Nihonkai Math. J.
Vol.23(2012), 1–10

PROPERTIES ON C^*-ALGEBRAS WITH OR WITHOUT RESIDUAL DIMENSION FINITE

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ABSTRACT. We study and develope a general theory for C^*-algebras by considering their residual dimension. For this we introduce a renewed notion that is equivalent to RFD, to divide C^*-algebras into two classes to be clarified as a main purpose.

1. Introduction

We study and develope a general theory for (Banach or) C^*-algebras by considering their residual dimension. For this we introduce a renewed notion RDF (residual dimension finite) to divide C^*-algebras into two classes to be clarified as a main purpose. The notion for C^*-algebras is equivalent to their being residually finite dimensional (RFD), but this time we would like to emphasize being RFD as an independent role of dimension and to rename it to having RDF, slightly grammatically different and possibly more convenient in usage, and to focus on it to study its general properties in a systematic way to obtain a panorama of C^*-algebras.

This paper is organized as follows. In Section 2, we consider C^*-algebras with RDF. In Section 3, we consider C^*-algebras without RDF. In Section 4, we introduce yet another notion ARDF (approximate residual dimension finite). In these sections we obtain several basic results on those C^*-algebras, which could be useful for further study on this topic.

2. C^*-algebras with RDF

Definition 2.1. Let $\mathfrak{A}$ be a (Banach or) C^*-algebra. We say that $\mathfrak{A}$ has residual dimension finite (RDF) if for any nonzero $a \in \mathfrak{A}$, there is a finite dimensional representation π of $\mathfrak{A}$ such that $\pi(a) \neq 0$.

2010 *Mathematics Subject Classification.* Primary 46L05, 46L80.
Key words and phrases. C^*-algebra, residual dimension, irreducible representation.

Remark. We may assume that π is irreducible. It has been said that such an algebra is residually finite dimensional (RFD) in the literature. This is also equivalent to say that $\mathfrak{A}$ has a separating family of finite dimensional representations of $\mathfrak{A}$.

Proposition 2.2. *If $\mathfrak{A}$ is a C^*-algebra with RDF, then a C^*-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ has RDF. In particular, a closed ideal of $\mathfrak{A}$ has RDF.*

Proof. Let $\mathfrak{B} \ni b \neq 0$. Then there is a finite dimensional representation π of $\mathfrak{A}$ such that $\pi(b) \neq 0$. Note that the restriction of π to $\mathfrak{B}$ is also a finite dimensional representation of $\mathfrak{B}$. □

Proposition 2.3. *If $\mathfrak{A}$ is an extension of a closed ideal $\mathfrak{I}$ with RDF by a quotient $\mathfrak{D}$ with RDF, then it has RDF.*

Proof. Let $a \neq 0 \in \mathfrak{A}$. If $a \in \mathfrak{I}$, then there is a finite dimensional representation π of $\mathfrak{I}$ such that $\pi(a) \neq 0$, and π corresponds to that of $\mathfrak{A}$ on the same Hilbert space. Note that as a fact in C^*-algebra theory, every irreducible representation π of $\mathfrak{A}$ with $\pi(\mathfrak{I}) \neq \{0\}$ is identified with that of $\mathfrak{I}$ by restriction (see [4, Section 2.11] or [6, Section 5.5]). A point of the proof for this as in [6] is that the state associated to an irreducible representation of $\mathfrak{I}$ can be extended to a state of $\mathfrak{A}$ from which one can construct an irreducible representation of $\mathfrak{A}$ via the GNS construction (as in [6]), and this extension preserves irreducibility of representation.

If $a \notin \mathfrak{I}$, then $q(a) \neq 0$, where q is the quotient map from $\mathfrak{A}$ to $\mathfrak{D}$. Then there is a finite dimensional representation ρ of $\mathfrak{D}$, such that $\rho(q(a)) \neq 0$, with $\rho \circ q$ a finite dimensional representation of $\mathfrak{A}$. □

Proposition 2.4. *If $\mathfrak{A}$ is a C^*-algebra with RDF and it has a split quotient $\mathfrak{D}$, then $\mathfrak{D}$ has RDF.*

Proof. Let $\mathfrak{D} \ni d \neq 0$. Then there is $a \in \mathfrak{A}$ such that $\pi(a) = d$, where $\pi : \mathfrak{A} \to \mathfrak{D}$ is the quotient map. Since $a \neq 0$, there is a finite dimensional representation π of $\mathfrak{A}$ such that $\pi(a) \neq 0$. Let $\rho : \mathfrak{D} \to \mathfrak{A}$ be the split homomorphism. Then $\pi \circ \rho$ is a finite dimensional representation of $\mathfrak{D}$ with $\pi \circ \rho(d) = \pi(a) \neq 0$. □

Remark. The same statement with $\mathfrak{D}$ a quotient of $\mathfrak{A}$ is not true in general. For example, it is known that the full C^*-algebra $C^*(F_2)$ of the free group F_2 with two generators is RFD ([2]), but there is a quotient which is not RFD. Such a quotient can be taken by universality as the irrational rotation algebra generated by two unitaries with a certain commutation relation, which is an infinite dimensional, simple C^*-algebras, so that it has no finite dimensional irreducible representations. In the propositions above (without using C^*-algebra technique), $\mathfrak{A}$ may be a Banach algebra.

Let $\mathfrak{A}$ be a (Banach or) C^*-algebra. We denote by $\mathfrak{A}^\wedge$ the set of all unitary equivalence classes of irreducible representations of $\mathfrak{A}$ and by $\mathfrak{A}_f^\wedge$ the set of all unitary equivalence classes of finite dimensional irreducible representations of $\mathfrak{A}$. It is well known that $\mathfrak{A}^\wedge$ separates elements of $\mathfrak{A}$ (see [6]).

Lemma 2.5. *A commutative C^*-algebra $\mathfrak{A}$ and a finite dimensional C^*-algebra $\mathfrak{B}$ have RDF.*

Proof. Note that $\mathfrak{A}^\wedge = \mathfrak{A}_f^\wedge$ with dimension 1 and $\mathfrak{B}^\wedge = \mathfrak{B}_f^\wedge$. $\square$

Proposition 2.6. *A C^*-algebra $\mathfrak{A}$ has RDF if and only if $\mathfrak{A}$ can be embedded into, i.e. be viewed as a C^*-subalgebra of, a direct product C^*-algebra $\Pi_j M_{n_j}(\mathbb{C})$ of some matrix algebras $M_{n_j}(\mathbb{C})$ over $\mathbb{C}$ with size $n_j \geq 1$, where if $\mathfrak{A}$ is separable, then the direct product can be a countable product.*

Proof. Consider the direct product representation $\Phi = \Pi_{\pi \in \mathfrak{A}_f^\wedge} \pi$ of $\mathfrak{A}$ into the direct product C^*-algebra $\Pi_{\pi \in \mathfrak{A}_f^\wedge} \pi(\mathfrak{A})$. If $\mathfrak{A}$ has RDF, then Φ is injective, so that $\mathfrak{A}$ can be embedded in the direct product. Conversely, if $\mathfrak{A}$ is a C^*-subalgebra of a direct product C^*-algebra of some matrix algebras over $\mathbb{C}$, it has RDF by considering projections to direct product factors. Note that any nonzero element of $\mathfrak{A}$ has a nonzero component of some direct product factor.

If $\mathfrak{A}$ is a separable C^*-algebra with RDF, let $\{x_j\}$ be a countable dense subset of $\mathfrak{A}$. Take a corresponding subset $\{\pi_j\}$ of $\mathfrak{A}_f^\wedge$. Let $\Phi = \Pi_j \pi_j$. Then Φ is injective on the dense subset, and extends to $\mathfrak{A}$. Indeed, let $a \neq 0 \in \mathfrak{A}$. Take $0 < \varepsilon < \|a\|$. Since $a = \lim_{k\to\infty} x_{j(k)}$ for some subsequence $\{x_{j(k)}\}_k$ of $\{x_j\}$ by density, we compute the following norm:

$$\begin{aligned}
\|\Phi(a)\| &= \|\Phi(a - x_{j(k)}) + \Phi(x_{j(k)})\| \\
&\geq \|\Phi(x_{j(k)})\| - \|\Phi(a - x_{j(k)})\| \\
&= \|x_{j(k)}\| - \|\Phi(a - x_{j(k)})\|
\end{aligned}$$

where the last equality follows from the injectiveness of Φ on the dense subset (see the proof of [6, Theorem 3.1.5]), and the first term $\|x_{j(k)}\|$ goes to $\|a\|$ as $k \to \infty$ and the second term goes to zero as $k \to \infty$ since $\|\Phi(a - x_{j(k)})\| \leq \|a - x_{j(k)}\|$, and hence $\|\Phi(a)\| > \varepsilon > 0$. This shows that Φ is injective on $\mathfrak{A}$. $\square$

Remark. A C^*-subalgebra of such a countably infinite direct product C^*-algebra has RDF, but not necessarily separable, because the direct product itself is non-separable.

Proposition 2.7. *Let $\mathfrak{A}$ be a continuous field C^*-algebra over a locally compact Hausdorff space X with fibers matrix algebras $M_{n_x}(\mathbb{C})$ over $\mathbb{C}$ with $n_x \geq 1$ for $x \in X$ (vanishing at infinity). Then $\mathfrak{A}$ has RDF.*

Proof. Note that $\mathfrak{A}^\wedge$ is identified with X (see [4, Chapter 10]), and $\mathfrak{A}$ can be embedded into $\Pi_{x\in X} M_{n_x}(\mathbb{C})$. $\square$

Example 2.8. A direct sum of some matrix algebras $M_{n_j}(\mathbb{C})$ for $j \in \mathbb{N}$, which is also a continuous field C^*-algebra over $\mathbb{N}$ with fibers $M_{n_j}(\mathbb{C})$ (vanishing at infinity) has RDF. The group C^*-algebra of a compact group G has RDF, because it is isomorphic to such a continuous field C^*-algebra over the discrete dual group of G with fibers matrix algebras over $\mathbb{C}$.

Moreover,

Proposition 2.9. *If $\mathfrak{A}^\wedge = \mathfrak{A}^\wedge_f$, then $\mathfrak{A}$ has RDF.*

In particular, n-homogeneous or n-subhomogeneous C^*-algebras with n finite have RDF.

Remark. Recall that a C^*-algebra is n-homogenous if its irreducible representations are all n-dimensional, and a C^*-algebra is n-subhomogeneous if it is a C^*-subalgebra of an n-homogeneous C^*-algebra.

Definition 2.10. We say that a C^*-algebra $\mathfrak{A}$ has irreducible representation dimension finite (IRDF) if $\mathfrak{A}^\wedge = \mathfrak{A}^\wedge_f$.

Remark. The class of C^*-algebras with IRDF is closed under taking quotients, subalgebras, and extensions. This is deduced from several facts in the representation theory of C^*-algebras (see [4]).

Proposition 2.11. *If $\mathfrak{A}$ has RDF and $\mathfrak{B}$ is a finite dimensional C^*-algebra, then their tensor product $\mathfrak{A} \otimes \mathfrak{B}$ has RDF.*

Proof. Note that $\mathfrak{A} \otimes \mathfrak{B} \cong \oplus_{j=1}^{l} (\mathfrak{A} \otimes M_{n_j}(\mathbb{C}))$ for some $n_j \geq 1$ and $l \geq 1$. Note also that any $\pi \in \mathfrak{A}^\wedge$ corresponds to $\pi \otimes \mathrm{id} \in (\mathfrak{A} \otimes M_{n_j}(\mathbb{C}))^\wedge$, under which the spectrums are homeomorphic. $\square$

Theorem 2.12. *Let $\mathfrak{A}$ be a C^*-algebra with RDF and F be a finite abelian group. Then the crossed product $\mathfrak{A} \rtimes_\alpha F$ of $\mathfrak{A}$ by an action α of F has RDF.*

Proof. The Takai dualtiy theorem implies that

$$(\mathfrak{A} \rtimes_\alpha F) \rtimes_{\alpha^\wedge} F^\wedge \cong \mathfrak{A} \otimes M_{|F|}(\mathbb{C}),$$

where $F^\wedge$ is the dual group of F and $\alpha^\wedge$ is the dual action of $F^\wedge$, and $|F|$ is the order of F (see [10] or [7]). Since $F^\wedge \cong F$ is finite, $\mathfrak{A} \rtimes_\alpha F$ is viewed as a C^*-subalgebra of the tensor product above with RDF. $\square$

Corollary 2.13. *The crossed product of $C^*(F_2)$ by the flip action of $\mathbb{Z}_2$ on two unitary generators has RDF.*

Corollary 2.14. *Let $\mathfrak{A}$ be a C^*-algebra with RDF. Then the crossed product $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ with α reduced to an action of $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ has RDF.*

Proof. Recall that the crossed product is isomorphic to the mapping torus on $\mathfrak{A} \rtimes_\alpha \mathbb{Z}_n$, and we have

$$0 \to C_0(\mathbb{R}) \otimes (\mathfrak{A} \rtimes_\alpha \mathbb{Z}_n) \to \mathfrak{A} \rtimes_\alpha \mathbb{Z} \to \mathfrak{A} \rtimes_\alpha \mathbb{Z}_n \to 0$$

(see [1]). By the theorem above, the closed ideal and the quotient have RDF, so does $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$. □

Theorem 2.15. *Let Γ be a finitely generated, two-step nilpotent group and $C^*(\Gamma)$ its group C^*-algebra. Then $C^*(\Gamma)$ has RDF.*

Proof. It is known that $C^*(\Gamma)$ can be viewed as a continuous field over the dual group $Z^\wedge$ of the center Z of Γ with fibers $C^*(\Gamma)_\lambda$ for $\lambda \in Z^\wedge$ that are also viewed as a continuous field over the dual group of the quotient $C(\lambda)/Z$ of the centralizer $C(\lambda)$ in Γ with respect to λ by Z, where

$$C(\lambda) = \{g \in \Gamma \,|\, \lambda(ghg^{-1}h^{-1}) = 1 \quad \text{for any } h \in \Gamma\},$$

with fibers either matrix algebras over $\mathbb{C}$ or matrix algebras over a simple non-commutative tori, which is obtained as a successive crossed product C^*-algebra by actions of $\mathbb{Z}$, and is viewed as a sort of quantization to the ordinary tori. This indeed follows from the irreducible representaion theory and the structure of the primitive ideal space of $C^*(\Gamma)$ (cf. [8]). If $0 \neq a \in C^*(\Gamma)$, then $0 \neq \pi_\lambda(a) \in C^*(\Gamma)_\lambda$ for some $\lambda \in Z^\wedge$ whose fiber is $M_n(\mathbb{C})$ for some $n \geq 1$, where $\pi_\lambda : C^*(\Gamma) \to C^*(\Gamma)_\lambda$ is the quotient map. Note that if $Z \cong \mathbb{Z}^k$ for some $k \geq 1$, we have $Z^\wedge \cong \mathbb{T}^k$, so that the set of points corresponding to rational rotations is dense in $Z^\wedge$, and if Z contains a torsion part, it corresponds to some direct summands of $C^*(\Gamma)$. Let $p_\lambda : C^*(G)_\lambda \to M_n(\mathbb{C})$ be the quotient map. Then $p_\lambda \circ \pi_\lambda$ is finite dimensional with $p_\lambda \circ \pi_\lambda(a) \neq 0$. □

Example 2.16. Let $H_3^{\mathbb{Z}}$ be the discrete Heisenberg group of rank 3, with center $Z = \mathbb{Z}$. Then $C^*(H_3^{\mathbb{Z}})$ is viewed as a continuous field over $\mathbb{T} \cong \mathbb{Z}^\wedge$ with fibers the rational or irrational rotation C^*-algebras $\mathfrak{A}_\theta$ which correspond respectively to rational or irrational rotations θ on $\mathbb{T}$, also called noncommutative 2-tori. If θ is rational, then $\mathfrak{A}_\theta$ is viewed as a continuous field over $\mathbb{T}^2 = (C(\theta)/Z)^\wedge$ with fibers a matrix algebra over $\mathbb{C}$, and if θ is irrational, then $\mathfrak{A}_\theta$ is simple with $(C(\theta)/Z)^\wedge = \{1\}$ (cf. [8]). It follows that $C^*(H_3^{\mathbb{Z}})$ has RDF, since the rational rotation C^*-algebras are finite homogeneous.

Example 2.17. The author [9] has defined the discrete $ax + b$ group of rank 2 and the discrete Mautner group of rank 4 and their generalizations as the semi-direct products $\mathbb{Z}^n \rtimes \mathbb{Z}$ $(n \geq 1)$ and $\mathbb{Z}^n \rtimes \mathbb{Z}^n$ $(n \geq 2)$ with diagonal actions respectively, which are non-nilpotent, solvable discrete groups, and shown that their group C^*-algebras have finite composition of closed ideals such that their subquotients are finite homogeneous, so that the group C^*-algebras have RDF. Moreover, the discrete Dixmier group of rank 7 and similar semi-direct products defined as $(\mathbb{Z}^2 \times \mathbb{Z}^2) \rtimes H_3^{\mathbb{Z}}$ and $\mathbb{Z}^{2n} \rtimes H_3^{\mathbb{Z}}$ $(n \geq 1)$, with $H_3^{\mathbb{Z}}$ the discrete Heisenberg group of rank 3, have finite composition series such that subquotients can be viewed as continuous field C^*-algebras as in the case of $C^*(H_3^{\mathbb{Z}})$, but with fibers which may be viewed as noncommutative disjoint tori, so that their group C^*-algebras have RDF, because those continuous field C^*-algebras have RDF as does $C^*(H_3^{\mathbb{Z}})$ in the example above.

Proposition 2.18. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital C^*-algebras and $\mathfrak{A} \otimes \mathfrak{B}$ be their minimal or maximal tensor product C^*-algebra. Then $\mathfrak{A} \otimes \mathfrak{B}$ has RDF if and only if $\mathfrak{A}$ and $\mathfrak{B}$ have RDF.*

Proof. Note that any representation of $\mathfrak{A} \otimes \mathfrak{B}$ gives by restriction those of $\mathfrak{A}$ and $\mathfrak{B}$. Also, representations of $\mathfrak{A}$ and $\mathfrak{B}$ extends to that of $\mathfrak{A} \otimes \mathfrak{B}$ by continuity (see [6] and [11]). $\square$

Remark. It is shown by Exel and Loring [5] that for two C^*-algebras, their full free product has RDF if and only if both of them have RDF, and if they are unital, their unital full free product has RDF if and only if both of them have RDF.

3. C^*-algebras without RDF

Lemma 3.1. *An infinite dimensional, simple C^*-algebra does not have RDF.*

Proof. Because $\mathfrak{A}_f^\wedge = \emptyset$. $\square$

Let $\mathbb{K} = \mathbb{K}(H)$ denote the C^*-algebra of all compact operators on an infinite dimensional Hilbert space H.

Lemma 3.2. *For any C^*-algebra $\mathfrak{A}$, its tensor product with $\mathbb{K}$ does not have RDF. In particular, a stable C^*-algebra $\mathfrak{A}$, i.e. $\mathfrak{A} \cong \mathfrak{A} \otimes \mathbb{K}$, does not have RDF.*

Proof. Note that any $\pi \in \mathfrak{A}^\wedge$ corresponds to $\pi \otimes \mathrm{id} \in (\mathfrak{A} \otimes \mathbb{K})^\wedge$, under which the spectrums are homeomorphic. $\square$

Proposition 3.3. *If a unital C^*-algebra $\mathfrak{A}$ has a tensor factor which is an infinite dimensional simple nuclear C^*-algebra, then it does not have RDF.*

Proof. Note that in this case, such a tensor factor is a C^*-subalgebra of $\mathfrak{A}$ without RDF. $\square$

Theorem 3.4. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{A} \rtimes_\alpha G$ be the crossed product of $\mathfrak{A}$ by an action of a compact abelian group G that is not finite. Then $\mathfrak{A} \rtimes_\alpha G$ does not have RDF.*

Proof. The Takai dualtiy theorem implies that

$$(\mathfrak{A} \rtimes_\alpha G) \rtimes_{\alpha^\wedge} G^\wedge \cong \mathfrak{A} \otimes \mathbb{K},$$

where $\mathbb{K} = \mathbb{K}(L^2(G))$ on the Hilbert space $L^2(G)$. Note that the dual group $G^\wedge$ of G compact is discrete, Thus $\mathfrak{A} \rtimes_\alpha G$ is viewed as a C^*-subalgebra of the tensor product above, and its representation theory is identified with that tensored with $\mathbb{K}$. □

On the other hand,

Example 3.5. The crossed product $C^*(\mathbb{Z}) \rtimes_\alpha \mathbb{Z}$ with α the shift is isomorphic to $\mathbb{K}$, so that it does not have RDF, while $C^*(\mathbb{Z}) \cong C(\mathbb{T})$ by the Fourier transform has RDF, where $\mathbb{T}$ is the 1-torus. The crossed product $C^*(\mathbb{R}) \rtimes_\alpha \mathbb{R}$ with α the shift is isomorphic to $\mathbb{K}$, while $C^*(\mathbb{R}) \cong C_0(\mathbb{R})$ by the Fourier transform.

Let $\mathfrak{A}$ be a C^*-algebra. Denote by $\mathfrak{A}^\wedge_\infty$ the subspace of all infinite dimensional irreducible representations of $\mathfrak{A}$ in $\mathfrak{A}^\wedge$.

Lemma 3.6. *If $\mathfrak{A}^\wedge = \mathfrak{A}^\wedge_\infty$, then $\mathfrak{A}$ does not have RDF.*

In particular, an ∞-homogeneous C^*-algebra does not have RDF.

Remark. Recall that a C^*-algebra is ∞-homogenous if its images under irreducible representations are all $\mathbb{K}$.

Definition 3.7. We say that a C^*-algebra $\mathfrak{A}$ has irreducible representation dimension infinite (IRDI) if $\mathfrak{A}^\wedge = \mathfrak{A}^\wedge_\infty$.

Remark. The class of C^*-algebras with IRDI is closed under taking closed ideals, quotients, and extensions.

Example 3.8. Let G be a connected real semi-simple Lie group and $C_r^*(G)$ be the reduced group C^*-algebra of G. It is known that $C_r^*(G)$ has IRDI and hence does not have RDF.

Theorem 3.9. *Let G be a connected solvable Lie group that is not commutative. Then the group C^*-algebra $C^*(G)$ of G does not have RDF.*

Proof. Recall that any irreducible representation of G is either one or infinite dimensional, and the spectrum of G is identified with that of $C^*(G)$. It follows that we have the following short exact sequence:

$$0 \to \mathfrak{I} \to C^*(G) \to C_0(G_1^\wedge) \to 0,$$

where $G_1^\wedge$ is the space of all one-dimensional representations of G, and $\mathfrak{I}$ is the closed ideal corresponding to $C^*(G)^\wedge_\infty$, so that $\mathfrak{I}^\wedge = \mathfrak{I}^\wedge_\infty$. Since $\mathfrak{I}$ does not have RDF, so does not $C^*(G)$. $\square$

Example 3.10. If G is a connected, simply connected Lie group of dimension n, then G is a successive semi-direct product by $\mathbb{R}$, i.e. $G \cong \mathbb{R} \rtimes \mathbb{R} \cdots \rtimes \mathbb{R}$, and then $C^*(G) \cong C^*(\mathbb{R}) \rtimes \mathbb{R} \cdots \rtimes \mathbb{R}$ a successive crossed product by $\mathbb{R}$.

Let G be the real 3-dimensional Heisenberg Lie group, with $G \cong \mathbb{R}^2 \rtimes \mathbb{R}$. It is well known that $C^*(G)$ has the following decomposition:

$$0 \to C_0(\mathbb{R} \setminus \{0\}) \otimes \mathbb{K} \to C^*(G) \to C_0(\mathbb{R}^2) \to 0,$$

and the closed ideal has IRDI. Thus $C^*(G)$ does not have RDF.

Let $\mathfrak{T}$ be the Toeplitz algebra generated by a proper isometry. It is well known that $\mathfrak{T}$ has the following short exact sequence:

$$0 \to \mathbb{K} \to \mathfrak{T} \to C(\mathbb{T}) \to 0.$$

Hence $\mathfrak{T}$ does not have RDF.

Let $\mathbb{B}$ be the C^*-algebra of all bounded operators on an infinite dimensional Hilbert space. Then we have the following short exact sequence:

$$0 \to \mathbb{K} \to \mathbb{B} \to Q \to 0$$

where Q is the Calkin algebra $\mathbb{B}/\mathbb{K}$. Thus $\mathbb{B}$ does not have RDF, and so does not Q simple.

Lemma 3.11. *Let $\mathfrak{A}$ be an infinite dimensional C^*-algebra and $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ be the crossed product of $\mathfrak{A}$ by a minimal action α of $\mathbb{Z}$ in the sense that the C^*-closure of any orbit under α is $\mathfrak{A}$. Then $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ does not have RDF.*

Proof. In this case, the crossed product is simple (cf. [3]). $\square$

4. C^*-algebras with ARDF

Definition 4.1. Let $\mathfrak{A}$ be a (Banach or) C^*-algebra. We say that $\mathfrak{A}$ has approximate residual dimension finite (ARDF) if for any nonzero $a \in \mathfrak{A}$ and $\varepsilon > 0$, there is a C^*-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, a finite dimensional representation π of $\mathfrak{B}$, and $b \in \mathfrak{B}$ such that $\pi(b) \neq 0$ and $\|a - b\| < \varepsilon$.

Remark. A C^*-algebra with RDF has ARDF, but the converse is not true. For instance, let $\mathbb{K}$ be the C^*-algebra of all compact operators on an infinite dimensional Hilbert space, which is an inductive limit of finite dimensional C^*-algebras so that $\mathbb{K}$ has ARDF, but it is simple so that it does not have RDF.

Proposition 4.2. *It $\mathfrak{A}$ is an inductive limit of C^*-algebras with RDF, then $\mathfrak{A}$ has ARDF.*

Proof. Note that $\mathfrak{A}$ is the C^*-closure of the union of C^*-subalgebras with RDF. $\square$

Proposition 4.3. *If $\mathfrak{A}$ is a C^*-algebra that has a composition series of closed ideals $\mathfrak{I}_j$ such that each subquotient $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ has RDF, then $\mathfrak{A}$ has ARDF.*

Proof. If $\mathfrak{A}$ is separable, then such a composition series can be taken to be countable, and each closed ideal $\mathfrak{I}_n$ is a finite extension by the subquotients $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ $(1 \leq j \leq n)$, so that $\mathfrak{I}_n$ has RDF. Since $\mathfrak{A}$ is the C^*-closure of the union of $\mathfrak{I}_n$, $\mathfrak{A}$ has ARDF.

If $\mathfrak{A}$ is non-separable, we use the transfinite induction method similarly. $\square$

Remark. Recall that a C^*-algebra of type I has a composition series of closed ideals such that subquotients have continuous trace (see [4] or [7]). The subquotients of type I may have IRDF or IRDI.

Proposition 4.4. *If $\mathfrak{A}$ is an extension of a closed ideal $\mathfrak{I}$ with ARDF by a quotient $\mathfrak{D}$ with ARDR, then it has ARDF.*

Proof. This follows by the similar way as in the case of extensions by C^*-algebras with RDF in the section 2 above. $\square$

Proposition 4.5. *If two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ have ARDF, then their minimal and maximal tensor products and their full free product have ARDF.*

Proof. This follows from the RDF property for tensor products and free products of C^*-algebras, obtained in the section 2 above. $\square$

Question. A C^*-subalgebra of a C^*-algebra with ARDF has ARDF ?

Probably, it is ture, but the proof seems to be non-trivial.

Acknowledgement. The author would like to thank the referee for reading the manuscript carefully and pointing out several mistakes or misprints.

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Received June 3, 2011
Revised April 11, 2012