since the Rankine-Hugoniot condition (3.14) for $\xi_0 = 0$ is not satisfied, the limit function $(u(\xi), v(\xi))$ is not a solution of the Riemann problem (1.1), (1.2). To this end, we must consider the weak limit of $(u_\epsilon(\xi), v_\epsilon(\xi))$. For $\xi_- < 0 < \xi_+$, we take

$\phi(\xi) \in C_0^\infty(\xi_-, \xi_+)$ with $\phi(\xi) = \phi(0)$ on $N(\lambda) = [-\lambda, \lambda]$ for sufficiently small $\lambda > 0$. Then, by (3.32), we have

$$\lim_{\epsilon \to 0+} \int_{\xi_-}^{\xi_+} u_\epsilon(\xi)(\phi(\xi) + \xi\phi'(\xi)) - f(v_\epsilon(\xi))\phi'(\xi)\, d\xi = 0. \tag{3.34}$$

Since $\phi(\xi) = \phi(0)$ for $\xi \in N(\lambda)$, we get

$$\begin{aligned}
&\lim_{\epsilon \to 0+} \int_{\xi_-}^{\xi_+} (\xi u_\epsilon(\xi) - f(v_\epsilon(\xi)))\phi'(\xi) d\xi \\
&= \int_{\xi_-}^{-\lambda} (\xi u_- - f(v_-))\phi'(\xi) d\xi + \int_{\lambda}^{\xi_+} (\xi u_+ - f(v_+))\phi'(\xi) d\xi \\
&= \big(f(v_+) - f(v_-) - \lambda(u_- + u_+)\big)\phi(0) - \int_{\xi_-}^{-\lambda} u_-\phi(\xi) d\xi - \int_{\lambda}^{\xi_+} u_+\phi(\xi) d\xi. \qquad (3.35)
\end{aligned}$$

Therefore, letting $\lambda \to 0+$, (3.34) and (3.35) yield that

$$\lim_{\epsilon \to 0+} \int_{\xi_-}^{\xi_+} u_\epsilon(\xi)\phi(\xi) d\xi = \big(f(v_-) - f(v_+)\big)\phi(0) + \int_{\xi_-}^{\xi_+} H_u(\xi)\phi(\xi) d\xi. \tag{3.36}$$

By the approximation process, equation (3.36) holds for all $\phi(\xi) \in C_0^\infty(\xi_-, \xi_+)$. Thus, we see that $u_\epsilon(\xi)$ converges weakly to $H_u(\xi) + s\delta(\xi)$. Similarly, we can prove that $v_\epsilon(\xi)$ converges weakly to $H_v(\xi)$.

Finally, we show that the limit function $(H_u(\xi) + s\delta(\xi), H_v(\xi))$ is a solution of the Riemann problem (1.1), (1.2). For $\xi_- < 0 < \xi_+$, we take $\phi(\xi) \in C_0^\infty(\xi_-, \xi_+)$. Then we see that

$$\begin{aligned}
&\int_{\xi_-}^{\xi_+} (H_u(\xi) + s\delta(\xi))(\phi(\xi) + \xi\phi'(\xi)) - f(H_v(\xi))\phi'(\xi) d\xi \\
&= \int_{\xi_-}^{0} u_-(\phi(\xi) + \xi\phi'(\xi)) - f(v_-)\phi'(\xi) d\xi \\
&\quad + \int_{0}^{\xi_+} u_+(\phi(\xi) + \xi\phi'(\xi)) - f(v_+)\phi'(\xi) d\xi + s\phi(0) \\
&= \big(f(v_+) - f(v_-) + s\big)\phi(0) = 0, \qquad (3.37)
\end{aligned}$$

and

$$
\begin{aligned}
&\int_{\xi_-}^{\xi_+} H_v(\xi)(\phi(\xi)+\xi\phi'(\xi)) - \frac{1}{2}(H_v(\xi))^2\phi'(\xi)d\xi \\
&= \int_{\xi_-}^{0} v_-(\phi(\xi)+\xi\phi'(\xi)) - \frac{1}{2}(v_-)^2\phi'(\xi)d\xi \\
&\quad + \int_{0}^{\xi_+} v_+(\phi(\xi)+\xi\phi'(\xi)) - \frac{1}{2}(v_+)^2\phi'(\xi)d\xi \\
&= \frac{1}{2}(v_- + v_+)(v_+ - v_-)\phi(0) = 0. \qquad (3.38)
\end{aligned}
$$

Thus the limit function $(H_u(\xi)+s\delta(\xi), H_v(\xi))$ is a solution of the Riemann problem (1.1), (1.2) and the proof of Theorem 3.1 is complete.

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