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QUASIASYMPTOTICS IN EXPONENTIAL DISTRIBUTIONS BY WAVELET ANALYSIS

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ABSTRACT. We investigate the quasiasymptotics of exponential distributions at a point or infinity via its multiresolution expansion. We also analyse the boundedness of the wavelet transform and wavelet coefficients of quasiasymptotically bounded exponential distributions.

1. Introduction

While the notion of the value of a continuous function at a point is clear, the notion of the value of a distribution at a point is somewhat complicated since distributions are not defined pointwise, but are the elements of certain dual spaces. It is therefore interesting to study the notion of the value of a distribution at a point, introduced by Lojasiewicz in [5], that reduces to the usual one for distributions locally equal to continuous functions.

The concept of the value of a distribution at a point in the sense of Lojasiewicz has many applications in areas such as Abelian and Tauberian type theorems for integral transform [8, 24], pointwise convergence of wavelet series of distributions [16, 29], convergence of Fourier series and integrals [3, 20, 21] and the boundry behavior of solutions of partial differential equations [27].

In general, not all distributions have a value at a point (see [5]), so there is a limitation for applying the value of a distribution at a point to Abelian and Tauberian type theorems for integral transforms such as Fourier, Laplace, Stieltjes and Mellin transforms. In order to avoid such a difficulty, we need a more general notion than the value of distribution at a point. The more general notion, namely, the quasiasymptotics at a point, was introduced by B.I. Zavialov [31]. This notion was rediscovered by Y. Meyer in his work on pointwise weak scaling exponents [7].

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The notion of quasiasymptotics has many applications in areas such as quantum field theory [25] and P.D.E. theory [26]. Especially, many authors have applied wavelet theory to investigate the quasiasymptotics of tempered distributions at some point or infinity, see [7, 9, 10, 11, 12, 13, 14, 15, 23].

The purpose of this paper is to investigate the quasiasymptotics of exponential distributions at a point via their multiresolution expansions. Let $\{V_j, j \in \mathbb{Z}\}$ be a multiresolution analysis of the space $L^2(\mathbb{R})$, f a exponential distribution, and f_j its projection to $V_j, j \in \mathbb{Z}$. We prove that if f has quasiasymptotics at a point or infinity related to a continuous function, then so does each $f_j, j \in \mathbb{Z}$, provided that the scaling function is sufficiently regular. Moreover, with an additional condition, we prove that the converse statement also holds. Also we analyse the boundedness of the wavelet transform and wavelet coefficients of the quasiasymptotically bounded exponential distributions.

2. Exponential Distributions and its Multiresolution Expansion

Throughout the paper, we consider the domain of functions as the set of real numbers, $\mathbb{R}$, and omit the suffix, therefore, in what follows, $\mathcal{H}$ means $\mathcal{H}(\mathbb{R})$. Also, we mean by $\mathbb{N}$ and $\mathbb{Z}$ the set of natural numbers and integers, respectively.

2-1. The spaces $\mathcal{K}'$ of distributions of exponential growth

Let $\mathcal{K}$ be the space of all C^∞ - functions φ in $\mathbb{R}$ such that

$$\|\varphi\|_{\mathcal{K}}^k = \sup_{x\in\mathbb{R},\ 0\le\alpha\le k} e^{k|x|} \left| \frac{d^\alpha}{dx^\alpha}\varphi \right|, \quad k = 1, 2, \dots, \quad \alpha \in \mathbb{N},$$

are finite. The topology in $\mathcal{K}$ is defined by the family of the semi-norms $\|\cdot\|_{\mathcal{K}}^k$. Then $\mathcal{K}$ is a Fréchet space and $\mathcal{D} \hookrightarrow \mathcal{K} \hookrightarrow \mathcal{S} \hookrightarrow \mathcal{E}$ are continuous and dense inclusions; here $\mathcal{D}$ denotes the spaces of all C^∞ - functions with compact supports, $\mathcal{S}$ the spaces of polynomially decreasing functions (Schwartz functions) and $\mathcal{E}$ the space of all C^∞ - functions. The dual space of $\mathcal{K}$ is denoted by $\mathcal{K}'$.

Definition 2.1. We say that the elements of $\mathcal{K}'$ are **exponential distributions**.

The spaces $\mathcal{K}'$ were introduced by K. Yoshinaga [30] and M. Hasumi [4], independently, and were characterized by S. Sznajder and Z.Zielezny [17, 18].

We define $\mathcal{K}_r$, $r \in \mathbb{N}$, to be the space of all C^r - functions ϕ in $\mathbb{R}$ such that

$$\|\phi\|_{\mathcal{K}_r} = \sup_{x\in\mathbb{R},\ 0\le\alpha\le r} e^{r|x|}\left|\frac{d^\alpha}{dx^\alpha}\phi(x)\right| < \infty, \quad \alpha\in\mathbb{N},$$

and

$$\lim_{|x|\to\infty}\sup_{0\le\alpha\le r} e^{r|x|}\left|\frac{d^\alpha}{dx^\alpha}\phi(x)\right| = 0, \quad \alpha\in\mathbb{N}.$$

The topology of $\mathcal{K}_r$ is defined by $\|\cdot\|_{\mathcal{K}_r}$. The dual space of $\mathcal{K}_r$ is denoted by $\mathcal{K}'_r$. Clearly, $\mathcal{K}$ is the projective limit of $\mathcal{K}_r$ when $r\to\infty$ and $\mathcal{K}' = \cup_{r\in\mathbb{N}}\mathcal{K}'_r$.

We also define $\widetilde{\mathcal{K}}_r$, $r\in\mathbb{N}$, to be the space of all C^r - functions ψ in $\mathbb{R}$ such that

$$\|\psi\|^l_{\widetilde{\mathcal{K}}_r} = \sup_{x\in\mathbb{R},\ 0\le\alpha\le r} e^{l|x|}\left|\frac{d^\alpha}{dx^\alpha}\psi(x)\right| < \infty, \quad l=1,2,\ldots \quad \alpha\in\mathbb{N}.$$

The topology of $\widetilde{\mathcal{K}}_r$ is defined by the family of semi-norms $\|\cdot\|^l_{\widetilde{\mathcal{K}}_r}$. The dual space of $\widetilde{\mathcal{K}}_r$ is denoted by $\widetilde{\mathcal{K}}'_r$. Obviously, $\widetilde{\mathcal{K}}_r \subset \mathcal{K}_r$.

Let ϕ and sequence $\{\phi_n\}_{n\in\mathbb{N}}$ be given in $\mathcal{K}_{r+1}$ such that $\{\frac{d^\alpha}{dx^\alpha}\phi_n\}_{n\in\mathbb{N}}$ converges uniformly to $\frac{d^\alpha}{dx^\alpha}\phi$ on every compact set $K\subset\mathbb{R}$ and for $\alpha=0,1,\ldots,r$. Then there exists N such that for arbitrary $\epsilon>0$ and $K\subset\mathbb{R}$.

$$\sup_{x\in K} e^{r|x|}\left|\frac{d^\alpha}{dx^\alpha}(\phi_n-\phi)(x)\right| < \epsilon, \quad n\ge N. \tag{1}$$

In addition, if the sequence $\{\phi_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{K}_{r+1}$, we can take a positive number $M>0$ and a compact set K such that $|x|>M$ when $x\notin K$ and

$$\begin{aligned}&\sup_{x\notin K} e^{r|x|}\left|\frac{d^\alpha}{dx^\alpha}(\phi_n-\phi)(x)\right| \\ &\le e^{-M}\sup_{x\notin K} e^{(r+1)|x|}\left(\left|\frac{d^\alpha}{dx^\alpha}\phi_n(x)\right| + \left|\frac{d^\alpha}{dx^\alpha}\phi(x)\right|\right) < \epsilon.\end{aligned} \tag{2}$$

From (1) and (2), we have

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}} e^{r|x|}\left|\frac{d^\alpha}{dx^\alpha}(\phi_n-\phi)(x)\right| = 0, \quad 0\le\alpha\le r.$$

Hence we have a lemma that will be used later.

Lemma 2.1. *Let ϕ and sequence $\{\phi_n\}_{n\in\mathbb{N}}$ be given in $\mathcal{K}_{r+1}$ such that $\{\frac{d^\alpha}{dx^\alpha}\phi_n\}_{n\in\mathbb{N}}$ converges uniformly to $\frac{d^\alpha}{dx^\alpha}\phi$ on every compact set $K \subset \mathbb{R}$ and for $\alpha = 0, 1, \ldots, r$. If $\{\phi_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{K}_{r+1}$, then the sequence $\{\phi_n\}_{n\in\mathbb{N}}$ converges to ϕ in $\mathcal{K}_r$.*

2.2 Mutiresolution expansion of exponential distributions

Let $\psi \in \widetilde{\mathcal{K}}_r$ (or $\mathcal{K}_r$). In order for it to qualify as a scaling function, there must be associated with ψ a multiresolution analysis (MRA) of L^2, i.e., a nested sequence of closed subspaces $\{V_n\}_{n\in\mathbb{Z}}$ such that

(i) $\{\psi(t-n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of V_0,

(ii) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2$,

(iii) $f \in V_n \Leftrightarrow f(2\cdot) \in V_{n+1}$,

(iv) $\cap_n V_n = \{0\}$, $\overline{\cup_n V_n} = L^2$.

Definition 2.2. We say that a multiresolution analysis $V_j, j \in \mathbb{Z}$, is an **exp-r-regular** MRA of L^2 if the scaling function ψ is in $\widetilde{\mathcal{K}}_r$.

Corollary 5.5.3 in [2] states that it is impossible that the scaling function ψ has exponential decay and $\psi \in C^\infty$, with all derivatives bounded, unless $\psi = 0$. So we will restrict our attention to $\widetilde{\mathcal{K}}_r$ or $\mathcal{K}_r$. From the remark in [2, page 152] or Example 4 in [28, page 36], Battle-Lemarié's wavelets are in $\mathcal{K}_r$ for some $r \in \mathbb{N}$, but not in $\widetilde{\mathcal{K}}_r$ even if they have exponential decay and smoothness. In [2], I. Daubechies showed that for an arbitrary nonnegative integer r, there exists an exp-r-regular MRA of L^2 such that the scaling function ψ has compact support. More details about the relations between wavelet and function spaces can be found in [6].

Let V_j be an exp-r-regular MRA of L^2 and let ψ be a scaling function. The reproducing kernel [28] of V_0 is given by

$$q_0(x,y) = \sum_{n\in\mathbb{Z}} \psi(x-n)\overline{\psi(y-n)}.$$

The series and its derivatives with respect to x or y of order $\leq r$ converge uniformly on $\mathbb{R}$ because of the regularity of $\psi \in \widetilde{\mathcal{K}}_r$. The reproducing kernel of the projection operator onto V_j is

$$q_j(x,y) = 2^j q_0(2^j x, 2^j y), \quad x, y \in \mathbb{R},$$

and the projection of $f \in L^2$ onto V_j is given by

$$q_j f(x) = \langle f(y), q_j(x,y)\rangle = \int f(y) q_j(x,y)dy, \quad x \in \mathbb{R}. \tag{3}$$

Definition 2.3. The sequence $\{q_j f\}_{j\in\mathbb{Z}}$ in (3) is called the **multiresolution expansion** of f.

For a given $f \in \mathcal{K}_r'$, the multiresolution expansion of f by the sequence $\{q_j\}_{j\in\mathbb{Z}}$ is defined by

$$\langle q_j f, \phi\rangle = \langle f, q_j\phi\rangle, \quad \phi \in \mathcal{K}_r. \tag{4}$$

We deduce the following properties of the reproducing kernel q_0 with scaling function $\psi \in \widetilde{\mathcal{K}}_r$:

(a) $q_0(x,y) = q_0(y,x)$ and $q_0(x+k, y+k) = q_0(x,y)$ for all $k \in \mathbb{Z}$.

(b) For every $l \in \mathbb{N}$ and $0 \le \alpha, \beta \le r$, there exists $C_l > 0$ such that

$$\begin{aligned} \left|\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\beta}{\partial y^\beta} q_0(x,y)\right| &\le \sum_j \left|\frac{d^\alpha}{dx^\alpha}\psi(x-j)\right|\left|\frac{d^\beta}{dy^\beta}\psi(y-j)\right| \\ &\le \sum_j C_{l+1} e^{-(l+1)|x-j|} e^{-(l+1)|y-j|} \\ &\le C_{l+1} e^{-l|x-y|} \sum_j e^{-|x-j|} e^{-|y-j|} \\ &\le C_{l+1} e^{-l|x-y|}. \end{aligned}$$

(c) $\int_{-\infty}^{\infty} q_0(x,y) y^\alpha dy = x^\alpha, \quad y \in \mathbb{R},\ 0 \le \alpha \le r.$

Let V_j be an exp-r-regular MRA of L^2. We fix a function $g \in \mathcal{D}$ with $\int g = 1$. We let g_j denote the function $2^j g(2^j x)$ and let G_j denote the operation of convolution by g_j. For each fixed x, we consider the function $\partial_x^\alpha q_0(x,y)$ of the variable y. From (c), we have

$$\int \partial_x^\alpha q_0(x,y) y^\beta dy = 0, \tag{5}$$

for $0 \le \beta < \alpha$, whereas

$$\int \partial_x^\alpha q_0(x,y) y^\alpha dy = \alpha!. \tag{6}$$

Now, it follows from integration by parts that the kernal $g(x-y)$ of the operator G shares these properties (5) and (6) with $q_0(x,y)$.

Let

$$R^\alpha(x,y) = \partial_x^\alpha q_0(x,y) - \partial_x^\alpha g(x-y).$$

From (b) and the fact that $g \in \mathcal{D} \subset \mathcal{K}_r$,

$$|R^\alpha(x,y)| \leq c_k e^{-k|x-y|}, \quad x, y \in \mathbb{R},\ k \in \mathbb{N}, \tag{7}$$

and these functions also satisfy

$$\int R^\alpha(x,y) dy = 0$$

identically in x for every $\alpha = 1, 2, \ldots, r$. They, for every $j \in \mathbb{Z}$ and $f \in C^r$ with at most exponential growth, define operator R_j^α by

$$R_j^\alpha f(x) = 2^j \int R^\alpha(2^j x, 2^j y) f(y) dy$$

which are such that

$$q_j \frac{d^\alpha}{dx^\alpha} f(x) = G_j \frac{d^\alpha}{dy^\alpha} f(y) + R_j^\alpha \frac{d^\alpha}{dy^\alpha} f(y), \tag{8}$$

i.e.,

$$\int q_j(x,y) \frac{d^\alpha}{dy^\alpha} f(y) dy = 2^j \int g(2^j(x-y)) \frac{d^\alpha}{dy^\alpha} f(y) dy + 2^j \int R^\alpha(2^j x, 2^j y) \frac{d^\alpha}{dy^\alpha} f(y) dy.$$

From the Theorem 1.1 in [1], we have

$$\lim_{j\to\infty} G_j \frac{d^\alpha}{dy^\alpha} f(y) dy = \frac{d^\alpha}{dx^\alpha} f(x), \quad x \in \mathbb{R}, \quad \alpha \geq 0, \tag{9}$$

uniformly on compact sets. In addition, we let f be in C^r such that the corresponding derivatives $\frac{d^\alpha}{dx^\alpha} f$ are bounded by a exponential when $|x| \to \infty$, for every $\alpha = 0, 1, \ldots, r$. Then, if $|y-x| \le c$, we have

$$\left| \frac{d^\alpha}{dy^\alpha} f(y) - \frac{d^\alpha}{dy^\alpha} f(y)\Big|_{y=x} \right| \le e_\alpha(y-x),$$

where $e_r(x)$ is a continuous function with exponential growth and $e_r(0) = 0$. From (7), given a compact set K,

$$\begin{aligned}
&2^j \left| \int R^\alpha(2^j x, 2^j y) \frac{d^\alpha}{dy^\alpha} f(y) dy \right| \\
&\le 2^j \int \left| R^\alpha(2^j x, 2^j y) \left(\frac{d^\alpha}{dy^\alpha} f(y) - \frac{d^\alpha}{dy^\alpha} f(y)\Big|_{y=x} \right) \right| dy \\
&\le 2^j \int c_k e^{-2^j k|x-y|} |e_\alpha(y-x)| dy \\
&= 2^j c_k \int e^{-2^j k y} |e_\alpha(y)| dy,
\end{aligned}$$

for large enough j and $x \in K$. Since k can be chosen arbitrary, we obtain by the dominated convergence theorem,

$$\begin{aligned}
&\lim_{j\to\infty} 2^j \left| \int R^\alpha(2^j x, 2^j y) \frac{d^\alpha}{dy^\alpha} f(y) dy \right| \\
&\le 2^j \int \lim_{j\to\infty} c_k e^{-2^j k|x-y|} |e_\alpha(y-x)| dy \\
&= c_k \int \lim_{j\to\infty} 2^j e^{-2^j k y} |e_\alpha(y)| dy
\end{aligned}$$

uniformly for $x \in K$. From (8) and (9), we have a lemma that will be used later.

Lemma 2.2. *Let $f \in C^r$ and let $q_j f$, given by (3), be the projection of f onto an exp-r-regular MRA of L^2 such that the corresponding derivatives $\frac{d^\alpha}{dx^\alpha} f$ are bounded by a exponential when $|x| \to \infty$, for every $\alpha = 0, 1, \ldots, r$. Then the sequence $\{\frac{d^\alpha}{dx^\alpha} q_j f\}_{j\in\mathbb{N}}$ converges uniformly on compact sets to $\frac{d^\alpha}{dx^\alpha} f$ as $j \to \infty$, for every $\alpha = 0, 1, \ldots, r$.*

We now show that the multiresolution expansion of $\phi \in \mathcal{K}_{r+1}$ converges in $\phi \in \mathcal{K}_r$, provided that the MRA is exp-r-regular.

Proposition 2.1. *Let $\phi \in \mathcal{K}_{r+1}$ and let $q_j\phi(x)$, given by (3), be a projection of ϕ onto an exp-r-regular MRA of L^2. Then the sequence $\{q_j\phi(x)\}$ converges to $\phi(x)$ in $\mathcal{K}_r$ as $j \to \infty$.*

Proof. Let g and R^α be given in (8) such that $g \in \mathcal{D}$ and $\int g(x)dx = 1$. From Lemma 2.1, Lemma 2.2 and (8), it suffices to show that

$$\begin{aligned}
&\sup_{x\in\mathbb{R}} e^{(r+1)|x|}\frac{1}{h}\left|\int q_0(\frac{x}{h},\frac{y}{h})\phi(y)dy\right| \\
&\quad = \sup_{x\in\mathbb{R}} e^{(r+1)|x|}\frac{1}{h}\left|\int g(\frac{x-y}{h})\frac{d^\alpha}{dy^\alpha}\phi(y)dy + \int R^\alpha(\frac{x}{h},\frac{y}{h})\frac{d^\alpha}{dy^\alpha}\phi(y)dy\right|
\end{aligned}$$

is bounded for every $\alpha \in \{0,1,\ldots,r\}$ and $h > 0$. Since

$$\begin{aligned}
&\sup_{x\in\mathbb{R}} e^{(r+1)|x|}\frac{1}{h}\left|\int g(\frac{x-y}{h})\frac{d^\alpha}{dy^\alpha}\phi(y)dy\right| \\
&\quad \le \sup_{x\in\mathbb{R}} \frac{1}{h}\left|\int g(\frac{x-y}{h})e^{(r+1)|x|}e^{-(r+1)|y|}dy\right| \\
&\quad \le \sup_{x\in\mathbb{R}} \frac{1}{h}\left|\int g(\frac{x-y}{h})e^{(r+1)|x-y|}dy\right| \le C,
\end{aligned}$$

we have only to show that

$$K = \sup_{x\in\mathbb{R}} e^{(r+1)|x|}\frac{1}{h}\left|\int R^\alpha(\frac{x}{h},\frac{y}{h})\frac{d^\alpha}{dy^\alpha}\phi(y)dy\right| \le C, \quad x \in \mathbb{R}$$

for every $\alpha \in \{0,1,\ldots,r\}$ and $h > 0$. Let $S_1 = \{y : |x-y| \le 1\}$, $S_2 = \{y : |x-y| \ge 1\}$. Then, by (7),

$$\begin{aligned}
K &= \sup_{x\in\mathbb{R}} e^{(r+1)|x|}\frac{1}{h}\left|\int R^\alpha(\frac{x}{h},\frac{y}{h})\frac{d^\alpha}{dy^\alpha}\phi(y)dy\right| \\
&\le c_l \sup_{x\in\mathbb{R}} e^{(r+1)|x|}\frac{1}{h}\int e^{-l\left|\frac{x-y}{h}\right|}e^{-(r+1)|y|}dy \\
&= c_l \sup_{x\in\mathbb{R}} e^{(r+1)|x|}\frac{1}{h}\int_{S_1} e^{-l\left|\frac{x-y}{h}\right|}e^{-(r+1)|y|}dy \\
&\quad + c_l \sup_{x\in\mathbb{R}} e^{(r+1)|x|}\frac{1}{h}\int_{S_2} e^{-l\left|\frac{x-y}{h}\right|}e^{-(r+1)|y|}dy \\
&= c_l(K_1 + K_2).
\end{aligned}$$

By a simple change of variable,

$$
\begin{aligned}
K_1 &= \sup_{x\in\mathbb{R}} e^{(r+1)|x|}\frac{1}{h}\int_{S_1} e^{-l\left|\frac{x-y}{h}\right|}e^{-(r+1)|y|}dy \\
&\leq \sup_{x\in\mathbb{R}} e^{(r+1)|x|}2\int_0^1 \frac{1}{h}e^{-l\left|\frac{t}{h}\right|}e^{-(r+1)|x-t|}dt \\
&\leq 2e^{(r+1)}\int_0^1 \frac{1}{h}e^{-l\left|\frac{t}{h}\right|}dt = 2e^{(r+1)}\int_0^{\frac{1}{h}} e^{-lu}du \leq C_1.
\end{aligned}
$$

and

$$
\begin{aligned}
K_2 &= \sup_{x\in\mathbb{R}} e^{(r+1)|x|}\frac{1}{h}\int_{S_2} e^{-l\left|\frac{x-y}{h}\right|}e^{-(r+1)|y|}dy \\
&\leq \sup_{x\in\mathbb{R}} \frac{1}{h}\int_{S_2} e^{-\frac{l}{h}|x-y|}e^{(r+1)(|x-y|)}dy \\
&\leq \sup_{x\in\mathbb{R}} \frac{1}{h}\int_{S_2} e^{\left(-\frac{l}{h}+(r+1)\right)|x-y|}dy \leq C_2
\end{aligned}
$$

for sufficiently large l. □

3. Quasiasymptotics of exponential distributions at a point

It is possible to expand a certain class of tempered distributions in orthogonal wavelets from L^2. In [28, 29], G. G. Walter considered the expansion of a distribution or function in regular orthogonal wavelets and showed that the expansion of a distribution converges pointwisely to the value of the distribution in the sense of Lojasiewicz. More generally, we expanded the exponential distributions in terms of orthgonal wavelets and showed the pointwise convergence of wavelet expansions of exponential distributions as follows [16]:

Definition 3.1. [5]. f is said to have **a value** γ **of order** r **at** x_0 **in** $\mathcal{K}'$ if there exists a continuous function $F(x)$ of exponential growth such that its distributional derivative $\frac{d^r}{dx^r}F(x) = f(x)$ and

$$\lim_{x\to x_0} \frac{F(x)}{(x-x_0)^r} = \frac{\gamma}{r!},$$

or equivalently

f has **the value** $\gamma = f(x_0)$ **at the point** x_0 **in** $\mathcal{K}'$ if

$$\lim_{\epsilon\to 0} \langle f(x_0+\epsilon x), \phi(x)\rangle = \gamma \int_{-\infty}^{\infty} \phi(x)dx, \quad \phi(x) \in \mathcal{K}.$$

Theorem 3.1. *Let* $f \in \widetilde{\mathcal{K}}'_r$ *and have a value* γ *of order* $\alpha \le r$ *at* x_0*. Then*

$$q_j f(x_0) \to \gamma \quad as \quad j \to \infty,$$

where q_j *is given in (3).*

In general, a distribution does not have to admit a value at a point, as seen in [5]. We need more general notion than the distributional value at a point in order to analyze the point behavior of distributions. The more general notion, namely the quasiasymptotics at some point or infinity, was introduced by B. I. Zavialov [31] and turned out to be appropriate for the Abelian and Tauberian type theorems for several integral tranforms [9, 10, 25, 26].

Definition 3.2. [22, 31]. Let $f \in \mathcal{K}'$ and let $c(\epsilon), x \in (0,a), a > 0$, be a continuous function. We say that f has **the quasiasymptotics at** x_0 **in** $\mathcal{K}'$ **related to** $c(\epsilon)$ **equal to** g, if there exists $g \in \mathcal{K}', g \neq 0$, such that

$$\lim_{\epsilon\to 0^+} \left\langle \frac{f(\epsilon x + x_0)}{c(\epsilon)}, \varphi(x) \right\rangle = \langle g(x), \varphi(x)\rangle, \quad \varphi(x) \in \mathcal{K}.$$

One can show that the comparison function c must be a regularly varying function in the sense of Karamata and g must be a homogeneous distribution (see [11] for details). Obviously, if we put $c(\epsilon) = 1$, we can obtain the value of exponential distributions at x_0 in the sense of Lojasiewicz.

In this section, by following the approach of Pilipović, A. Takaći and N. Teofanov from [9], we show that if f has the quasiasymptotics at x_0 in $\mathcal{K}'$ related to $c(\epsilon)$, then so does $q_j f, j \in \mathbb{Z}$, given in (3) and prove the opposite statement with an additional Tauberian condition. In order to investigate the properties of the kernel of the integral transform, we may allow j in (3) to be real number.

Lemma 3.1. *Let* V_j *be an exp-r-regular MRA of* L^2 *and let* q_0 *be the reproducing kernel of* V_0*. If we let* $q_j^\epsilon(x,y) = 2^j\epsilon q_0(2^j\epsilon x, 2^j\epsilon y)$*, then given any fixed* $\varphi \in \mathcal{K}$*, the set*

$$\{\langle q_j^\epsilon(x+\frac{x_0}{\epsilon}, y+\frac{x_0}{\epsilon}), \varphi(x)\rangle, \quad \epsilon \in (0,1)\ ;\ \varphi \in \mathcal{K}\}$$

is bounded in $\mathcal{K}$ for every $j \in \mathbb{Z}$ and is uniformly bounded in ϵ.

Proof. Let $\varphi \in \mathcal{K}$ be given. We show that

$$\sup_{y\in\mathbb{R}} e^{l|y|}\left|\frac{d^p}{dy^p}\int_{-\infty}^{\infty} q_j^{\epsilon}(x+\frac{x_0}{\epsilon}, y+\frac{x_0}{\epsilon})\varphi(x)dx\right|,\ p=0,1,\dots,r,$$

is uniformly bounded in ϵ.

$$\begin{aligned}
I &= e^{l|y|}\Big|\frac{d^p}{dy^p}\int_{-\infty}^{\infty} q_j^{\epsilon}(x+\frac{x_0}{\epsilon}, y+\frac{x_0}{\epsilon})\varphi(x)dx\Big| \\
&\le e^{l|y|}\Big|\frac{d^p}{dy^p}\int_{-\infty}^{y-c} q_j^{\epsilon}(x+\frac{x_0}{\epsilon}, y+\frac{x_0}{\epsilon})\varphi(x)dx\Big| \\
&\quad + e^{l|y|}\Big|\frac{d^p}{dy^p}\int_{y-c}^{y+c} q_j^{\epsilon}(x+\frac{x_0}{\epsilon}, y+\frac{x_0}{\epsilon})(\varphi(x)-\varphi(y))dx\Big| \\
&\quad + e^{l|y|}\Big|\frac{d^p}{dy^p}\int_{y+c}^{\infty} q_j^{\epsilon}(x+\frac{x_0}{\epsilon}, y+\frac{x_0}{\epsilon})\varphi(x)dx\Big| \\
&\quad + e^{l|y|}\Big|\frac{d^p}{dy^p}\int_{y-c}^{y+c} q_j^{\epsilon}(x+\frac{x_0}{\epsilon}, y+\frac{x_0}{\epsilon})\varphi(y)dx\Big| \\
&= I_1+I_2+I_3+I_4.
\end{aligned}$$

First, we estimate $I_i, i=2,3,4$.

$$\begin{aligned}
I_2 &= e^{l|y|}\Big|\frac{d^p}{dy^p}\int_{y-c}^{y+c} q_j^{\epsilon}(x+\frac{x_0}{\epsilon}, y+\frac{x_0}{\epsilon})(\varphi(x)-\varphi(y))dx\Big| \\
&\le e^{l|y|}\Big|\frac{d^p}{dy^p}\int_{y-c}^{y+c} q_j^{\epsilon}(x+\frac{x_0}{\epsilon}, y+\frac{x_0}{\epsilon})\varphi'(\xi)(x-y)dx\Big| \\
&\le e^{l|y|}c|\varphi'(\xi)|\int_{y-c}^{y+c}\Big|\frac{d^p}{dy^p}q_j^{\epsilon}(x+\frac{x_0}{\epsilon}, y+\frac{x_0}{\epsilon})\Big|dx \\
&\le cc_{1,s}e^{l|y|}e^{-s|y-c|}\int_{y-c}^{y+c}\Big|\frac{d^p}{dy^p}q_j^{\epsilon}(x+\frac{x_0}{\epsilon}, y+\frac{x_0}{\epsilon})\Big|dx, \qquad (10)
\end{aligned}$$

where $\xi \in (y-c, y+c)$.

$$\begin{aligned}
I_3 &= e^{l|y|}\Big|\frac{d^p}{dy^p}\int_{y+c}^{\infty} q_j^{\epsilon}(x+\frac{x_0}{\epsilon}, y+\frac{x_0}{\epsilon})\varphi(x)dx\Big| \\
&\le c_{2,s}e^{l|y|}e^{-s|y+c|}\int_{y+c}^{\infty}\Big|\frac{d^p}{dy^p}q_j^{\epsilon}(x+\frac{x_0}{\epsilon}, y+\frac{x_0}{\epsilon})\Big|dx. \qquad (11)
\end{aligned}$$

The estimation of I_1 can be showed in a similar way.

$$\begin{aligned} I_4 &= e^{l|y|}\Big|\frac{d^p}{dy^p}\int_{y-c}^{y+c} q_j^\epsilon(x+\frac{x_0}{\epsilon}, y+\frac{x_0}{\epsilon})\varphi(y)dx\Big| \\ &\leq c_{3,s}e^{l|y|}e^{-s|y|}\int_{y-c}^{y+c}\Big|\frac{d^p}{dy^p}q_j^\epsilon(x+\frac{x_0}{\epsilon}, y+\frac{x_0}{\epsilon})\Big|dx \\ &\quad + c'_{3,s}e^{l|y|}e^{-s|y|}\int_{y-c}^{y+c}\Big|q_j^\epsilon(x+\frac{x_0}{\epsilon}, y+\frac{x_0}{\epsilon})\Big|dx. \end{aligned} \tag{12}$$

Since we can choose s to be arbitrary in (10),(11) and (12), to prove the uniform boundedness of I in ϵ, it suffices to show that

$$I_5 = \int_{y+c}^{\infty}\Big|\frac{d^p}{dy^p}q_j^\epsilon(x+\frac{x_0}{\epsilon}, y+\frac{x_0}{\epsilon})\Big|dx,\ \ p=0,1,\ldots,r,$$

is uniformly bounded in ϵ. Since $q_j^\epsilon(x+\frac{x_0}{\epsilon}, y+\frac{x_0}{\epsilon}) = 2^j\epsilon q_0(2^j\epsilon x+2^jx_0, 2^j\epsilon y+2^jx_0)$, after the change of variables $t=2^j\epsilon x$,

$$\begin{aligned} I_5 &= \int_{2^j\epsilon(y+c)}^{\infty}\Big|\frac{d^p}{dy^p}q_0(t+2^jx_0, 2^j\epsilon y+2^jx_0)\Big|dt \\ &\leq \sum_{k\in\mathbb{Z}}\Big|\frac{d^p}{dy^p}\varphi(2^j\epsilon y+2^jx_0-k)\Big|\int_{2^j\epsilon(y+c)}^{\infty}|\varphi(t+2^jx_0-k)|dt \\ &\leq C_s\sum_{k\in\mathbb{Z}}|2^j\epsilon|^p e^{-s|2^j\epsilon y+2^jx_0-k|}\int_{2^j\epsilon(y+c)}^{\infty}e^{-(s+1)|t+2^jx_0-k|}dt \\ &\leq C_s|2^j\epsilon|^p\sum_{k\in\mathbb{Z}}e^{-s|2^j\epsilon y+2^jx_0-k|}\ e^{-s|2^j\epsilon(y+c)+2^jx_0-k|}\int_{2^j\epsilon(y+c)+2^jx_0-k}^{\infty}e^{-|u|}du \\ &\leq C'_s|2^j\epsilon|^p\sum_{k\in\mathbb{Z}}e^{-s|2^j\epsilon y+2^jx_0-k|}\ e^{-s|2^j\epsilon(y+c)+2^jx_0-k|}. \end{aligned}$$

Since we can choose s arbitrary, the last series is uniformly bounded in x and y, for fixed j and $p=0,1,\ldots,r$, hence I_5 is uniformly bounded in ϵ. □

Now, we are ready to characterize the quasiasymptotic behavoir of exponential distributions at a point via its multiresolution expansion.

Theorem 3.2. *Let $f, g(\neq 0), g_j(\neq 0)\in\mathcal{K}'$ and let q_jf, given in (3), be a projection of f onto an exp-r-regular MRA of L^2. If f has the quasiasymptotics at x_0 in $\mathcal{K}'$ related to $c(\epsilon)$ equal to $g(x)$, then $q_jf, j\in\mathbb{R}$, has the quasiasymptotics at x_0 in $\mathcal{K}'$ related to $c(\epsilon)$ equal to $g(x)$. Moreover, if $q_jf, j\in\mathbb{R}$, has the quasiasymptotics at x_0*

in $\mathcal{K}'$ related to $c(\epsilon)$ equal to $g_j(x)$ such that $g_j(x) \to g(x)$ pointwisely as $j \to \infty$ and the family $\left\{\frac{f(\epsilon x + x_0)}{c(\epsilon)} : \epsilon \in (0,1)\right\}$ is bounded in $\mathcal{K}$, then f has the quasiasymptotics at x_0 in $\mathcal{K}'$ related to $c(\epsilon)$ equal to $g(x)$.

Proof. For the first statement, we let $q_j^\epsilon(x,y) = 2^j \epsilon q_0(2^j \epsilon x, 2^j \epsilon y)$ and

$$\lim_{\epsilon \to 0^+} \left\langle \frac{f(\epsilon x + x_0)}{c(\epsilon)}, \varphi(x) \right\rangle = \langle g(x), \varphi(x) \rangle, \quad \varphi(x) \in \mathcal{K}.$$

If we use the equivalence of weak and strong convergence in $\mathcal{K}'$, the Lemma 2.1, Lemma 2.2, and Lemma 3.1 (or, Corollary 8.3 in [28]), then

$$\begin{aligned}
&\lim_{\epsilon \to 0^+} \left\langle \frac{q_j f(\epsilon x + x_0)}{c(\epsilon)}, \varphi(x) \right\rangle \\
&= \lim_{\epsilon \to 0^+} \left\langle \frac{1}{c(\epsilon)} \langle f(y), q_j(\epsilon x + x_0, y) \rangle, \varphi(x) \right\rangle \\
&= \lim_{\epsilon \to 0^+} \left\langle \frac{f(\epsilon y + x_0)}{c(\epsilon)}, \langle \epsilon q_j(\epsilon x + x_0, \epsilon y + x_0), \varphi(x) \rangle \right\rangle \\
&= \lim_{\epsilon \to 0^+} \left\langle \frac{f(\epsilon y + x_0)}{c(\epsilon)}, \langle 2^j \epsilon q_0(2^j \epsilon (x + \frac{x_0}{\epsilon}), 2^j \epsilon (y + \frac{x_0}{\epsilon})), \varphi(x) \rangle \right\rangle \\
&= \lim_{\epsilon \to 0^+} \left\langle \frac{f(\epsilon y + x_0)}{c(\epsilon)}, \langle q_j^\epsilon (x + \frac{x_0}{\epsilon}, y + \frac{x_0}{\epsilon}), \varphi(x) \rangle \right\rangle \\
&= \langle g(x), \varphi(x) \rangle.
\end{aligned}$$

For the second statement, we let $q_{\frac{1}{\epsilon}}^\epsilon(x,y) = 2^{\frac{1}{\epsilon}} \epsilon q_0(2^{\frac{1}{\epsilon}} \epsilon x, 2^{\frac{1}{\epsilon}} \epsilon y)$ and

$$\lim_{\epsilon \to 0^+} \left\langle \frac{q_j f(\epsilon x + x_0)}{c(\epsilon)}, \varphi(x) \right\rangle = \langle g_j(x), \varphi(x) \rangle, \quad \varphi(x) \in \mathcal{K}.$$

Once we have showed that $\langle q_{\frac{1}{\epsilon}}^\epsilon (x + \frac{x_0}{\epsilon}, y + \frac{x_0}{\epsilon}) - \delta(x-y), \varphi(x) \rangle$ tends to 0 in $\mathcal{K}$ as $\epsilon \to 0$ for Dirac distribution δ, we have

$$
\begin{aligned}
&\lim_{\epsilon\to 0^+}\Big\langle\frac{f(\epsilon y+x_0)}{c(\epsilon)},\varphi(y)\Big\rangle\\
&=\lim_{\epsilon\to 0^+}\Big(\Big\langle\frac{f(\epsilon y+x_0)}{c(\epsilon)},\varphi(y)\Big\rangle\\
&\quad+\Big\langle\frac{f(\epsilon y+x_0)}{c(\epsilon)},\langle q^{\epsilon}_{\frac{1}{\epsilon}}(x+\frac{x_0}{\epsilon},y+\frac{x_0}{\epsilon})-\delta(x-y),\varphi(x)\rangle\Big\rangle\Big)\\
&=\lim_{\epsilon\to 0^+}\Big\langle\frac{\langle f(\epsilon y+x_0),q^{\epsilon}_{\frac{1}{\epsilon}}(x+\frac{x_0}{\epsilon},y+\frac{x_0}{\epsilon})\rangle}{c(\epsilon)},\varphi(x)\Big\rangle\\
&=\lim_{\epsilon\to 0^+}\Big\langle\frac{\langle f(y+\frac{x_0}{\epsilon}),q^{\epsilon}_{\frac{1}{\epsilon}}(\epsilon x+x_0,y+\frac{x_0}{\epsilon})\rangle}{c(\epsilon)},\varphi(x)\Big\rangle\\
&=\lim_{\epsilon\to 0^+}\Big\langle\frac{q^{\epsilon}_{\frac{1}{\epsilon}}f(\epsilon x+x_0)}{c(\epsilon)},\varphi(x)\Big\rangle=\langle g(y),\varphi(y)\rangle,\quad \varphi(x)\in\mathcal{K}.
\end{aligned}
$$

It remains to show that

$$
\sup_{y\in\mathbb{R}}e^{l|y|}\left|\frac{d^p}{dy^p}\langle q^{\epsilon}_{\frac{1}{\epsilon}}(x+\frac{x_0}{\epsilon},y+\frac{x_0}{\epsilon})-\delta(x-y),\varphi(x)\rangle\right|
$$

tends to 0 as $\epsilon\to 0$. Since $\frac{d^p}{dy^p}q^{\epsilon}_{\frac{1}{\epsilon}}(x,y)=\frac{d^p}{dx^p}q^{\epsilon}_{\frac{1}{\epsilon}}(y,x)$,

$$
\begin{aligned}
J&=e^{l|y|}\Big|\int_{-\infty}^{\infty}\frac{d^p}{dy^p}q^{\epsilon}_{\frac{1}{\epsilon}}(x+\frac{x_0}{\epsilon},y+\frac{x_0}{\epsilon})\varphi(x)dx\\
&\qquad-(-1)^p\varphi^{(p)}(y)\int_{-\infty}^{\infty}q^{\epsilon}_{\frac{1}{\epsilon}}(x+\frac{x_0}{\epsilon},y+\frac{x_0}{\epsilon})dx\Big|\\
&=e^{l|y|}\Big|(-1)^p\Big(\int_{-\infty}^{\infty}q^{\epsilon}_{\frac{1}{\epsilon}}(x+\frac{x_0}{\epsilon},y+\frac{x_0}{\epsilon})\frac{d^p}{dx^p}\varphi(x)dx\\
&\qquad-\varphi^{(p)}(y)\int_{-\infty}^{\infty}q^{\epsilon}_{\frac{1}{\epsilon}}(x+\frac{x_0}{\epsilon},y+\frac{x_0}{\epsilon})dx\Big)\Big|\\
&\le e^{l|y|}\Big|\int_{-\infty}^{y-c}q^{\epsilon}_{\frac{1}{\epsilon}}(x+\frac{x_0}{\epsilon},y+\frac{x_0}{\epsilon})\frac{d^p}{dx^p}\varphi(x)dx\Big|\\
&\qquad+e^{l|y|}\Big|\int_{y-c}^{y+c}q^{\epsilon}_{\frac{1}{\epsilon}}(x+\frac{x_0}{\epsilon},y+\frac{x_0}{\epsilon})(\varphi^{(p)}(x)-\varphi^{(p)}(y))dx\Big|\\
&\qquad+e^{l|y|}\Big|\int_{y+c}^{\infty}q^{\epsilon}_{\frac{1}{\epsilon}}(x+\frac{x_0}{\epsilon},y+\frac{x_0}{\epsilon})\frac{d^p}{dx^p}\varphi(x)dx\Big|\\
&\qquad+e^{l|y|}\Big|\varphi^{(p)}(y)\int_{-\infty}^{y-c}q^{\epsilon}_{\frac{1}{\epsilon}}(x+\frac{x_0}{\epsilon},y+\frac{x_0}{\epsilon})dx\Big|\\
&\qquad+e^{l|y|}\Big|\varphi^{(p)}(y)\int_{y+c}^{\infty}q^{\epsilon}_{\frac{1}{\epsilon}}(x+\frac{x_0}{\epsilon},y+\frac{x_0}{\epsilon})dx\Big|\\
&=J_1+J_2+J_3+J_4+J_5.
\end{aligned}
$$

Consider J_3.

$$\begin{aligned}
J_3 &= e^{l|y|}\Big|\int_{y+c}^{\infty} q^{\epsilon}_{\frac{1}{\epsilon}}(x+\frac{x_0}{\epsilon}, y+\frac{x_0}{\epsilon})\frac{d^p}{dx^p}\varphi(x)dx\Big| \\
&\leq c_s e^{l|y|}e^{-s|y+c|}\int_{y+c}^{\infty}\Big|q^{\epsilon}_{\frac{1}{\epsilon}}(x+\frac{x_0}{\epsilon}, y+\frac{x_0}{\epsilon})\Big|dx \\
&\leq c\sum_{k\in\mathbb{Z}}|\varphi(2^{\frac{1}{\epsilon}}\epsilon(y+\frac{x_0}{\epsilon}-k))|\int_{y+c}^{\infty}2^{\frac{1}{\epsilon}}\epsilon|\varphi(2^{\frac{1}{\epsilon}}\epsilon(x+\frac{x_0}{\epsilon}-k))|dx \\
&\leq c'\sum_{k\in\mathbb{Z}}e^{-2l|2^{\frac{1}{\epsilon}}\epsilon(y+\frac{x_0}{\epsilon}-k)|}\int_{2^{\frac{1}{\epsilon}}\epsilon(y+\frac{x_0}{\epsilon}+c)}^{\infty}e^{-(2l+1)|x-k|}dx \\
&\leq c'\sum_{k\in\mathbb{Z}}e^{-2l|2^{\frac{1}{\epsilon}}\epsilon(y+\frac{x_0}{\epsilon}-k)|}e^{-2l|2^{\frac{1}{\epsilon}}\epsilon(y+\frac{x_0}{\epsilon}+c-k)|}\int_{2^{\frac{1}{\epsilon}}\epsilon(y+\frac{x_0}{\epsilon}+c)}^{\infty}e^{-|x-k|}dx \\
&\leq c'e^{-l|2^{\frac{1}{\epsilon}}\epsilon c|}\sum_{k\in\mathbb{Z}}e^{-l|2^{\frac{1}{\epsilon}}\epsilon(y+\frac{x_0}{\epsilon}-k)|}e^{-l|2^{\frac{1}{\epsilon}}\epsilon(y+\frac{x_0}{\epsilon}+c-k)|} \\
&\leq c''e^{-l|2^{\frac{1}{\epsilon}}\epsilon c|}.
\end{aligned}$$

It follows that $J_3 \to 0$ as $\epsilon \to 0$. In the similar way, we have

$$(J_1 + J_4 + J_5) \to 0 \quad as \quad \epsilon \to 0.$$

Now, since

$$\begin{aligned}
J_2 &= e^{l|y|}\Big|\int_{y-c}^{y+c} q^{\epsilon}_{\frac{1}{\epsilon}}(x+\frac{x_0}{\epsilon}, y+\frac{x_0}{\epsilon})(\varphi^{(p)}(x)-\varphi^{(p)}(y)dx\Big| \\
&\leq e^{l|y|}c_1|\varphi^{(p+1)}(\xi)|\int_{y-c}^{y+c}\Big|q^{\epsilon}_{\frac{1}{\epsilon}}(x+\frac{x_0}{\epsilon}, y+\frac{x_0}{\epsilon})\Big|dx \\
&\leq c_1'e^{l|y|}e^{-s|y-c|}\int_{y-c}^{\infty}\Big|q^{\epsilon}_{\frac{1}{\epsilon}}(x+\frac{x_0}{\epsilon}, y+\frac{x_0}{\epsilon})\Big|dx,
\end{aligned}$$

where $\xi \in (y-c, y+c)$, it follow from the similar way of estimation of J_3 that $J_2 \to 0$ as $\epsilon \to 0$. □

Remark 3.1. It should be noticed that K. Saneva and J. Vindas have recently shown in [15] that the wavelet expansions of tempered distributions are convergent when viewed in the quotient space $\mathcal{S}'$ modulo polynomials. They have used this result to characterize quasiasymptotics at finite points in terms of size estimates for wavelet coefficients. Their approach depends upon the existence of mother wavelets in $\mathcal{S}$. For instance, Lemarié-Meyer wavelets are of this kind. Since it is impossible to have mother wavelets in $\mathcal{K}$, their results seem to have no analogs in the context of exponential distributions

4. Quasiasymptotics of exponential distributions at infinity

As mentioned in [10], a function may have the quasaiasymptotics different from its classical asymptotic behavior. For example, even if e^{ix} does not have classical asymptotic behavior at infinity [5], it has the quasaiasymptotics at infinity. In this section, by following the approach of Pilipović and N. Teofanov from [10], we characterize the quasaiasymptotics of a exponential distribution f at infinity via its multiresolution expansion $q_j f$, and vice versa, with an additional assumption.

Definition 4.1. [19]. Let $f \in \mathcal{K}'$ and let $c(x), x \in (0,a), a > 0$, be a continuous function. we say that f has **the quasaiasymptotics at infinity in $\mathcal{K}'$ related to $c(k)$ equal to** g, if there exists $g \in \mathcal{K}', g \neq 0$, such that

$$\lim_{k\to\infty} \left\langle \frac{f(kx)}{c(k)}, \varphi(x) \right\rangle = \langle g(x), \varphi(x)\rangle, \quad \varphi(x) \in \mathcal{K}.$$

Now, we are ready to characterize the quasaiasymptotics of exponential distributions at infinity via its multiresolution expansion and vice versa, with an additional assumption.

It follows from a results of J. Vindas that if $f \in \mathcal{K}'$ has quasiasymptotics at infinity in $\mathcal{K}'$, then it must be a tempered distribution and f automatically has the same quasiasymptotic behavior in $\mathcal{S}'$ (see Remark 3.1 in [19]). Now, if we combine this fact with the results from [10], we obtain at once the ensuing theorem. It characterizes the quasiasymptotics of exponential distributions at infinity in terms of multiresolution expansions.

Theorem 4.1. *Let $f \in \mathcal{K}'$ be of order r_0, let $r \geq r_0$ and let $g(\neq 0), g_j(\neq 0) \in \mathcal{K}'$. Also, let $q_j f$, given in (3), be a projection of f onto an exp-r-regular MRA of L^2. If f has the quasaiasymptotics at infinity in $\mathcal{K}'$ related to $c(k)$ equal to $g(x)$, then $q_j f, j \in \mathbb{R}$, has the quasaiasymptotics at infinity in $\mathcal{K}'$ related to $c(\epsilon)$ equal to $g(x)$. Moreover, if $q_j f, j \in \mathbb{R}$, has the quasaiasymptotics at infinity in $\mathcal{K}'$ related to $c(k)$ equal to $g_j(x)$ such that $g_j \to g$ in $\mathcal{K}'_r$ as $j \to \infty$ and the family $\left\{\frac{f(kx)}{c(k)} : k \in \mathbb{N}\right\}$ is bounded in $\mathcal{K}'_r$, then f has the quasaiasymptotics at infinity in $\mathcal{K}'$ related to $c(k)$ equal to $g(x)$.*

5. Boundedness of the wavelet transform and wavelet coefficients of the quasiasymptotically bounded exponential distributions at 0

In [13, 14] , K. Saneva analyzed the boundedness of the wavelet transform and wavelet coefficients at 0 of the distributions in $\mathcal{S}'$ with respect to the wavelet in the space of highly time-frequence localized functions. In this section, we follow the approach of K. Saneva from [13, 14] to analyse the boundedness of the wavelet transform and wavelet coefficients of the quasiasymptotically bounded exponential distributions $f \in \mathcal{K}_r'$ at 0 with respect to the wavelet $\phi \in \mathcal{K}_r$. See [3, 23] for a complete wavelet analysis of asymptotic properties of tempered distributions via orthogonal wavelet expansions and wavelet transforms.

The wavelet transform of $F \in L^2$ with respect to the wavelet $\Phi \in L^2$ is defined by

$$\mathcal{W}_\Phi F(a,b) = \int_{-\infty}^{\infty} F(x)\bar{\Phi}_{b,a}(x)dx,\ b \in \mathbb{R},\ a > 0,$$

where $\Phi_{b,a}(x) = \frac{1}{a}\Phi\left(\frac{x-b}{a}\right)$. We can define the wavelet transform of the exponential distribution $f \in \mathcal{K}_r'$ with respect to the wavelet $\phi \in \mathcal{K}_r$ by

$$\mathcal{W}_\phi f(a,b) = \langle f(x), \bar{\phi}_{b,a}(x)\rangle, \quad x \in \mathbb{R},$$

where $\phi_{b,a}(x) = \frac{1}{a}\phi\left(\frac{x-b}{a}\right)$. We will give a definition of quasiasymptotic boundedness of exponential distributions at 0.

Definition 5.1. [20]. Let $f \in \mathcal{K}_r'$ and $c(\epsilon), \epsilon \in (0,a), a > 0$ be a continuous positive function. We say that f is **quasiasymptotically bounded at** 0 **related to** $c(\epsilon)$ **in** $\mathcal{K}_r'$ if there exists $M > 0$ such that

$$\left|\left\langle \frac{f(\epsilon x)}{c(\epsilon)}, \phi(x)\right\rangle\right| \leq M\|\phi\|_{\mathcal{K}_r},\ 0 < \epsilon < 1,$$

for every $\phi \in \mathcal{K}_r$.

For properties of quasiasymptotically bounded tempered distributions, we refer to [20]. Obviously, if f has a quasiasymptotics at 0 related to $c(\epsilon)$ in $\mathcal{K}_r'$, then f is quasiasymptotically bounded at 0 related to $c(\epsilon)$ in $\mathcal{K}_r'$.

We will analyze the boundedness of the wavelet transform $\mathcal{W}_\phi f$ of the quasi-asymptotically bounded exponential distributions at 0 with respect to the wavelet $\phi \in \mathcal{K}_r$.

Theorem 5.1. *Let $f \in \mathcal{K}'_r$ and $c(\epsilon), \epsilon \in (0, \epsilon')$ be a continuous positive function. If f is quasiasymptotically bounded at 0 related to $c(\epsilon)$, then there exists $C > 0$ such that*

$$|\mathcal{W}_\phi f(b,a)| \leq C \frac{1}{a^{r+1}} e^{r\left|\frac{b}{a}\right|}, \quad 0 < a < 1,$$

for every $\phi \in \mathcal{K}_r$.

Proof. Let $\phi_{b,a}(x) = \frac{1}{a}\phi\left(\frac{x-b}{a}\right)$ and let $\phi \in \mathcal{K}_r$. By the change of variables $x = \epsilon t$ and Definition 5.1, we obtain that for $0 < a < 1$ and $0 < \epsilon < 1$, there exists $M > 0$ such that

$$\begin{aligned}
\left|\frac{\mathcal{W}_\phi f(\epsilon b, \epsilon a)}{c(\epsilon)}\right| &= \left|\left\langle \frac{f(x)}{c(\epsilon)}, \bar{\phi}_{\epsilon b, \epsilon a}(x)\right\rangle\right| \\
&= \left|\left\langle \frac{f(\epsilon t)}{c(\epsilon)}, \bar{\phi}_{b,a}(t)\right\rangle\right| \\
&\leq M \left\|\bar{\phi}_{b,a}\right\|_{\mathcal{K}_r} \\
&= M \left\|\frac{1}{a}\bar{\phi}\left(\frac{t-b}{a}\right)\right\|_{\mathcal{K}_r} \\
&= M \sup_{t\in\mathbb{R},\ 0\leq\alpha\leq r} e^{r|t|}\left|\frac{1}{a}\bar{\phi}\left(\frac{t-b}{a}\right)^{(\alpha)}\right| \\
&\leq \frac{M}{a^{r+1}} \sup_{t\in\mathbb{R}} e^{r|t|} e^{-r\left|\frac{t-b}{a}\right|} \\
&\leq \frac{M}{a^{r+1}} e^{r\left|\frac{b}{a}\right|}.
\end{aligned}$$

If we choose $0 < \epsilon_0 < 1$ and put $\epsilon_0 b = u$ and $\epsilon_0 a = v (0 < v < 1)$, we obtain that there exists $C > 0$ such that

$$|\mathcal{W}_\phi f(u,v)| \leq M \epsilon_0^{r+1} c(\epsilon_0) \frac{1}{v^{r+1}} e^{r\left|\frac{u}{v}\right|} \leq C \frac{1}{v^{r+1}} e^{r\left|\frac{u}{v}\right|}.$$

□

Now we will analyze the boundedness of the wavelet coefficients of the quasi-asymptotically bounded exponential distributions at 0 with respect to the wavelet in $\mathcal{K}_r$.

According to [2], we define the discrete wavelet transform of $F \in L^2$ with respect to the wavelet $\Psi \in L^2$ as a double-indexed sequence

$$c_{m,n} = \langle F, \Psi_{m,n} \rangle = \int_{-\infty}^{\infty} F(x)\bar{\Psi}_{m,n}(x)dx,$$

where $\Psi_{m,n}(x) = 2^{m/2}\Psi(2^m x - n), m, n \in \mathbb{Z}$. If the wavelet Ψ is orthonormal, then every $F \in L^2$ can be written as

$$F(x) = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} c_{m,n}\Psi_{m,n}(x)$$

with convergence in L^2-norm.

The wavelet expansion of $f \in \mathcal{K}_r'$ with respect to the orthonormal wavelet $\psi \in \mathcal{K}_r$ is defined by

$$\left\langle \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \langle f, \psi_{m,n} \rangle \psi_{m,n}, \phi \right\rangle = \left\langle f, \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \langle \phi, \psi_{m,n} \rangle \psi_{m,n} \right\rangle, \phi \in \mathcal{K}_r, \tag{13}$$

where $\psi_{m,n}(x) = 2^{m/2}\psi(2^m x - n), m, n \in \mathbb{Z}$.

Definition 5.2. We say that $c_{m,n} = \langle f, \psi_{m,n} \rangle, m, n \in \mathbb{Z}$ in (13) are **wavelet coefficients** of $f \in \mathcal{K}_r'$ with respect to $\psi \in \mathcal{K}_r$.

Theorem 5.2. *Let $f \in \mathcal{K}_r'$ and $c(\epsilon), \epsilon \in (0, \epsilon')$ be a continuous positive function. If f is quasiasymptotically bounded at 0 related to $c(\epsilon), 0 < \epsilon < a$, then there exists $C > 0$ such that for the wavelet coefficients $c_{m,n}, m, n \in \mathbb{Z}$ of $f \in \mathcal{K}_r'$ and $\psi \in \mathcal{K}_r$*

$$c_{m,n} \le C \ 2^{m(r+\frac{1}{2})} \ e^{r|n|}, \quad m > 0, n \in \mathbb{Z}.$$

Proof. Let $\psi_{m,n}(x) = 2^{m/2}\psi(2^m x - n)$ and let $\psi \in \mathcal{K}_r$. Take $\eta(m, \epsilon) = m + \log_2(1/\epsilon)$ for $m > 0$ and $0 < \epsilon < 1$. It follows from the change of variables $x = \epsilon t$ and Definition 5.2 that

$$
\begin{aligned}
\left|\frac{c_{\eta(m,\epsilon),n}}{\sqrt{\epsilon}c(\epsilon)}\right| &= \left|\left\langle \frac{f(x)}{\sqrt{\epsilon}c(\epsilon)}, \psi_{\eta(m,\epsilon),n}(x)\right\rangle\right| \\
&= \left|\left\langle \frac{f(x)}{\sqrt{\epsilon}c(\epsilon)}, \frac{2^{m/2}}{\sqrt{\epsilon}}\psi\left(\frac{2^m x}{\epsilon}-n\right)\right\rangle\right| \\
&= \left|\left\langle \frac{f(\epsilon t)}{c(\epsilon)}, \psi_{m,n}(t)\right\rangle\right| \\
&\leq M\|\psi_{m,n}\|_{\mathcal{K}_r} \\
&= M \sup_{t\in\mathbb{R},0\leq\alpha\leq r} e^{r|t|}\left|2^{m/2}\psi\left(2^m t-n\right)^{(\alpha)}\right| \\
&= M2^{m/2} \sup_{t\in\mathbb{R},0\leq\alpha\leq r} e^{r|t|}2^{m\alpha}\left|\psi^{(\alpha)}\left(2^m t-n\right)\right| \\
&\leq M2^{m(r+1/2)} \sup_{t\in\mathbb{R}} e^{r|t|}e^{-r|2^m t-n|} \\
&\leq M2^{m(r+1/2)}e^{r|n|}.
\end{aligned}
$$

If we choose $0<\epsilon_0<1$ and put $l=\eta(m,\epsilon_0)=m+\log_2(1/\epsilon_0)=m-\log_2\epsilon_0>0$, we obtain that there exists $C>0$ such that

$$
|c_{l,n}|=\sqrt{\epsilon}_0 c(\epsilon_0)M2^{(l+\log_2\epsilon_0)(r+1/2)}e^{r|n|}\leq C2^{l(r+1/2)}e^{r|n|}.
$$

□

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